CHAPTER 10

Exercises 10.2 (page 587)

2. $y = \dfrac{(e^{10}-1)e^{x}+(1-e^{2})e^{5x}}{e^{10}-e^{2}}$

4. $y = 2\sin 3x$

6. No solution

8. $y = e^{x-1} + xe^{x-1}$

10. $\lambda_n = \dfrac{(2n-1)^2}{4}$ and $y_n = c_n \cos\left[\dfrac{(2n-1)x}{2}\right]$, where $n = 1, 2, 3, \ldots$ and the c_n's are arbitrary.

12. $\lambda_n = 4n^2$ and $y_n = c_n \cos(2nx)$, where $n = 0, 1, 2, \ldots$ and the c_n's are arbitrary.

14. $\lambda_n = n^2 + 1$ and $y_n = c_n e^{x} \sin(nx)$, where $n = 1, 2, 3, \ldots$ and the c_n's are arbitrary.

16. $u(x,\,t) = e^{-27t}\sin 3x + 5e^{-147t}\sin 7x - 2e^{-507t}\sin 13x$

18. $u(x,\,t) = e^{-48t}\sin 4x + 3e^{-108t}\sin 6x - e^{-300t}\sin 10x$

20. $u(x,\,t) = -\left(\dfrac{2}{9}\right)\sin 9t\sin 3x + \left(\dfrac{3}{7}\right)\sin 21t\sin 7x - \left(\dfrac{1}{30}\right)\sin 30t\sin 10x$

22. $u(x,\,t) = \cos 3t\sin x - \cos 6t\sin 2x + \cos 9t\sin 3x + \left(\dfrac{2}{3}\right)\sin 9t\sin 3x - \left(\dfrac{7}{15}\right)\sin 15t\sin 5x$

24. $u(x,\,t) = \displaystyle\sum_{n=1}^{\infty}\left\{\left(\frac{1}{n^2}\right)\cos 4nt + \left[\frac{(-1)^{n+1}}{4n^2}\right]\sin 4nt\right\}\sin nx$

Exercises 10.3 (page 603)

2. Even

4. Neither

6. Odd

10. $f(x) \sim \dfrac{\pi}{2} - \dfrac{4}{\pi}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos(2k+1)x$

12. $f(x) \sim \dfrac{\pi^2}{6} + \displaystyle\sum_{n=1}^{\infty}\left\{\frac{2(-1)^n}{n^2}\cos nx + \left[\frac{2(-1)^n - n^2\pi^2(-1)^n - 2}{n^3\pi}\right]\sin nx\right\}$

14. $f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{n} \sin 2nx$

16. $f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right] \sin nx$

18. The 2π periodic function $g(x)$, where $g(x) = |x|,\ -\pi \le x \le \pi$

20. The 2π periodic function $g(x)$, where $g(x) = \begin{cases} 0 & -\pi < x \le 0 \\ x^2 & 0 \le x < \pi \\ \frac{\pi^2}{2} & x = \pm\pi \end{cases}$

22. The 2π periodic function $g(x)$, where $g(x) = \begin{cases} x + \pi & -\pi < x < 0 \\ x & 0 < x < \pi \\ \frac{\pi}{2} & x = 0, \pm\pi \end{cases}$

24. The 2π periodic function $g(x)$, where $g(x) = \begin{cases} 0 & -\pi \le x < \frac{-\pi}{2} \\ -\frac{1}{2} & x = -\frac{\pi}{2} \\ -1 & \frac{-\pi}{2} < x < 0 \\ 0 & x = 0 \\ 1 & 0 < x < \frac{\pi}{2} \\ \frac{1}{2} & x = \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x \le \pi \end{cases}$

30. $a_0 = \frac{1}{2},\ a_1 = 0,\ a_2 = \frac{5}{8}$

Exercises 10.4 (page 611)

2. **a.** The π periodic function $\tilde{f}(x) = \sin 2x$ for $x \neq k\pi$ where k is an integer.

b. The 2π periodic function $f_0(x) = \sin 2x$ for $x \neq k\pi$ where k is an integer.

