Laguerre polynomials

Equation $\quad xy'' + (1-x)y' + ny = 0, \quad n \in N_0$

Definition $\quad L_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(k!)^2 (n-k)!} x^k, \quad x > 0, n \in N_0$

G.F $\quad \frac{e^{-\frac{xt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} L_n(x) t^n$

Rodrigue's Formula $\quad L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}), \quad n \in N_0$

Recurrence Relations ($n \in N_0$)

(**R1**) $(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$

(**R2**) $L_n'(x) = L_{n-1}'(x) - L_{n-1}(x)$

(**R3**) $xL_n'(x) = nL_n(x) - nL_{n-1}(x)$

(**R4**) $L_n'(x) = -\sum_{m=0}^{n-1} L_m(x)$

Orthogonality $\quad \int_0^{\infty} e^{-x} L_n(x) L_m(x) dx = \delta_{n,m}$

The First Few Polynomials

$L_0(x) = 1, \qquad L_1(x) = 1-x, \qquad L_2(x) = \frac{1}{2}(x^2 - 4x + 2),$

$L_3(x) = \frac{1}{6}(-x^3 + 9x^2 - 18x + 6), \quad L_4(x) = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$

Hermite polynomials

Equation $$y'' - 2xy' + 2ny = 0, \quad n \in N_0$$

Definition $$H_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^m}{m!(n-2m)!}(2x)^{n-2m}, \quad -\infty < x < \infty, n \in N_0$$

G.F $$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)t^n$$

Rodrigue's Formula $$H_n(x) = \frac{(-1)^n}{n!} e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n \in N_0$$

Recurrence Relations ($n \in N_0$)

(**R1**) $(n+1)H_{n+1}(x) = 2xH_n(x) - 2H_{n-1}(x)$

(**R2**) $H_n'(x) = 2H_{n-1}(x)$

Orthogonality $$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = \frac{2^n\sqrt{\pi}}{n!}\delta_{n,m}$$

The First Few Polynomials

$$H_0(x) = 1, \qquad H_1(x) = 2x, \qquad H_2(x) = 2x^2 - 1,$$

$$H_3(x) = \tfrac{1}{3}(4x^3 - 6x), \quad H_4(x) = \tfrac{1}{6}(4x^4 - 12x^2 + 3)$$

Legendre polynomials

Equation $$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n \in N_0$$

Definition $$P_n(x) = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m\,(2n-2m)!}{2^n m!\ (n-2m)!(n-m)!} x^{n-2m}, \quad |x| < 1, n \in N_0$$

G.F $$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

Rodrigue's Formula $$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n \in N_0$$

Recurrence Relations ($n \in N_0$)

(R1) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

(R2) $P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x)$

(R3) $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$

(R4) $xP'_n(x) - P'_{n-1}(x) = nP_n(x)$

(R5) $P'_n(x) - xP'_{n-1}(x) = nP_{n-1}(x)$

(R6) $(x^2-1)P'_n(x) = (n+1)P_{n+1}(x) - (n+1)xP_n(x)$

Orthogonality $$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m}$$

The First Few Polynomials

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2-1),$$

$$P_3(x) = \tfrac{1}{2}(5x^3-3x), \quad P_4(x) = \tfrac{1}{8}(35x^4-30x^2+3)$$

Associated Legendre Polynomials $$P_{n(m)}(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_n(x), \quad n,m \in N$$

Orthogonality $$\int_{-1}^{1} P_{n(m)}(x)P_{k(m)}(x)dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n,k}$$

Bessel Functions

Equation $$x^2y'' + xy' + (x^2 - n^2)y = 0, \quad n \geq 0$$

1^{st} kind definition $$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n}, \quad x > 0,\ n \geq 0$$

2^{nd} kind definition $$Y_n(x) = \frac{\cos n\pi J_n(x) - J_{-n}(x)}{\sin n\pi},$$

G.F $$e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Recurrence Relations ($\forall n$)

(R1) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

(R2) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

(R3) $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$

(R4) $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

(R5) $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

(R6) $J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x)$

Spherical Bessel Functions ($m \in N_0$)

$$j_m(x) = \sqrt{\frac{\pi}{2x}}\, J_{m+\frac{1}{2}}(x)$$

$$y_m(x) = \sqrt{\frac{\pi}{2x}}\, Y_{m+\frac{1}{2}}(x)$$

where

$$J_{m+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\ (-1)^m\ x^{m+\frac{1}{2}} \left(\frac{d}{xdx}\right)^m \frac{\sin x}{x}$$

$$J_{-m-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\ x^{m+\frac{1}{2}} \left(\frac{d}{xdx}\right)^m \frac{\cos x}{x}$$

Integral Representation $\quad J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)d\theta,\ \ n$ is a positive integer

where

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{n \text{ even}} \cos n\theta\ J_n(x)$$

$$\sin(x\sin\theta) = 2\sum_{n \text{ odd}} \sin n\theta\ J_n(x)$$

Parametric Bessel's Equation

$$x^2y'' + xy' + (\lambda^2x^2 - n^2)y = 0, \quad n \geq 0, \quad \lambda \text{ is a parameter}$$

Orthogonality $\quad \int_0^a xJ_n(\lambda x)J_n(\mu x)dx = a\ \frac{\mu J_n(\lambda a)J_n'(\mu a) - \lambda J_n(\mu a)J_n'(\lambda a)}{\lambda^2 - \mu^2}, \quad \text{if } \lambda \neq \mu$

$$\int_0^a x[J_n(\lambda x)]^2dx = \frac{a^2}{2}\left[J_n'(\lambda a)\right]^2 + \frac{1}{2}\left(a^2 - \frac{n^2}{\lambda^2}\right)[J_n(\lambda a)]^2, \quad \text{if } \lambda = \mu$$