

Research Article

Subordination for Higher-Order Derivatives of Multivalent Functions

Rosihan M. Ali,¹ Abeer O. Badghaish,^{1,2} and V. Ravichandran³

¹ School of Mathematical Sciences, Universiti Sains Malaysia (USM), 11800 Penang, Malaysia

² Mathematics Department, King Abdul Aziz University, P.O. Box 581, Jeddah 21421, Saudi Arabia

³ Department of Mathematics, University of Delhi, Delhi 110 007, India

Correspondence should be addressed to Rosihan M. Ali, rosihan@cs.usm.my

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Differential subordination methods are used to obtain several interesting subordination results and best dominants for higher-order derivatives of p -valent functions. These results are next applied to yield various known results as special cases.

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1. Motivation and preliminaries

For a fixed $p \in \mathbb{N} := \{1, 2, \dots\}$, let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (1.1)$$

which are p -valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A} := \mathcal{A}_1$. Upon differentiating both sides of (1.1) q -times with respect to z , the following differential operator is obtained:

$$f^{(q)}(z) = \lambda(p; q) z^{p-q} + \sum_{k=1}^{\infty} \lambda(k+p; q) a_{k+p} z^{k+p-q}, \quad (1.2)$$

where

$$\lambda(p; q) := \frac{p!}{(p-q)!} \quad (p \geq q; p \in \mathbb{N}; q \in \mathbb{N} \cup \{0\}). \quad (1.3)$$

Several researchers have investigated higher-order derivatives of multivalent functions, see, for example, [1–10]. Recently, by the use of the well-known Jack's lemma [11, 12], Irmak and Cho [5] obtained interesting results for certain classes of functions defined by higher-order derivatives.

Let f and g be analytic in $\mathbb{U}$. Then f is *subordinate* to g , written as $f(z) < g(z)$ ($z \in \mathbb{U}$) if there is an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. In particular, if g is univalent in $\mathbb{U}$, then f subordinate to g is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$. A p -valent function $f \in \mathcal{A}_p$ is *starlike* if it satisfies the condition $(1/p)\Re(zf'(z)/f(z)) > 0$ ($z \in \mathbb{U}$). More generally, let $\phi(z)$ be an analytic function with positive real part in $\mathbb{U}$, $\phi(0) = 1$, $\phi'(0) > 0$, and $\phi(z)$ maps the unit disc $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes $S_p^*(\phi)$ and $C_p(\phi)$ consist, respectively, of p -valent functions f *starlike* with respect to ϕ and p -valent functions f *convex* with respect to ϕ in $\mathbb{U}$ given by

$$f \in S_p^*(\phi) \iff \frac{1}{p} \frac{zf'(z)}{f(z)} < \phi(z), \quad f \in C_p(\phi) \iff \frac{1}{p} \left(1 + \frac{zf''(z)}{f'(z)} \right) < \phi(z). \quad (1.4)$$

These classes were introduced and investigated in [13], and the functions $h_{\phi,p}$ and $k_{\phi,p}$, defined, respectively, by

$$\begin{aligned} \frac{1}{p} \frac{zh'_{\phi,p}}{h_{\phi,p}} &= \phi(z) \quad (z \in \mathbb{U}, h_{\phi,p} \in \mathcal{A}_p), \\ \frac{1}{p} \left(1 + \frac{zk''_{\phi,p}}{k'_{\phi,p}} \right) &= \phi(z) \quad (z \in \mathbb{U}, k_{\phi,p} \in \mathcal{A}_p), \end{aligned} \quad (1.5)$$

are important examples of functions in $S_p^*(\phi)$ and $C_p^*(\phi)$. Ma and Minda [14] have introduced and investigated the classes $S^*(\phi) := S_1^*(\phi)$ and $C(\phi) := C_1(\phi)$. For $-1 \leq B < A \leq 1$, the class $S^*[A, B] = S^*((1 + Az)/(1 + Bz))$ is the class of Janowski starlike functions (cf. [15, 16]).

In this paper, corresponding to an appropriate subordinate function $Q(z)$ defined on the unit disk $\mathbb{U}$, sufficient conditions are obtained for a p -valent function f to satisfy the subordination

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z). \quad (1.6)$$

In the particular case when $q = 1$ and $p = 1$, and $Q(z)$ is a function with positive real part, the first subordination gives a sufficient condition for univalence of analytic functions, while the second subordination implication gives conditions for convexity of functions. If $q = 0$ and $p = 1$, the second subordination gives conditions for starlikeness of functions. Thus results obtained in this paper give important information on the geometric properties of functions satisfying differential subordination conditions involving higher-order derivatives.

