بسم الله الرحمن الرحيم

Lectures 16-21

Amplitude at the Resonance Peak

The amplitude at the resonant frequency, which we call A_{max} , is

$$A_{\max} = \frac{F_0 / m}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$$

When the damping is weak:

If $\gamma << \omega_0$, *the resonant frequency* ω_r *, the freely running damped oscillator frequency wd , and the natural frequency* ω_0 *of the oscillator are essentially* ***identical***. I.e;

$$\omega_r \approx \omega_d \approx \omega_0$$

When the damping is strong:

If $\gamma >> \omega_0^2/2$, **no amplitude resonance occurs**, because the amplitude then becomes a totally decreasing function of ω_0

Sharpness of the Resonance

Let us consider the case of *weak damping* $(\omega_r \approx \omega_d \approx \omega_0)$.
Then, the expression for steady-state amplitudeis

$$A(\omega) \approx \frac{A_{\max}\gamma}{\sqrt{(\omega_0 - \omega)^2 + \gamma^2}}$$

The above equation shows that when $|\omega_0 - \omega| = \gamma$, or equivalently, if $\omega = \omega_0 \pm \gamma$,then

$$A^2 = \tfrac{1}{2} A_{\max}^2$$

This means that 2γ *is a measure of the width of the resonance curve* $\Delta\omega$ *at half-energy points.*

since the *quality factor* Q for *weak damping case* is given by*;*

$$Q = \frac{\omega_d}{2\gamma} \approx \frac{\omega_0}{2\gamma}$$

Thus, the width of the resonance curve $\Delta\omega$ is;

$$\boxed{\Delta\omega = 2\gamma \approx \frac{\omega_0}{Q}}$$

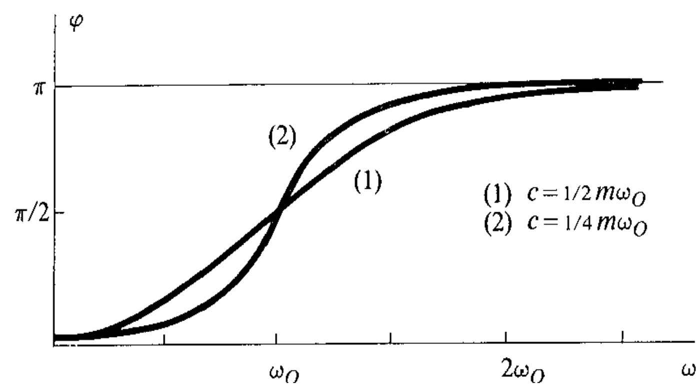
Phase Angle

The phase difference between the applied driving force and the steady-state response is given by

$$\boxed{\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}}$$

The plot of this relation shows ϕ *as a function of the driving* frequency ω.

1- For small ω :
the phase difference is **zero** ($\phi = 0$) and remains small. So; *the response is **in phase** with the driving force*.



2- Near the resonance frequency:
Actually at $\omega = \omega_r$, the phase angle ϕ increases to $\pi/2$ and so; the response is **90° out of phase** at this frequency.

3- For large values of ω:
the value of ϕ *approaches* π, hence; *the motion of the system is just 180° **out of phase** with the driving force*

Electrical-Mechanical Analogs

There is an exact analogy between a moving **mechanical system** of *masses* and *springs* with frictional forces and an **electric circuit** containing; *inductance, capacitance,* and *resistance.*

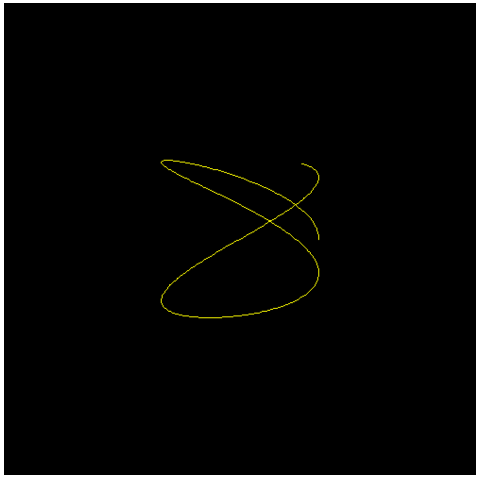
Mechanical	Electrical
Displacement x	Charge q
Velocity dx/dt	Current dq/dt
Mass m	Inductance L
Stiffness constant k	Reciprocal of capacitance C^{-1}
Damping constant c	Resistance R
Force F	Potential difference V

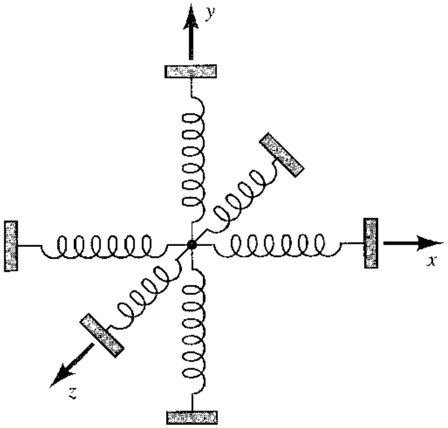
The Harmonic Oscillator in 2D and 3D

Consider the motion of a particle attached to a set of elastic springs as shown in the Figure. This is the three-dimensional generalization of the linear oscillator studied earlier.

