بسم الله الرحمن الرحيم

Lectures 6-10

Newton's laws of Motion

In 1687, Isaac Newton laid down three fundamental laws of motion, which are:

1st Law
The Law of Inertia

Every body continues in its state of rest, or of uniform motion in a straight line, unless it is forced to change that state by forces impressed upon it.

2nd Law
The Law of acceleration

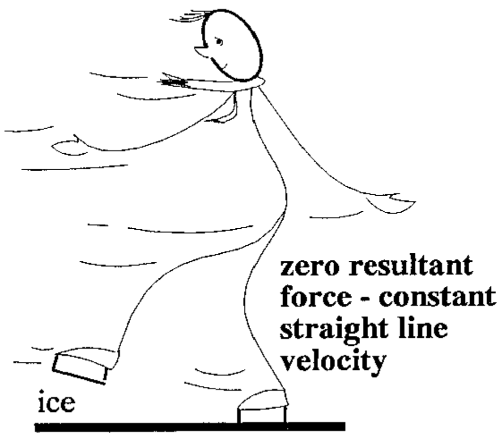
The change of motion is proportional to the net force impressed and is made in the direction of that force.

3rd Law
The Law of action & reaction

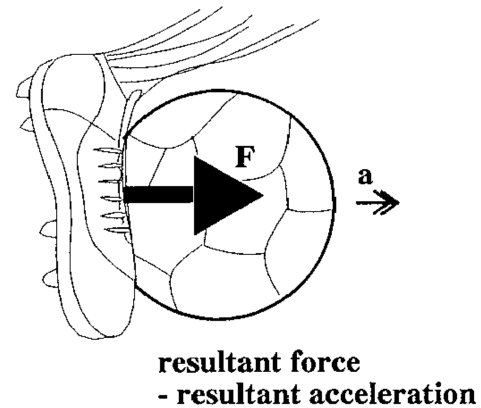
To every action there is always an equal reaction.

Newton's laws of Motion

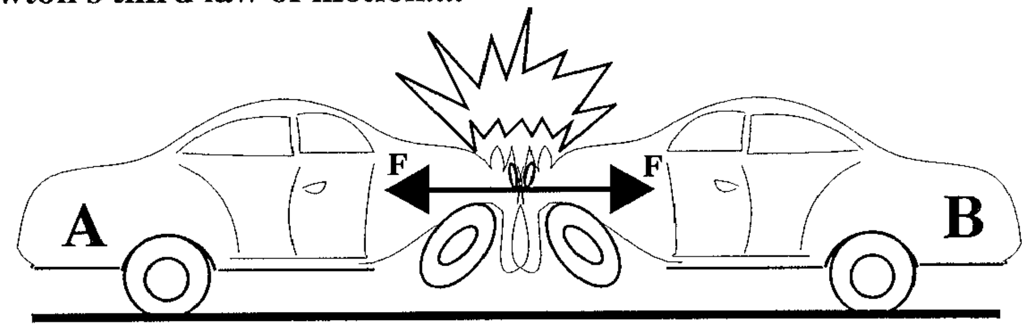
Newton's first law of motion....



Newton's second law of motion....



Newton's third law of motion....



Force of A on B is equal and opposite to the force of B on A.

Newton's 1st law

The first law describes a common property of matter, known as ***inertia***.

What is Inertia?

It is the resistance of a matter to change its state of motion.

This means that; in the absence of applied forces, matter simply continues in its current velocity state-forever.

Inertial Frames of Reference

-A mathematical description of the motion of a particle requires the selection of a *frame of reference*, or *a set of coordinates* according to which the position, velocity, and acceleration of the particle can be specified.

- Uniformly moving frames of reference (i.e. those considered *at 'rest' or moving with constant velocity in a straight line*) are called ***inertial frames of reference***.

- If we can neglect the effect of the earth's rotations, a frame of reference fixed in the earth is an *inertial reference frame*.