The following lemmas are needed to prove our main results.

Lemma 1.1 (see [12, page 135, Corollary 3.4h.1]). *Let Q be univalent in $\mathbb{U}$, and φ be analytic in a domain D containing $Q(\mathbb{U})$. If $zQ'(z) \cdot \varphi[Q(z)]$ is starlike, and P is analytic in $\mathbb{U}$ with $P(0) = Q(0)$ and $P(\mathbb{U}) \subset D$, then*

$$zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)] \implies P < Q, \quad (1.7)$$

and Q is the best dominant.

Lemma 1.2 (see [12, page 135, Corollary 3.4h.2]). *Let Q be convex univalent in $\mathbb{U}$, and let θ be analytic in a domain D containing $Q(\mathbb{U})$. Assume that*

$$\Re \left[\theta'[Q(z)] + 1 + \frac{zQ''(z)}{Q'(z)} \right] > 0. \quad (1.8)$$

If P is analytic in $\mathbb{U}$ with $P(0) = Q(0)$ and $P(\mathbb{U}) \subset D$, then

$$zP'(z) + \theta[P(z)] < zQ'(z) + \theta[Q(z)] \implies P < Q, \quad (1.9)$$

and Q is the best dominant.

2. Main results

The first four theorems below give sufficient conditions for a differential subordination of the form

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z) \quad (2.1)$$

to hold.

Theorem 2.1. *Let $Q(z)$ be univalent and nonzero in $\mathbb{U}$, $Q(0) = 1$, and let $zQ'(z)/Q(z)$ be starlike in $\mathbb{U}$. If a function $f \in \mathcal{A}_p$ satisfies the subordination*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < \frac{zQ'(z)}{Q(z)} + p - q, \quad (2.2)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad (2.3)$$

and Q is the best dominant.

Proof. Define the analytic function $P(z)$ by

$$P(z) := \frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}}. \quad (2.4)$$

Then a computation shows that

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} = \frac{zP'(z)}{P(z)} + p - q. \quad (2.5)$$

The subordination (2.2) yields

$$\frac{zP'(z)}{P(z)} + p - q < \frac{zQ'(z)}{Q(z)} + p - q, \quad (2.6)$$

or equivalently

$$\frac{zP'(z)}{P(z)} < \frac{zQ'(z)}{Q(z)}. \quad (2.7)$$

Define the function φ by $\varphi(w) := 1/w$. Then (2.7) can be written as $zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)]$. Since $Q(z) \neq 0$, $\varphi(w)$ is analytic in a domain containing $Q(\mathbb{U})$. Also $zQ'(z) \cdot \varphi(Q(z)) = zQ'(z)/Q(z)$ is starlike. The result now follows from Lemma 1.1. $\square$

Remark 2.2. For $f \in \mathcal{A}_p$, Irmak and Cho [5, page 2, Theorem 2.1] showed that

$$\Re \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < p - q \implies |f^{(q)}(z)| < \lambda(p; q)|z|^{p-q-1}. \quad (2.8)$$

However, it should be noted that the hypothesis of this implication cannot be satisfied by any function in $\mathcal{A}_p$ as the quantity

$$\left. \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right|_{z=0} = p - q. \quad (2.9)$$

Theorem 2.1 is the correct formulation of their result in a more general setting.

Corollary 2.3. *Let $-1 \leq B < A \leq 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < \frac{z(A-B)}{(1+Az)(1+Bz)} + p - q, \quad (2.10)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < \frac{1 + Az}{1 + Bz}. \quad (2.11)$$

Proof. For $-1 \leq B < A \leq 1$, define the function Q by

$$Q(z) = \frac{1 + Az}{1 + Bz}. \quad (2.12)$$

Then a computation shows that

$$\begin{aligned} F(z) &:= \frac{zQ'(z)}{Q(z)} = \frac{(A - B)z}{(1 + Az)(1 + Bz)}, \\ h(z) &:= \frac{zF'(z)}{F(z)} = \frac{1 - ABz^2}{(1 + Az)(1 + Bz)}. \end{aligned} \quad (2.13)$$