The differential equation of the motion simply can be expressed as;

$$\boxed{m\frac{d^2\mathbf{r}}{dt^2} = -k\mathbf{r}}$$





2D Isotropic Oscillator:

First, we will consider the motion of the **isotropic** oscillator, in which the *restoring force is* **independent** of the *direction of the displacement*. In the case of motion in a single plane, **2D**, we get is two equations of motion;

$$m\ddot{x} = -kx \qquad m\ddot{y} = -ky$$

These are separated, so their solutions can be in the form

$$x = A\cos(\omega t + \alpha) \qquad y = B\cos(\omega t + \beta)$$

With $$\omega = \sqrt{k/m}$$

The constants *A, B,* α, and β are determined from the *initial conditions* .

To find the equation of the path, we eliminate the time *t* between the x & y equations.

$$y = B\cos(\omega t + \beta + \alpha - \alpha) = B\cos(\omega t + \alpha + \Delta)$$

$$= B\left[\cos(\omega t + \alpha)\cos\Delta - \sin(\omega t + \alpha)\sin\Delta\right]$$

$$\frac{y}{B} = \frac{x}{A}\cos\Delta - \left(1 - \frac{x^2}{A^2}\right)^{1/2}\sin\Delta$$

We then have a quadratic equation in x and y;

$$\frac{x^2}{A^2} - xy\frac{2\cos\Delta}{AB} + \frac{y^2}{B^2} = \sin^2\Delta$$

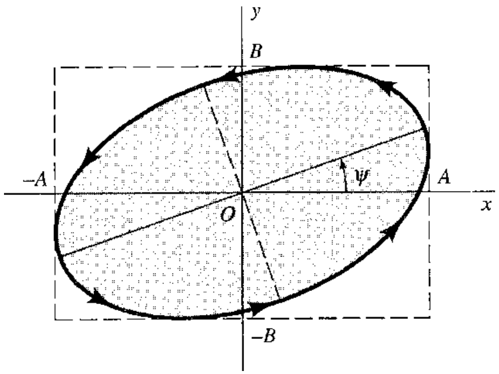
Recall: The general quadratic equation;

$$ax^2 + bxy + cy^2 + dx + ey = f$$

represents;

	An ellipse,	if	$b^2 - 4ac < 0$
	A parabola,	if	$b^2 - 4ac = 0$
or	A hyperbola,	if	$b^2 - 4ac > 0$

In our case the discriminant is equal to $-(2\sin\Delta/AB)^2$, which is **negative**, so the path is an **ellipse**. If the phase difference Δ is equal to $\pi/2$, then the equation of the path reduces to the equation



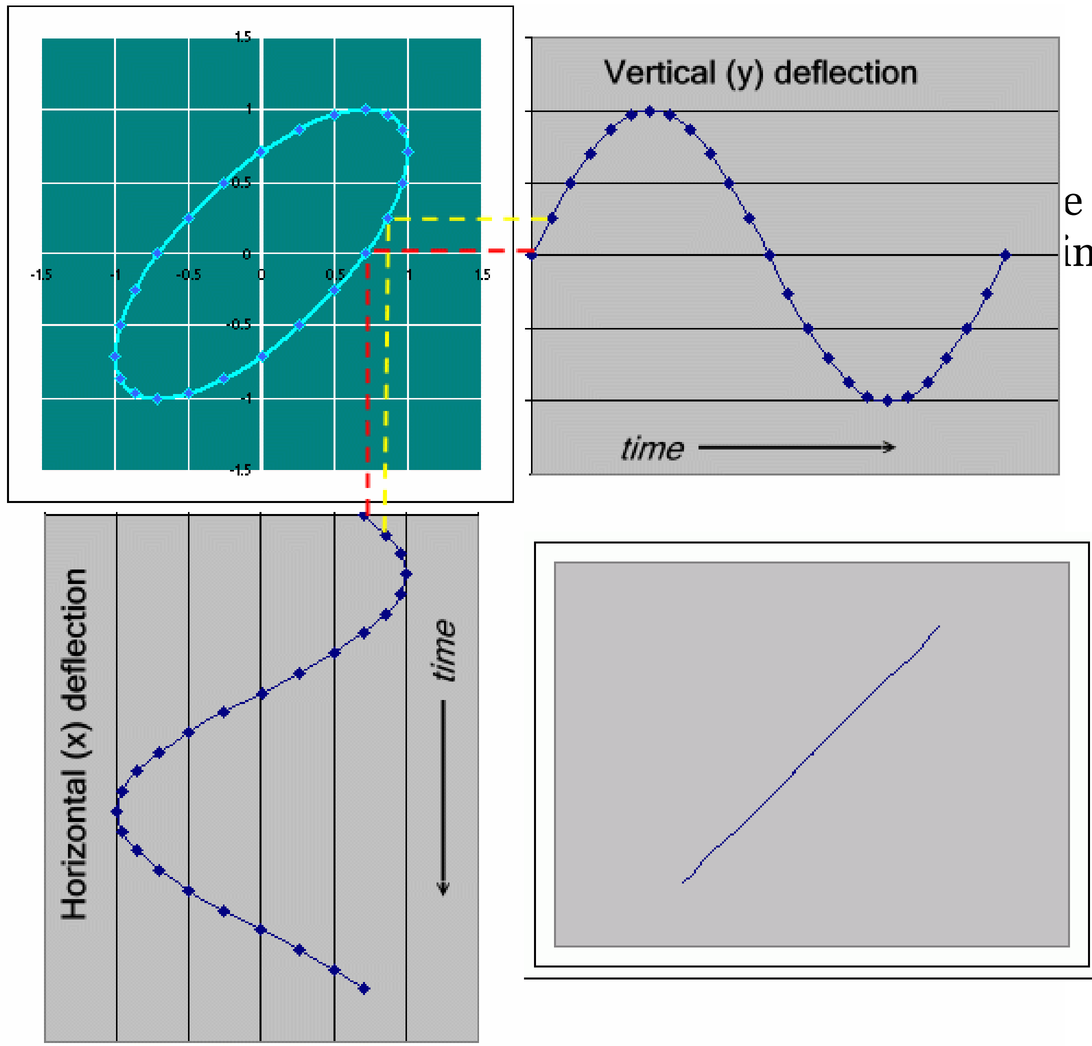
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

which is the equation of an **ellipse** whose axes coincide with the coordinate axes.