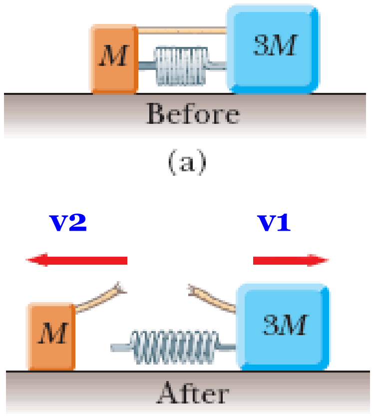
- Newton's laws are ***only applicable*** *at inertial reference frames*.

Newton's Second and Third Laws

- The physical quantity that measures inertia is called ***mass***.
- The more massive an object is, the more resistive it is to acceleration

Suppose we have two masses m_1, m_2 on a frictionless surface. Now imagine someone pushing the two masses together, and then suddenly releasing them so that they fly apart, achieving speeds v_1 and v_2. The ratio of the two masses can be expressed as;

$$\frac{m_2}{m_1} = \left|\frac{\mathbf{v}_1}{\mathbf{v}_2}\right|$$



Or;

$$\Delta(m_1\mathbf{v}_1) = -\Delta(m_2\mathbf{v}_2)$$

The (-) appears because the final velocities v_1 and v_2 are in opposite directions. If we divide by Δt and take limits as $\Delta t \sim 0$, we obtain

$$\frac{d}{dt}(m_1\mathbf{v}_1) = -\frac{d}{dt}(m_2\mathbf{v}_2) \qquad (1)$$

According to Newton's 2nd law, this "*change of motion*" is proportional to the *force* caused it;

$$\mathbf{F} \propto \frac{d}{dt}(m\mathbf{v})$$

Defining the unit in the SI system, Newton's 2nd law can be expressed in the familiar form:

Newton's 2nd Law

$$\mathbf{F}_{\text{net}} = \frac{d}{dt}(m\mathbf{v}) = m\mathbf{a}$$

Hence, equation (1) is equivalent to

Newton's 3rd Law

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2$$

Which is Newton's 3rd law, that states; *two interacting bodies exert equal and opposite forces upon one another.*

Linear Momentum

The product of mass and velocity, **mv**, is called *linear momentum*, ***P***, hence, the 2nd law can be rewritten as;

Newton's 2nd Law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

which means that;

The time rate of change of an object's linear momentum is proportional to the applied force, **F**.

Similarly,

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2$$

is equivalent to

Newton's 3rd Law

$$\frac{d}{dt}\left(\mathbf{p}_1 + \mathbf{p}_2\right) = 0$$

Or;

$$\left(p_1 + p_2\right) = cons.$$

In other words, Newton's 3rd law implies that *the total momentum of two mutually interacting bodies is a constant*.

Motion with Constant Force

Rectilinear Motion

When a moving particle remains on a *single straight line*, the motion is said to be *rectilinear*. In this case, we can choose the ***x*-axis** as the line of motion. The general equation of motion is then

$$F(x, \dot{x}, t) = m\ddot{x}$$

The simplest case is *when F is constant*. In this case *a is constant* ;

$$\ddot{x} = \frac{F}{m} = \textbf{constant} = a$$

Integrating with respect to time:

$$\dot{x} = v = at + v_0 \qquad (1)$$

$$x = \tfrac{1}{2}at^2 + v_0 t + x_0 \qquad (2)$$

where v_0 is the velocity and x_0 is the position at $t = 0$.