With $z = re^{i\theta}$, note that

$$\begin{aligned} \Re(h(re^{i\theta})) &= \Re \frac{1 - ABr^2 e^{2i\theta}}{(1 + Are^{i\theta})(1 + Bre^{i\theta})} \\ &= \frac{(1 - ABr^2)(1 + ABr^2 + (A + B)r \cos \theta)}{|(1 + Are^{i\theta})(1 + Bre^{i\theta})|^2}. \end{aligned} \quad (2.14)$$

Since $1 + ABr^2 + (A + B)r \cos \theta \geq (1 - Ar)(1 - Br) > 0$ for $(A + B) \geq 0$, and similarly, $1 + ABr^2 + (A + B)r \cos \theta \geq (1 + Ar)(1 + Br) > 0$ for $(A + B) \leq 0$, it follows that $\Re h(z) > 0$, and hence $zQ'(z)/Q(z)$ is starlike. The desired result now follows from Theorem 2.1. $\square$

Example 2.4. (1) For $0 < \beta < 1$, choose $A = \beta$ and $B = 0$ in Corollary 2.3. Since $w < \beta z / (1 + \beta z)$ is equivalent to $|w| \leq \beta |1 - w|$, it follows that if $f \in \mathcal{A}_p$ satisfies

$$\left| \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + \frac{\beta^2}{1 - \beta^2} \right| < \frac{\beta}{1 - \beta^2}, \quad (2.15)$$

then

$$\left| \frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} - 1 \right| < \beta. \quad (2.16)$$

(2) With $A = 1$ and $B = 0$, it follows from Corollary 2.3 that whenever $f \in \mathcal{A}_p$ satisfies

$$\Re \left\{ \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right\} < \frac{1}{2}, \quad (2.17)$$

then

$$\left| \frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} - 1 \right| < 1. \quad (2.18)$$

Taking $q = 0$ and $Q(z) = h_{\phi, p}/z^p$, Theorem 2.1 yields the following corollary.

Corollary 2.5 (see [13]). *If $f \in S_p^*(\phi)$, then*

$$\frac{f(z)}{z^p} < \frac{h_{\phi, p}}{z^p}. \quad (2.19)$$

Similarly, choosing $q = 1$ and $Q(z) = k'_{\phi, p}/pz^{p-1}$, Theorem 2.1 yields the following corollary.

Corollary 2.6 (see [13]). *If $f \in C_p^*(\phi)$, then*

$$\frac{f'(z)}{z^{p-1}} < \frac{k'_{\phi, p}}{z^{p-1}}. \quad (2.20)$$

Theorem 2.7. *Let $Q(z)$ be convex univalent in $\mathbb{U}$ and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) < zQ'(z), \quad (2.21)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad (2.22)$$

and Q is the best dominant.

Proof. Define the analytic function $P(z)$ by $P(z) := f^{(q)}(z)/\lambda(p; q)z^{p-q}$. Then it follows from (2.5) that

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) = zP'(z). \quad (2.23)$$

By assumption, it follows that

$$zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)], \quad (2.24)$$

where $\varphi(w) = 1$. Since $Q(z)$ is convex, and $zQ'(z) \cdot \varphi[Q(z)] = zQ'(z)$ is starlike, Lemma 1.1 gives the desired result. $\square$

Example 2.8. When

$$Q(z) := 1 + \frac{z}{\lambda(p; q)}, \quad (2.25)$$

Theorem 2.7 is reduced to the following result in [5, page 4, Theorem 2.4]. For $f \in \mathcal{A}_p$,

$$\left| f^{(q)}(z) \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) \right| \leq |z|^{p-q} \implies |f^{(q)}(z) - \lambda(p; q)z^{p-q}| \leq |z|^{p-q}. \quad (2.26)$$

In the special case $q = 1$, this result gives a sufficient condition for the multivalent function $f(z)$ to be close-to-convex.

Theorem 2.9. *Let $Q(z)$ be convex univalent in $\mathbb{U}$ and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{zf^{(q+1)}(z)}{\lambda(p; q)z^{p-q}} < zQ'(z) + (p - q)Q(z), \quad (2.27)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad (2.28)$$

and Q is the best dominant.