If the phase difference Δ is equal 0 or π, then the equation of the path reduces to that of straight line;

$$y = \pm \frac{B}{A} x$$

Si e place
tot in that
pl



3D Non-isotropic Oscillator:

If the magnitudes of the components of the *restoring force* **depend** on the *direction of the displacement, we have the case of the* **non-isotropic** oscillator. The differential equations for this case can be written as;

$$m\ddot{x} = -k_1 x \qquad m\ddot{y} = -k_2 y \qquad m\ddot{z} = -k_3 z$$

Here we have three different frequencies of oscillation, ω_1 , ω_2 and ω_3, and the motion is given by the solutions

$$x = A\cos(\omega_1 t + \alpha)$$
$$y = B\cos(\omega_2 t + \beta)$$
$$z = C\cos(\omega_3 t + \gamma)$$

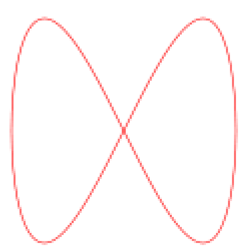
The resulting oscillation of the particle lies completely within a rectangular box (whose sides are *2A, 2B, and 2C) centered on the origin.*

Commensurate oscillator

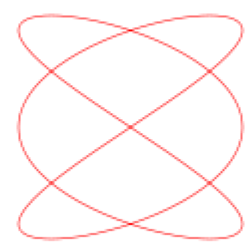
If ω_1 , ω_2 and ω_3 are **commensurate**-that is,

$$\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2} = \frac{\omega_3}{n_3}$$

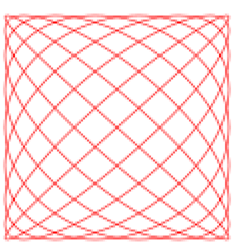
Where n_1, n_2 *and* n_3 *are integers, the path is* **closed**, i.e. the particle returns to its initial position and the motion is repeated. Such a path called *Lissajous figure*.



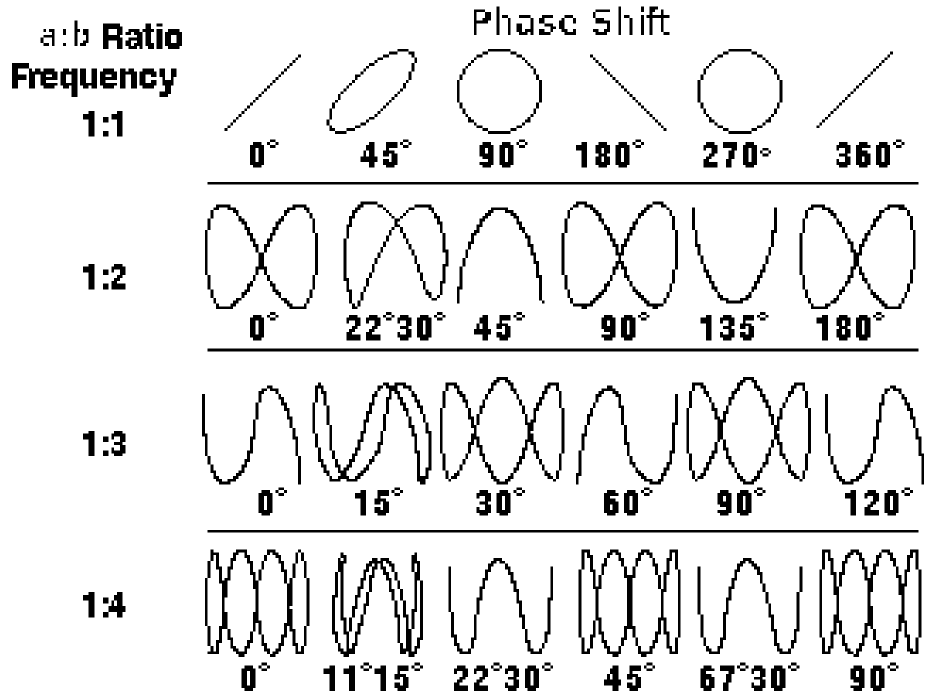
$\omega_1 : \omega_2 =$ 1:2



$\omega_1 : \omega_2 =$ 3:2



$\omega_1 : \omega_2 =$ 9:8



Non-Commensurate oscillator

On the other hand, If ω_1 , ω_2 and ω_3 are **not commensurate**, the path is **not closed** and the path will completely fill the rectangular box.