From (1) & (2) we obtain;

$$2a(x - x_0) = v^2 - v_0^{\,2} \qquad (3)$$

Forces that Depend on Position

The Concepts of Kinetic & Potential Energy

The Work

If the force is independent of velocity or time, then the differential equation for rectilinear motion is simply

$$F(x) = m\ddot{x} = m\left(\frac{dx}{dt}\frac{d\dot{x}}{dx}\right) = mv\frac{dv}{dx}$$

$$= \tfrac{1}{2}m\frac{d(v^2)}{dx} = \frac{dT}{dx} \qquad (4)$$

The quantity $T = \tfrac{1}{2}mv^2$ is called ***the kinetic energy*** of the particle. Taking the integral of (4):

$$W = \int_{x_0}^{x} F(x)\,dx = T - T_0$$

Where W is the work done on the particle by the impressed force $F(x)$. This work is equal to the change in the kinetic energy of the particle.

Examples

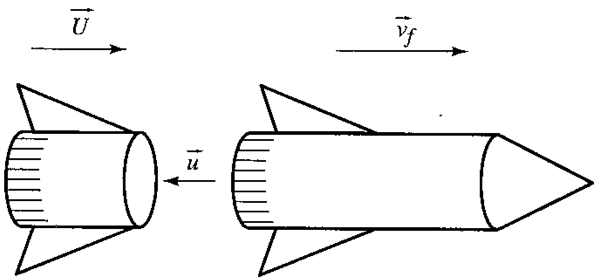
Linear Momentum

EXAMPLE 2.1.2

A spaceship of mass M is traveling in deep space with velocity v_i = 20 km/s relative to the Sun. It ejects a rear stage of mass 0.2 M with a relative speed u = 5 km/s .What then is the velocity of the spaceship?

Since the total *linear momentum* is conserved,
Then

$$P_i = P_f$$



Where,

$$P_i = M\, v_i$$

Let U be the velocity of the ejected rear stage and v_f be the velocity of the ship after ejection. The total momentum of the system after ejection is then

$$P_f = 0.2\, M\, U + 0.8\, M\, v_f$$

But $U = v_f - u$

Then $0.2\, M\, (v_f - u) + 0.8\, M\, v_f = M\, v_i$

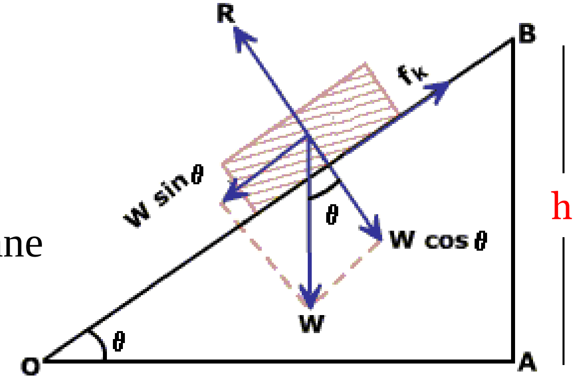
which gives us

$$v_f = v_i + 0.2\, u = 20 \text{ km/s} + 0.2\,(5 \text{ km/s}) = 21 \text{ km/s}$$

Motion with Constant Force

EXAMPLE 2.2.1

Consider a block that is free to slide down a smooth, frictionless plane that is inclined at an angle θ to the horizontal. If the height of the plane is h and the block is released from rest at the top ($v_0=0$), what will be its speed when it reaches the bottom?



Body accelerating down an inclined plane

We choose a coordinate system whose positive x-axis points down the plane and whose y-axis points "upward," ⊥ to the plane

The only force along the x direction is the component of gravitational force, $mg \sin \theta$, and it is constant. Then

$$\ddot{x} = a = \frac{F_g}{m} = g \sin\theta$$

and

$$x - x_0 = \frac{h}{\sin\theta}$$

Using;

$$2a(x - x_0) = v^2 - v_0^2$$

we obtain;