c. The 2π periodic function $f_e(x)$, where $f_e(x) = \begin{cases} \sin 2x & 0 < x < \pi \\ -\sin 2x & -\pi < x < 0 \end{cases}$

4. **a.** The π periodic function $\tilde{f}(x)$, where $\tilde{f}(x) = \pi - x,\ 0 < x < \pi$

b. The 2π periodic function $f_0(x)$, where $f_0(x) = \begin{cases} \pi - x & 0 < x < \pi \\ -\pi - x & -\pi < x < 0 \end{cases}$

c. The 2π periodic function $f_e(x)$, where $f_e(x) = \begin{cases} \pi - x & 0 < x < \pi \\ \pi + x & -\pi < x < 0 \end{cases}$

6. $f(x) \sim \sum_{k=1}^{\infty} \frac{8k}{\pi(4k^2 - 1)} \sin(2kx)$

8. $f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$

10. $f(x) \sim \sum_{n=1}^{\infty} \frac{2\pi n[1 - e(-1)^n]}{1 + \pi^2 n^2} \sin(n\pi x)$

12. $f(x) \sim 1 + \frac{\pi}{2} - \left(\frac{4}{\pi}\right) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos(2k+1)x$

14. $f(x) \sim 1 - e^{-1} + \sum_{n=1}^{\infty} \frac{2}{1 + \pi^2 n^2} [1 - (-1)^n e^{-1}] \cos(n\pi x)$

16. $f(x) \sim \frac{1}{6} - \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2} \cos(2k\pi x)$

18. $u(x, t) = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^3 \pi} e^{-5(2k+1)^2 t} \sin(2k+1)x$

Exercises 10.5 (page 624)

2. $u(x,\, t) = \sum_{n=1}^{\infty} \left[\frac{2\pi}{n} (-1)^{n+1} + \frac{4}{n^3 \pi} [(-1)^n - 1] \right] e^{-n^2 t} \sin nx$

4. $u(x,\, t) = \frac{1}{6} - \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2} e^{-8k^2 \pi^2 t} \cos 2\pi k x$

6. $u(x,\, t) = 1 - \frac{2}{\pi} + 2e^{-7t} \cos x + \sum_{k=1}^{\infty} \frac{4}{\pi(4k^2 - 1)} e^{-28k^2 t} \cos 2kx$

8. $u(x,\, t) = 3x + 6 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin nx$

10. $u(x,\, t) = \left(\frac{\pi^2}{18} \right) x - \left(\frac{1}{18} \right) x^3 + \left(\frac{1}{3} \right) e^{-3t} \sin x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{3n^3} e^{-3n^2 t} \sin nx$

12. $u(x,\, t) = \sum_{n=1}^{\infty} a_n e^{-\ {}_n^2 t} \sin\ {}_n x$, where $\{\ {}_n\}_{n=1}^{\infty}$ is the increasing sequence of positive real numbers that are solutions to $\tan\ {}_n \pi = -\ {}_n$, and $a_n = \frac{1}{\left[\frac{\pi}{2} - \frac{\sin(2\pi\ {}_n)}{4} \right]} \int_0^{\pi} f(x) \sin\ {}_n x\, dx$.

14. $u(x,\, t) = 1 + \left(\frac{5\pi}{6} \right) x - \left(\frac{5}{6} \right) x^2 - \frac{20}{3\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} e^{-3(2k+1)^2 t} \sin(2k+1)x$

16. $u(x,\, y,\, t) = e^{-2t} \cos x \sin y + 4e^{-5t} \cos 2x \sin y - 3e^{-25t} \cos 3x \sin 4y$

18. $u(x,\, y,\, t) = \left(\frac{\pi}{2} \right) e^{-t} \sin y - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{-[(2k+1)^2 + 1]t} \cos[(2k+1)x] \sin y$

Exercises 10.6 (page 636)