Energy Considerations

For the general 3D case, it is easy to verify that

$$V(x,y,z) = \tfrac{1}{2}k_1x^2 + \tfrac{1}{2}k_2y^2 + \tfrac{1}{2}k_3z^2$$

If $k_1 = k_2 = k_3 = k$, *we have the* **isotropic** case, and

$$V(x,y,z) = \tfrac{1}{2}k\left(x^2 + y^2 + z^2\right) = \tfrac{1}{2}kr^2$$

The total energy in the **isotropic** case then given by;

$$\boxed{E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kr^2}$$

Motion of Charged Particles in Electric & Magnetic Fields

When a charged particle is surrounded by other electric charges, it will experience a force. This force *F* is caused by the *electric field E*, which arises from these other charges. We write

$$\boldsymbol{F} = q\boldsymbol{E}$$

where *q is the electric charge*. The equation of motion of the particle is then

$$m\frac{d^2\mathbf{r}}{dt^2} = q\mathbf{E}$$

Let us consider a case of a **uniform constant electric field** which is directed along the z-axis. Then $E_x = E_y = 0$, *and* $E = E_z$. The differential equations of motion of a particle of charge q moving in this field are then

$$\ddot{x} = 0 \qquad \ddot{y} = 0 \qquad \ddot{z} = \frac{qE_z}{m} = cons.$$

These are of the same form as those for a projectile in a uniform gravitational field.

Therefore, the **path** is a *parabola, if* v_x *and* v_y *if are not both* zero initially. Otherwise, the path is a *straight line, as with a body falling vertically.*

According to the electromagnetic theory, if *E* is due to static charges then;

$$\boxed{\nabla \times \mathbf{E} = \mathbf{0}}$$

This means that motion in such a field is *conservative,* and that there exists *a potential function* F such that;

$$\boxed{\mathbf{E} = -\nabla\Phi}$$

Then, the force *F* caused by the electric field *E* may be rewritten as;

$$\mathbf{F} = q\mathbf{E} = -\nabla q\Phi$$

Then, the potential energy of a particle of charge *q* in such a field is then *qF*. The **total energy**, hence, is constant and is equal to

$$\boxed{\tfrac{1}{2}mv^2 + q\Phi}$$

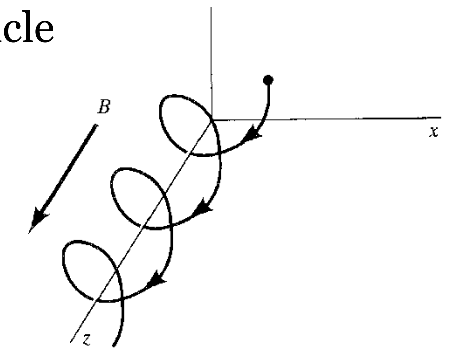
In the presence of a static magnetic field *B*, *called the magnetic induction, the force acting on a moving particle is conveniently expressed by means of the cross product,* namely,

$$\boxed{\mathbf{F} = q(\mathbf{v} \times \mathbf{B})}$$

where **v** is the velocity and *q* is the charge.

The differential equation of motion of a particle moving in a purely magnetic field is then

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{v}\times\mathbf{B})$$



This equation states that the **acceleration** of the particle is always $\perp$ to **the direction of motion**. This means that the ***tangential*** component of the ***acceleration*** is ***zero,*** and so the particle **moves with constant speed**.

The **path** is a *helix* , *and i*f there is no component of the velocity in the *z direction, the path is a circle.*

Constrained Motion of a Particle

When a moving particle is **restricted geometrically**, i.e. *it must stay on a certain definite surface or curve*, the motion is said to be ***constrained***.

Examples of constrained motion:
- A piece of ice sliding around a bowl. (one-sided constraint)
- A bead sliding on a wire. (complete constraint)

EXAMPLE 4.6.1

A particle is placed on top of a smooth sphere of radius a. *If the particle is slightly disturbed, at what point will it leave the sphere?*

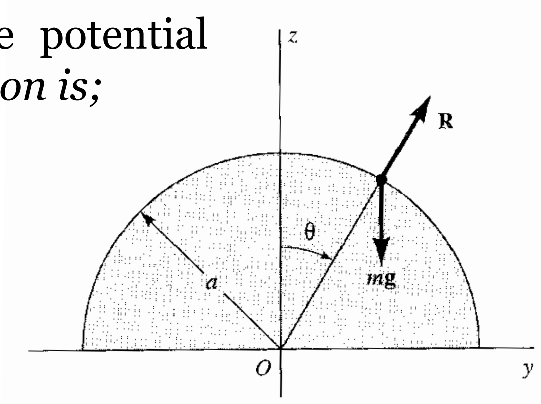
Solution:

The forces acting on the particle are the downward force of gravity $m\mathbf{g}$ and the reaction $\mathbf{R}$ of the spherical surface. The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + \mathbf{R}$$

According to the chosen coordinate the potential energy is then mgz, *and the energy equation is;*

$$\tfrac{1}{2}mv^2 + mgz = E$$



But from the initial conditions ($v = 0$ *for* $z = a$) *we have* $E = mga$*, so;*

$$\tfrac{1}{2}mv^2 + mgz = mga$$

As the particle slides down, its speed is given by;

$$v^2 = 2g(a - z)$$

$$-\frac{mv^2}{a} = mg\cos\theta + R = -mg\frac{z}{a} + R$$

Hence,

$$R = \frac{mg}{a}(3z - 2a)$$

Thus, R *vanishes when*

$$z = \tfrac{2}{3}a$$

At this point the particle leaves the sphere.