Proof. Define the function $P(z)$ by $P(z) = f^{(q)}(z) / \lambda(p; q)z^{p-q}$. It follows from (2.5) that

$$zP'(z) + (p - q)P(z) < zQ'(z) + (p - q)Q(z), \quad (2.29)$$

that is,

$$zP'(z) + \theta[P(z)] < zQ'(z) + \theta[Q(z)], \quad (2.30)$$

where $\theta(w) = (p - q)w$. The conditions in Lemma 1.2 are clearly satisfied. Thus $f^{(q)}(z) / \lambda(p; q)z^{p-q} < Q(z)$, and Q is the best dominant. $\square$

Taking $q = 0$, Theorem 2.9 yields the following corollary.

Corollary 2.10 (see [17, Corollary 2.11]). *Let $Q(z)$ be convex univalent in $\mathbb{U}$, and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{f'(z)}{z^{p-1}} < zQ'(z) + pQ(z), \quad (2.31)$$

then

$$\frac{f(z)}{z^p} < Q(z). \quad (2.32)$$

With $p = 1$, Corollary 2.10 yields the following corollary.

Corollary 2.11 (see [17, Corollary 2.9]). *Let $Q(z)$ be convex univalent in $\mathbb{U}$, and $Q(0) = 1$. If $f \in \mathcal{A}$ satisfies*

$$f'(z) < zQ'(z) + Q(z), \quad (2.33)$$

then

$$\frac{f(z)}{z} < Q(z). \quad (2.34)$$

Theorem 2.12. *Let $Q(z)$ be univalent and nonzero in $\mathbb{U}$, $Q(0) = 1$, and $zQ'(z)/Q^2(z)$ be starlike. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{\lambda(p; q)z^{p-q}}{f^{(q)}(z)} \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q \right) < \frac{zQ'(z)}{Q^2(z)}, \quad (2.35)$$

then

$$\frac{f^{(q)}(z)}{\lambda(p; q)z^{p-q}} < Q(z), \quad (2.36)$$

and Q is the best dominant.

Proof. Define the function $P(z)$ by $P(z) = f^{(q)}(z)/\lambda(p; q)z^{p-q}$. It follows from (2.5) that

$$\frac{\lambda(p; q)z^{p-q}}{f^{(q)}(z)} \cdot \left(\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - (p - q) \right) = \frac{1}{P(z)} \cdot \frac{zP'(z)}{P(z)} = \frac{zP'(z)}{P^2(z)}. \quad (2.37)$$

By assumption,

$$\frac{zP'(z)}{P^2(z)} < \frac{zQ'(z)}{Q^2(z)}. \quad (2.38)$$

With $\varphi(w) := 1/w^2$, (2.38) can be written as $zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)]$. The function $\varphi(w)$ is analytic in $\mathbb{C} - \{0\}$. Since $zQ'(z)\varphi[Q(z)]$ is starlike, it follows from Lemma 1.1 that $P(z) < Q(z)$, and $Q(z)$ is the best dominant. $\square$

The next four theorems give sufficient conditions for the following differential subordination

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z) \quad (2.39)$$

to hold.

Theorem 2.13. *Let $Q(z)$ be univalent and nonzero in $\mathbb{U}$, $Q(0) = 1$, $Q(z) \neq q - p + 1$, and $zQ'(z)/[Q(z)(Q(z) + p - q - 1)]$ be starlike in $\mathbb{U}$. If $f \in \mathcal{A}_p$ satisfies*

$$\frac{1 + (zf^{(q+2)}(z)/f^{(q+1)}(z)) - p + q + 1}{(zf^{(q+1)}(z)/f^{(q)}(z)) - p + q + 1} < 1 + \frac{zQ'(z)}{Q(z)(Q(z) + p - q - 1)}, \quad (2.40)$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z), \quad (2.41)$$

and Q is the best dominant.