2. $u(x,\ t)=\sum_{n=1}^{\infty}[a_n\cos 4nt+b_n\sin 4nt]\sin nx$, where

$$a_n=\begin{cases}0 & n\text{ even}\\ \dfrac{2}{\pi}\left[\dfrac{n}{n^2-4}-\dfrac{1}{n}\right] & n\text{ odd}\end{cases}$$

and

$$b_n=\begin{cases}\dfrac{-1}{4\pi(n^2-1)} & n\text{ even}\\ \dfrac{1}{\pi n^2} & n\text{ odd}\end{cases}$$

4. $u(x,\ t)=\cos 12t\sin 4x+7\cos 15t\sin 5x+\dfrac{4}{3\pi}\sum_{k=0}^{\infty}\dfrac{(-1)^k}{(2k+1)^3}\sin 3(2k+1)t\sin(2k+1)x$

6. $u(x,\ t)=\dfrac{2v_0L^3}{\pi^3\alpha a(L-a)}\sum_{n=1}^{\infty}\dfrac{1}{n^3}\sin\dfrac{n\pi a}{L}\sin\dfrac{n\pi x}{L}\sin\dfrac{n\pi\alpha t}{L}$

8. $u(x,\ t)=(\sin t-t\cos t)\sin x+\sum_{n=2}^{\infty}\dfrac{2(-1)^n}{n^2(n^2-1)}[\sin nt-n\sin t]\sin nx$

10. $u(x,\ t)=U_1+(U_2-U_1)\left(\dfrac{x}{L}\right)+\sum_{n=1}^{\infty}\left[a_n\cos\dfrac{n\pi\alpha t}{L}+b_n\sin\dfrac{n\pi\alpha t}{L}\right]\sin\dfrac{n\pi x}{L}$, where a_n 's and b_n 's are chosen that

$$f(x)-U_1-(U_2-U_1)\left(\frac{x}{L}\right)=\sum_{n=1}^{\infty}a_n\sin\frac{n\pi x}{L}$$

$$g(x)=\sum_{n=1}^{\infty}b_n\left[\frac{n\pi\alpha}{L}\right]\sin\frac{n\pi x}{L}$$

14. $u(x,\ t)=x^2+\alpha^2t^2$

16. $u(x,t)=\sin 3x\cos 3\alpha t+t$

18. $u(x,t)=\cos 2x\cos 2\alpha t+t-xt$

Exercises 10.7 (page 649)

2. $u(x,\ y) = \dfrac{\cos x \sinh(y-\pi)}{\sinh(-\pi)} - \dfrac{2\cos 4x \sinh(4y-4\pi)}{\sinh(-4\pi)}$

4. $u(x,\ y) = \dfrac{\sin x \sinh(y-\pi)}{\sinh(-\pi)} + \dfrac{\sin 4x \sinh(4y-4\pi)}{\sinh(-4\pi)}$

8. $u(r,\theta) = \dfrac{1}{2} + \dfrac{r^2}{8}\cos 2\theta$

12. $u(r,\theta) = \dfrac{27}{3^6-1}[r^3 - r^{-3}]\cos 3\theta + \dfrac{3^5}{3^{10}-1}[r^5 - r^{-5}]\cos 5\theta$

14. $u(r,\theta) = C + \sum_{n=1}^{\infty} r^{-n}[a_n \cos n\theta + b_n \sin n\theta]$, where C is arbitrary and for $n = 1, 2, \ldots$

$$a_n = \frac{-1}{n\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta$$

$$b_n = \frac{-1}{n\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta$$

16. $u(r,\theta) = \sum_{n=1}^{\infty} a_n \sinh\left[\dfrac{n\pi(\theta-\pi)}{\ln 2}\right]\sin\left[\dfrac{n\pi(\ln r - \ln \pi)}{\ln 2}\right]$, where

$$a_n = \frac{-2}{\ln 2 \sinh\left(\frac{n\pi^2}{\ln 2}\right)}\int_{\pi}^{2\pi} \sin r \sin\left[\frac{n\pi(\ln r - \ln \pi)}{\ln 2}\right]\frac{1}{r}\,dr\,.$$

24. $u(x,\ y) = \sum_{n=1}^{\infty} A_n e^{-ny}\sin nx$, where $A_n = \dfrac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx.$