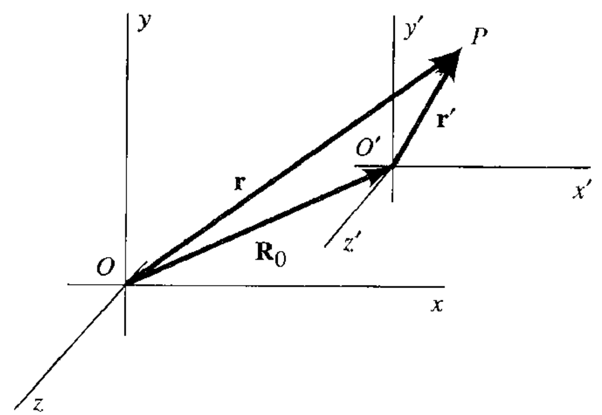
Accelerated Coordinate Systems:

Recall: Uniformly moving reference frames (e.g. those considered at 'rest' or moving with constant velocity in a straight line) are called *inertial reference frames*.

Sometimes it is necessary, to employ a coordinate system that is **not** *inertial*.

The **position vector** of a particle *P is denoted by* **r** in the fixed system and by **r'** in the moving system. The displacement *OO' of the moving origin is denoted by* $\mathbf{R_o}$. Thus, from the triangle *OO'P, we have*

$$\boldsymbol{r} = \boldsymbol{R}_o + \mathbf{r}'$$



Taking the first and second time derivatives gives

$$\mathbf{v} = \mathbf{V_o} + \mathbf{v}'$$
$$\mathbf{a} = \mathbf{A_o} + \mathbf{a}'$$

in which $\mathbf{V_o}$ and $\mathbf{A_o}$ are, respectively, the velocity and acceleration of the moving system, and **v'** and **a'** are the velocity and acceleration of the particle in the moving system.

If the moving system is **not accelerating**, i.e. it is also **inertial**, so that $\mathbf{A_o} = 0$, then

$$\mathbf{a} = \mathbf{a}'$$

In this case we cannot specify a unique coordinate system, because Newton's laws will be the same in both systems.
For example, Newton's second law in fixed system $\mathbf{F} = m\mathbf{a}$ becomes $\mathbf{F}' = m\mathbf{a}'$ in the moving system.

On the other hand if the moving system is **accelerating**, then Newton's second law becomes

$$\mathbf{F} = m\boldsymbol{A}_{\mathbf{o}} + m\boldsymbol{a}'$$

or

$$\boxed{\mathbf{F} - m_{\mathbf{o}} = \mathbf{F}'}$$

where $(-m\mathbf{A_o})$ is known as the inertial term or inertial force. Such "force" is not due to interactions with other bodies; rather, it happens as a result of the acceleration of the reference system

Rotating Coordinate Systems

Assume that the axes of the both coordinate systems have a common origin. Let the rotation of the rotated system takes place about some specific axis of rotation, whose direction is designated by a unit vector, **n**.

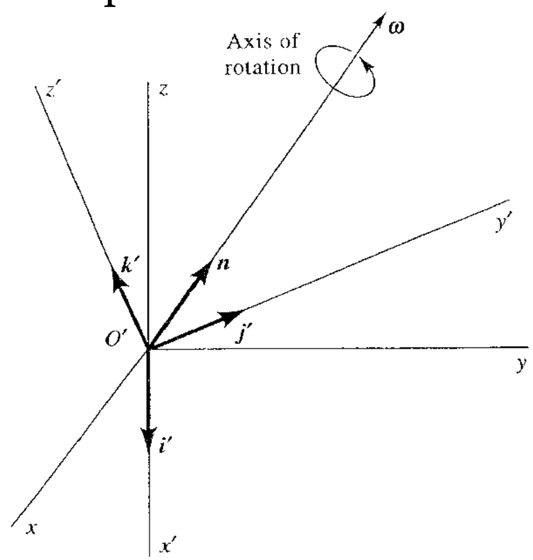
The *angular velocity* of the rotating system then is;

$$\boldsymbol{\omega} = \omega \mathbf{n}$$

The direction of the velocity vector is given by the right-hand rule.

Because the coordinate axes of the two systems have the same origin, the vector **r** in the fixed system equals the vector **r'** in the rotating system, that is,

$$\boxed{\mathbf{r} = \mathbf{r}'}$$



or;

$$\mathbf{i}x + \mathbf{j}y + \mathbf{k}z = \mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z'$$

When we differentiate with respect to time to find the velocity, we must keep in mind the fact that the unit vectors **i'**, **j'**, and **k'** are **not constant**. Thus, we can write the velocity vector **v** in the fixed system as;

$$\mathbf{v} = \mathbf{v}' + x'\frac{di'}{dt} + y'\frac{dj'}{dt} + z'\frac{dk'}{dt}$$

From the definition of the cross product, we can write;

$$\frac{d\,\mathbf{i}'}{dt} = \omega \times i' \qquad \frac{d\,\mathbf{j}'}{dt} = \omega \times j' \qquad \text{and} \qquad \frac{d\,\mathbf{k}'}{dt} = \omega \times k'$$

Hence;

$$x'\frac{d\mathbf{i}'}{dt} + y'\frac{d\mathbf{j}'}{dt} + z'\frac{d\mathbf{k}'}{dt} = x'(\boldsymbol{\omega} \times \mathbf{i}') + y'(\boldsymbol{\omega} \times \mathbf{j}') + z'(\boldsymbol{\omega} \times \mathbf{k}')$$
$$= \boldsymbol{\omega} \times \mathbf{r}'$$