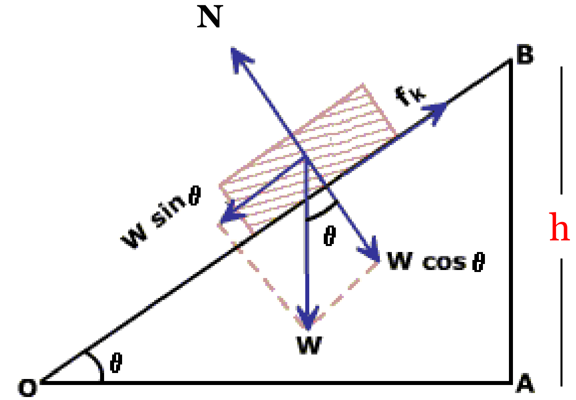
$$v^2 = 2g\sin\theta\left(\frac{h}{\sin\theta}\right) = 2gh$$

Motion with Constant Force

EXAMPLE 2.2.1

Suppose that, instead of being smooth, *the plane is rough* and ***its kinetic coefficient of friction*** is μ_k. Then the net force in the x direction, is equal to $mg \sin\theta - f_k$. Where;

$$f = \mu_k N = \mu_k mg \cos\theta$$



Body accelerating down an inclined plane

hence,

$$\ddot{x} = a = \frac{F_{net}}{m} = \frac{mg\sin\theta - \mu_k mg\cos\theta}{m}$$
$$= g(\sin\theta - \mu_k \cos\theta)$$

and

$$x - x_0 = \frac{h}{\sin\theta}$$

Using;

$$2a(x - x_0) = v^2 - v_0^{\ 2}$$

we obtain;

$$v^2 = 2g(\sin\theta - \mu_k \cos\theta)(\frac{h}{\sin\theta}) = 2gh\left(1 - \frac{\mu_k}{\tan\theta}\right)$$

Forces that Depend on Position

The Concepts of Kinetic & Potential Energy

The Work

If the force is independent of velocity or time, then the differential equation for rectilinear motion is simply

$$F(x) = m\ddot{x} = m\left(\frac{dx}{dt}\frac{d\dot{x}}{dx}\right) = mv\frac{dv}{dx}$$

$$= \tfrac{1}{2}m\frac{d(v^2)}{dx} = \frac{dT}{dx} \qquad (4)$$

The quantity $T = \tfrac{1}{2}mv^2$ is called ***the kinetic energy*** of the particle. Taking the integral of (4):

$$W = \int_{x_0}^{x} F(x)\,dx = T - T_0$$

Where W is the work done on the particle by the impressed force $F(x)$. This work is equal to the change in the kinetic energy of the particle.

Let us define another function $V(x)$ such that;

$$F(x) = -\frac{dV(x)}{dx}$$

The function $V(x)$ is called ***the potential energy***. Hence;

$$W = \int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} dV = -V(x) + V(x_0) = T - T_0$$

Or;

$$T_0 + V(x_0) = T + V(x) \equiv E$$

E is known as the *total mechanical energy* of the particle.

Note:

1- The sum of the kinetic and potential energies E is constant throughout the motion of the particle.

2- The force is a function only of the position x**.** Such a force is said to be ***conservative***.

3- when $v=0$ ➲ $T=0$ ➲ $V(x)= E$ **.** This point known as "***the turning point***"

Examples

Free Fall

EXAMPLE (2.3.1)

The Concept of Potential Energy

The motion of a *freely falling* body is an example of conservative motion. In this case:

Then

$$F = -\frac{dV}{dx} = -mg$$

$$V = mgx + C$$

We can choose $C = 0$, which means that $V = 0$ when $x = 0$.
The energy equation is then

$$\tfrac{1}{2}mv^2 + mgx = E$$

For instance, let the body be projected upward with initial speed v_0 from the origin $x=0$. These values give;

$$\tfrac{1}{2}mv^2 + mgx = \tfrac{1}{2}mv_0^2 + 0$$

so;

$$v^2 = v_0^2 - 2gx$$

The turning point of the motion, which is in this case *the maximum height*, is given by setting $v = 0$.
This gives

$$h = x_{\max} = \frac{v_0^2}{2g}$$

Other Solution

Using;

$$2a(x - x_0) = v^2 - v_0{}^2$$

we obtain;

$$h = x - x_0 = \frac{-v_0^2}{-2g} = \frac{v_0^2}{2g}$$

Morse Function

EXAMPLE (2.3.3):