Proof. Let the function $P(z)$ be defined by

$$P(z) = \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1. \quad (2.42)$$

Upon differentiating logarithmically both sides of (2.42), it follows that

$$\frac{zP'(z)}{P(z) + p - q - 1} = 1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)}. \quad (2.43)$$

Thus

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - p + q + 1 = \frac{zP'(z)}{P(z) + p - q - 1} + P(z). \quad (2.44)$$

The equations (2.42) and (2.44) yield

$$\frac{1 + (zf^{(q+2)}(z)/f^{(q+1)}(z)) - p + q + 1}{(zf^{(q+1)}(z)/f^{(q)}(z)) - p + q + 1} = \frac{zP'(z)}{P(z)(P(z) + p - q - 1)} + 1. \quad (2.45)$$

If $f \in \mathcal{A}_p$ satisfies the subordination (2.40), (2.45) gives

$$\frac{zP'(z)}{P(z)(P(z) + p - q - 1)} < \frac{zQ'(z)}{Q(z)(Q(z) + p - q - 1)}, \quad (2.46)$$

that is,

$$zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)] \quad (2.47)$$

with $\varphi(w) := 1/w(w + p - q - 1)$. The desired result is now established by an application of Lemma 1.1. $\square$

Theorem 2.13 contains a result in [18, page 122, Corollary 4] as a special case. In particular, we note that Theorem 2.13 with $p = 1$, $q = 0$, and $Q(z) = (1 + Az)/(1 + Bz)$ for $-1 \leq B < A \leq 1$ yields the following corollary.

Corollary 2.14 (see [18, page 123, Corollary 6]). *Let $-1 \leq B < A \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1 + (zf''(z)/f'(z))}{zf'(z)/f(z)} < 1 + \frac{(A - B)z}{(1 + Az)^2}, \quad (2.48)$$

then $f \in S^[A, B]$.*

For $A = 0$, $B = b$ and $A = 1$, $B = -1$, Corollary 2.14 gives the results of Obradović and Tuneski [19].

Theorem 2.15. *Let $Q(z)$ be univalent and nonzero in $\mathbb{U}$, $Q(0) = 1$, $Q(z) \neq q - p + 1$, and let $zQ'(z)/[Q(z) + p - q - 1]$ be starlike in $\mathbb{U}$. If $f \in \mathcal{A}_p$ satisfies*

$$1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} < \frac{zQ'(z)}{Q(z) + p - q - 1}, \quad (2.49)$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z), \quad (2.50)$$

and Q is the best dominant.

Proof. Let the function $P(z)$ be defined by (2.42). It follows from (2.43) and the hypothesis that

$$\frac{zP'(z)}{P(z) + p - q - 1} < \frac{zQ'(z)}{Q(z) + p - q - 1}. \quad (2.51)$$

Define the function φ by $\varphi(w) := 1/(w + p - q - 1)$. Then (2.51) can be written as

$$zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)]. \quad (2.52)$$

Since $\varphi(w)$ is analytic in a domain containing $Q(\mathbb{U})$, and $zQ'(z) \cdot \varphi[Q(z)]$ is starlike, the result follows from Lemma 1.1. $\square$

Theorem 2.16. Let $Q(z)$ be a convex function in $\mathbb{U}$, and $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \left[2 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right] < zQ'(z) + Q(z) + p - q - 1, \quad (2.53)$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z), \quad (2.54)$$

and Q is the best dominant.

Proof. Let the function $P(z)$ be defined by (2.42). Using (2.43), it follows that

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \left(1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right) = zP'(z), \quad (2.55)$$

and, therefore,

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \left(2 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right) = zP'(z) + P(z) + p - q - 1. \quad (2.56)$$

By assumption,

$$zP'(z) + P(z) + p - q - 1 < zQ'(z) + Q(z) + p - q - 1, \quad (2.57)$$

or

$$zP'(z) + \theta[P(z)] < zQ'(z) + \theta[Q(z)], \quad (2.58)$$

where the function $\theta(w) = w + p - q + 1$. The proof is completed by applying Lemma 1.2. $\square$

Theorem 2.17. Let $Q(z)$ be a convex function in $\mathbb{U}$, with $Q(0) = 1$. If $f \in \mathcal{A}_p$ satisfies

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \left[1 + \frac{zf^{(q+2)}(z)}{f^{(q+1)}(z)} - \frac{zf^{(q+1)}(z)}{f^{(q)}(z)} \right] < zQ'(z), \quad (2.59)$$

then

$$\frac{zf^{(q+1)}(z)}{f^{(q)}(z)} - p + q + 1 < Q(z), \quad (2.60)$$

and Q is the best dominant.

Proof. Let the function $P(z)$ be defined by (2.42). It follows from (2.43) that $zP'(z) \cdot \varphi[P(z)] < zQ'(z) \cdot \varphi[Q(z)]$, where $\varphi(w) = 1$. The result follows easily from Lemma 1.1. $\square$

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