And so, the velocity can be rewritten as

$$\boxed{\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'}$$

Coriolis acceleration

Taking the first time derivatives gives the acceleration in the fixed system in terms of the position, velocity, and acceleration in the rotating system;

$$\boxed{\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')}$$

If the moved system is undergoing both **translation and rotation**, the general equations for transforming from a fixed system to a moving and rotating system will be:

$$\boxed{\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \mathbf{V}_0}$$

And;

$$\boxed{\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0}$$

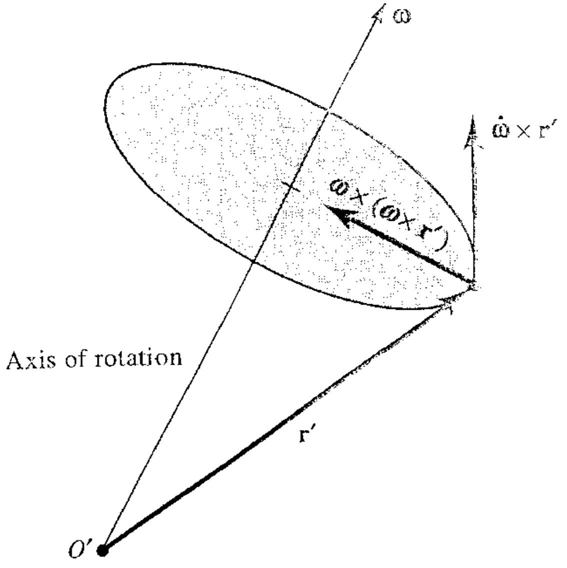
The term $2\omega \times \mathbf{v}'$ is known as ***the Coriolis acceleration***, which appears whenever a particle moves in a rotating coordinate system except when the velocity $\mathbf{v}'$ is parallel to the axis of rotation.

Centripetal acceleration

The term $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}')$ is called ***the centripetal acceleration***. which is the result of the particle being carried around a circular path in the rotating system. It is always **directed toward the axis of rotation** and is **perpendicular to the axis** as shown in the figure.

Transverse acceleration

The term $\dot{\boldsymbol{\omega}}\times\mathbf{r}'$ is called ***the transverse acceleration***, *because it is perpendicular to the position vector* $\mathbf{r}'$. It appears whenever the rotating system has an angular acceleration, i.e. **if the angular velocity vector is changing in either magnitude or direction, or both**.

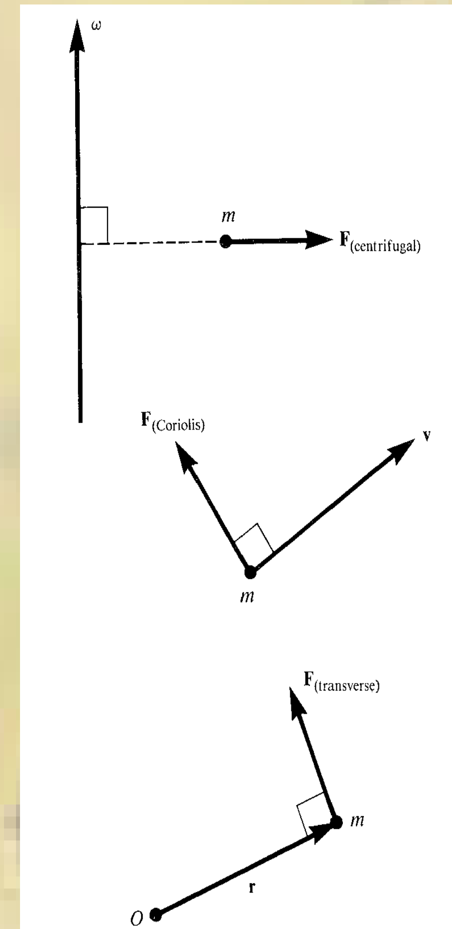


Dynamics of a Particle in a Rotating System

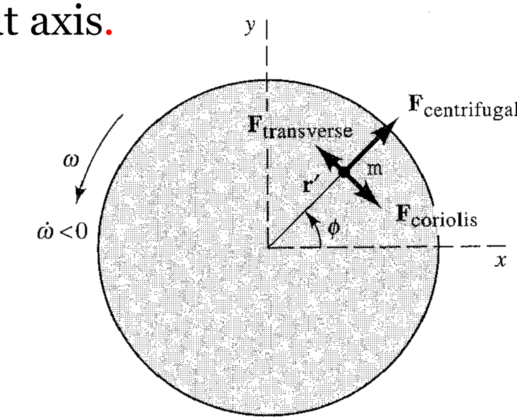
According to $\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \mathbf{A}_0$

The equation of motion of a particle in a noninertial frame of reference is ;

$$\boxed{\mathrm{F} - m\dot{\omega} \times \mathrm{r}' - 2m\omega \times \mathrm{v}' - m\omega \times (\omega \times \mathrm{r}') - m\mathrm{A}_0 = m\mathrm{a}'}$$



- The force - $m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times \mathbf{r}')$ is the centrifugal force, which is the result of the particle being carried around a circular path in the rotating system. It is directed **outward away** from the axis of rotation and is **perpendicular** to that axis.



- The force $-2m\boldsymbol{\omega}\times\mathbf{v}'$ is the Coriolis force, which appears whenever a particle moves in a rotating coordinate system. Its direction is always **perpendicular** to **v'**.

- The force $-m\,\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ is called the transverse force, because it is **perpendicular** to the position vector **r'**. It is present **only** if there is an angular acceleration (or deceleration) of the rotating coordinate system.

Effects of Earth's Rotation

- Consider a coordinate system that is moving with the Earth. Because the angular speed of Earth's rotation is 2π radians per day, the effects of such rotation is relatively small.