The potential energy of a vibrating diatomic molecule as a function of x is given by;

$$V(x) = V_0\left[1 - e^{-(x-x_0)/\delta}\right]^2 - V_0$$

Show that:

1- x_0 is the separation of the two atoms at equilibrium, i.e. when the *potential energy function* is **minimum**.

2- and that $V(x_0) = -V_0$.

Solution

$V(x)$ is min when its derivative (w.r.t) x is zero;

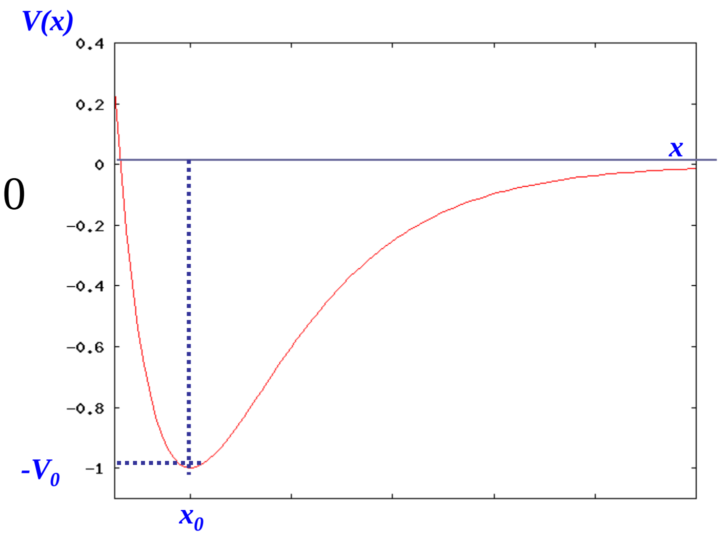
$$F(x) = -\frac{dV(x)}{dx} = 0$$

$$2\frac{V_0}{\delta}\left(1 - e^{-(x-x_0)/\delta}\right)\left(e^{-(x-x_0)/\delta}\right) = 0$$

$$1 - e^{-(x-x_0)/\delta} = 0$$

$$\ln(1) = -(x - x_0)/\delta$$

$$\therefore \quad x = x_0$$



Substituting in the main equation, the value of the min $V(x)$ can be found as;

$$V(x_0) = -V_0$$

Forces that Depend on Time

The Concept of Impulse

Forces of extremely short duration in time, such as those exerted by bodies undergoing collisions, are called impulsive forces.

If we confine our attention to one body, or particle, the differential equation of motion is

$$d(mv) = F\,dt.$$

Let us take the time integral over the interval t_1 to t_2 , the time during which the force is considered to act, then we have

$$\Delta(mv) = \int_{t_1}^{t_2} F\,dt = P$$

The *time integral of the force* is ***the impulse***. It is usually denoted by the symbol ***P***.

The Impulse

Note:

1- The ***work*** is equal to the change in the energy of the particle.

$$\Delta T = \int_{x_1}^{x_2} F\,dx = W$$

2- The ***impulse*** is equal to the change in the momentum of the particle.

$$\Delta(mv) = \int_{t_1}^{t_2} F\,dt = P$$

Forces that Depend on Velocity

Fluid Resistance and Terminal Velocity

The force acts on a body is often a function of its velocity.
For example, the viscous resistance exerted on a body moving through a fluid depends on its velocity. In such case, the differential equation of motion may be written in either of the two forms

$$F_0 + F(v) = m\frac{dv}{dt}$$

$$F_0 + F(v) = mv\frac{dv}{dx}$$

Here F_0 is any constant force that does not depend on v.