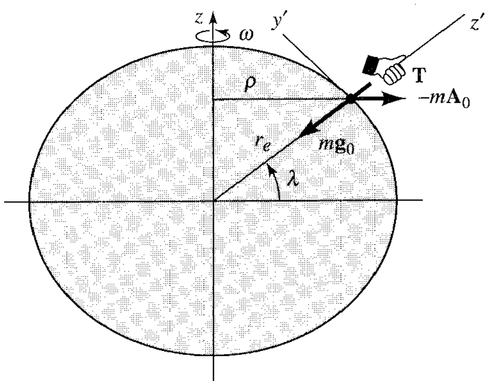
- Nevertheless, it is the spin of the Earth that makes the equatorial radius is some 21 km greater than the polar radius, i.e. *equatorial bulge*.

Static Effects:
The Plumb line

Let us describe the motion of the plumb bob in a local frame of reference whose origin is at the position of the bob. Our frame of reference is attached to the surface of the Earth, so it is undergoing translation as well as rotation.

- The **translation** of the frame takes place along a circle whose radius is $\rho = r_e \cos\lambda$, where r_e is the radius of the Earth and λ is the geocentric latitude of the plumb bob. Hence;

$$A_0 = \omega^2 \rho = \omega^2 r_e \cos\lambda$$



Its rate of **rotation** is w, the same as that of the Earth about its axis. Let us now examine the terms of

$$F - m\dot{\omega}\times r' - 2m\omega\times v' - m\omega\times(\omega\times r') - mA_0 = ma'$$

transverse force **zero**, because ω is **constant.** **zero**, because **v'**= 0.

Coriolis force **zero**, because the bob is at rest

centrifugal force **zero**, because **r'**= 0

Thus,

$$\mathbf{F} - m\mathbf{A_0} = 0$$

In other words, the bob does not hang on a line pointing toward the center of the Earth because the inertial force $-m\mathbf{A_o}$ throws it outward, away from Earth's axis of rotation. The magnitude of this force is;

$$\boxed{mA_0 = m\omega^2 r_e \cos\lambda}$$

The ***tension*** **T** in the string balances out the real *gravitational force* $m\mathbf{g}_0$ and the *inertial force* $-m\mathbf{A}_0$, i.e;

As can be seen the inertial reaction $-m\mathbf{A}_0$ causes the direction of the plumb line to **deviate** by a small angle ε away from the direction toward Earth's center, and

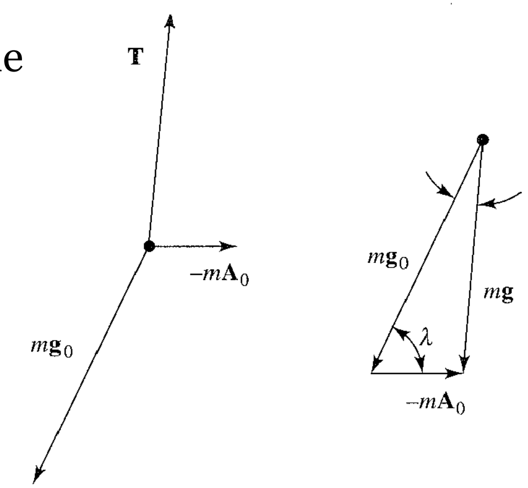
$$mg = mg_0 - mA_0$$

We can easily calculate the value of the angle ε. From the Figure we have;

$$\frac{\sin\varepsilon}{m\omega^2 r_e \cos\lambda} = \frac{\sin\lambda}{mg}$$

or, because ε is small

$$\boxed{\sin\varepsilon \approx \varepsilon = \frac{\omega^2 r_e}{2g}\sin 2\lambda}$$



(a) (b)

Thus, ε **vanishes** at the equator ($\lambda = 0$) and the poles ($\lambda = \pm 90$). The **maximum deviation** of the direction of the plumb line from the center of the Earth occurs at $\lambda = 45°$ where;

$$\varepsilon_{\max} = \frac{\omega^2 r_e}{2g} \approx 0.1^o$$

Kepler's laws and the motion of planets

One of the great intellectual events of the 16th and 17th centuries was the threefold realization;
1- that the earth is also a planet,
2- that all planets orbit the sun,
3- and that the apparent motions of the planets as seen from the earth can be used to determine the orbits of the planets precisely.

The first and second of these ideas were published by *Nicolaus Copernicus* in 1543. The determination of planetary orbits was carried out between 1609 and 1619 by the German astronomer and mathematician *Johannes Kepler*.

By trial and error, *Kepler* discovered *three observed laws* that accurately described the motions of all planets .

Two hundred years later, Newton discovered that each of Kepler's laws can be derived using Newton's laws of motion and the law of gravitation.

In fact, Kepler's laws provided one of the foundations of Newton's theory of *universal gravitation*.

First law:
Law of Ellipses

Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.

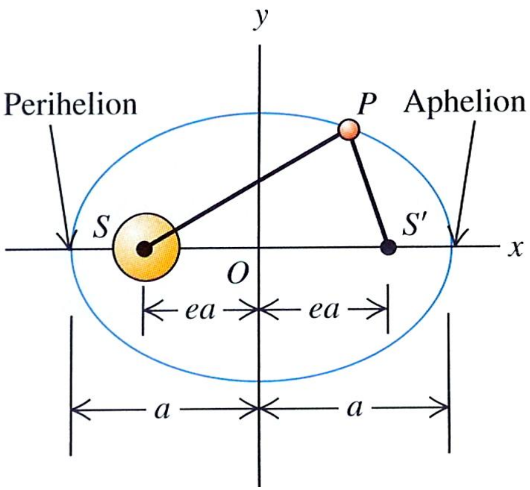
Main properties of the ellipse:

- The longest dimension $2a$ is the major axis, with half-length a known as the *semi-major axis*.