Since, $F(v)$ is a complex function and must be found through experimental measurements, it can be replaced by the following approximation :

$$F(v) = -c_1 v - c_2 v|v|$$

$$F(v) = -v(c_1 + c_2|v|)$$

where c_1 and c_2 are constants whose values depend on the size and shape of the body.

Linear or Quadratic ?

For spheres in air,

$$c_1 = 1.55 \times 10^{-4} D \qquad \& \qquad c_2 = 0.22\, D^2$$

where D is the diameter of the sphere in *meters*.
For *small* v the ***linear*** term in $F(v)$ can be used , while the ***quadratic*** term dominates at *large* v.

To decide whether the case is linear or quadratic, the ratio of the latter to the former usually used;

$$\frac{c_2 v|v|}{c_1 v} = \frac{0.22\, v|v| D^2}{1.55 \times 10^{-4} v D} = 1.4 \times 10^3 |v| D$$

If the value of v will make the ratio **exceeds 1** then it is a quadratic case, otherwise, it is a linear one.

Linear Resistance

(Exp.2.4.1)

Horizontal Motion through a Fluid

Suppose a block is projected with initial velocity v_0 on a smooth horizontal surface and that there is air resistance such that the ***linear term*** dominates.

Hence, $F_0 = 0$, and $F(v) = -c_1 v$.

The differential equation of motion is then; $$-c_1 v = m\frac{dv}{dt}$$

By integrating, $$t = -\int_{v_0}^{v}\frac{m\,dv}{c_1 v} = -\frac{m}{c_1}\ln\left(\frac{v}{v_0}\right)$$

Solving for v as a function of t gives; $$v = v_0 e^{-c_1 t/m}$$

A second integration gives $$x = \int_0^t v_0 e^{-c_1 t/m}\,dt$$

or $$x = \frac{m v_0}{c_1}\left(1 - e^{-c_1 t/m}\right)$$

Showing that after a long time ($t \sim \infty$) the block approaches a *limiting position* given by;

$$x_{\lim} = m v_0 / c_1$$

Horizontal Motion through a Fluid

Quadratic Resistance

(Exp.2.4.2)

The differential equation of motion in this case is;

$$-c_2v^2 = m\frac{dv}{dt}$$

Similarly we can get v and the position x as a function of time.

Exercise

Check the Book's results Using Maple

Vertical Fall through a Fluid

1- Linear Case

For an object falling vertically in a resisting fluid, the force F_0 in this case, is the weight of the object, *-mg*. For the linear case of fluid resistance, the differential equation of motion is;

$$-mg - c_1 v = m\frac{dv}{dt}$$

Integrating and solving for v, we get

$$v = -\frac{mg}{c_1} + \left(\frac{mg}{c_1} + v_0\right)e^{-c_1 t/m}$$

Terminal Velocity

After a sufficient time ($t \gg m/c_1$), the velocity approaches a limiting value ($-mg/c_1$). This limiting velocity of a falling body is called ***the terminal velocity*** (v_t). Hence *the terminal speed* is;

$$v_t = \frac{mg}{c_1}$$

The value of v_t/g *is known as* the ***characteristic time*** of the motion (τ). I.e ,

$$\tau = \frac{v_t}{g} = \frac{m}{c_1}$$

Note:

At the velocity v_t the force of resistance is just equal and opposite to the weight of the body so that the **net force** is **zero**, and so the **acceleration** is **zero**.

2- Quadratic case:

In this case $F(v) \propto v^2$ and the *differential equation* of motion is;

$$-mg - c_2 v^2 = m\frac{dv}{dt}$$

Similarly, ***the terminal speed*** *is ;*

$$v_t = \sqrt{\frac{mg}{c_2}}$$

And the ***characteristic time*** *is;*

$$\tau = \frac{v_t}{g} = \sqrt{\frac{m}{c_2 g}}$$

See **(Exp.2.4.3)**