- S and S' are the *foci* . The sun is at S, and the planet is at S'

- The point in the planet's orbit closest to the sun is the *perihelion*, and the point most distant from the sun is the *aphelion*.

- The distance of each focus from the center of the ellipse is ea, where e is a dimensionless number between 0 and 1 called the *eccentricity*. If **e = 0**, the ellipse is a circle. The actual orbits of the planets are somewhat circular; their eccentricities range from **0.007** for Venus to **0.248** for Pluto. The earth's orbit has **e = 0.017**



Second law: Law of Equal Areas

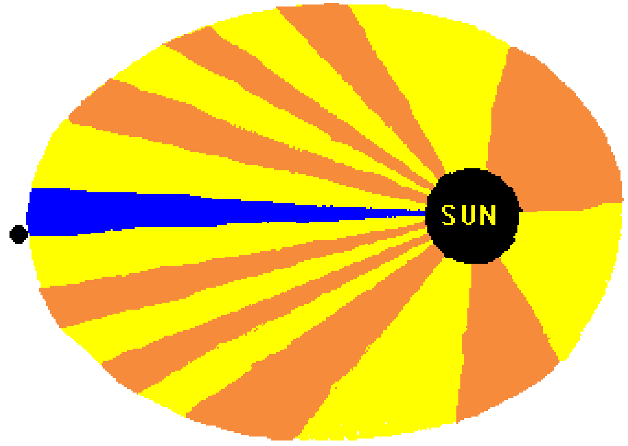
A line from the sun to a given planet sweeps out equal areas in equal times.

In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out is the

$$dA = \tfrac{1}{2} r^2 \, d\theta .$$

The rate at which area is swept out, dA/dt, is called the ***sector velocity***:

$$\boxed{\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}}$$



The real meaning of Kepler's second law is that *the sector velocity has the same value at all points in the orbit*. When the planet is close to the sun, r is small and $d\theta/dt$ is large; when the planet is far from the sun, r is large and $d\theta/dt$ is small.

Third law:
Harmonic Law

The ratio of the squares of the periods for two planets is equal to the ratio of the cubes of their semi-major axes.

This law can be expressed as;

$$\boxed{\frac{T_1^2}{T_2^2}=\frac{a_1^3}{a_2^3}} \quad \Rightarrow \quad \boxed{\frac{T_1^2}{a_1^3}=cons.}$$

convenient unit of measurement for periods is in Earth years, and a convenient unit of measurement for distances is the average separation of the Earth from the Sun, which is termed an *astronomical unit* and is abbreviated as AU.
If these units are used in Kepler's 3rd Law, the constant in the previous equation are numerically equal to unity and it may be written in the simple form:

$$\boxed{\frac{T_{yrs}^2}{a_{AU}^3}=1}$$

This Law (unlike the first two) ties together the motions of different planets.

$$\frac{T_{yrs}^{2}}{a_{AU}^{3}} = 1$$

	a in AU	T in yrs	T^2/a^3
Mercury	**0. 387**	**0. 241**	**1. 002**
Venus	**0. 723**	**0.615**	**1.001**
Earth	**1. 000**	**1.000**	**1.000**
Mars	**1. 524**	**1. 881**	**1. 000**
Jupiter	**5. 203**	**11. 862**	**0. 999**
Saturn	**9. 534**	**29.456**	**1.001**

Kepler's 3rd law can be then rewritten as

$$T = 2\pi \frac{a^{3/2}}{\sqrt{Gm_s}}$$

EXAMPLE (6.6.1): Find the period of a comet whose semi-major axis is 4 AU.

Solution:

With T measured in years and a in *astronomical units*, we have

$T^2 = a^3 = (4)^3 = 64 \text{ yrs}^2$ → $T = 8 \text{ yrs}$

Example : Comet Halley

Comet Halley moves in an elongated elliptical orbit around the sun. At perihelion, the comet is 8.75 × 107 km from the sun; at aphelion it is 5.26 × 109 km from the sun. Find the semi- major axis, eccentricity, and period of the orbit.

Solution:

- The length of the major axis is;

$$2a = 8.75 \times 10^7 + 5.26 \times 10^9$$

So; $a = 2.67 \times 10^9$ km

- Since the comet-sun distance at perihelion is given by

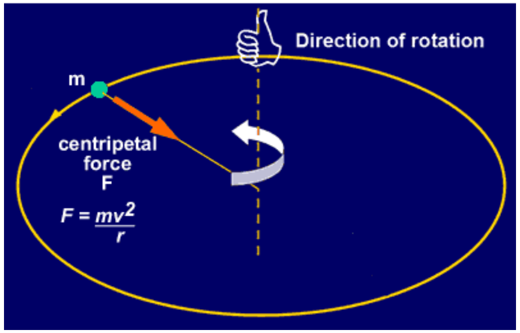
$$a - ea = a\,(1- e) = 8.75 \times 10^7$$

Then; $e = 0.967$

$$T = 2\pi \frac{(2.67\times10^{12})^{3/2}}{\sqrt{(6.67\times10^{-11})(1.99\times10^{30})}} = 2.38\times10^{9}\,s = 75.5 \text{ years}$$

Newton's Law of Gravitation

Circular motion



❍ During his study of the motions of the planets and of the moon, Newton found the fundamental charter of the gravitational attraction between any two bodies.

❍ Builds upon the idea that **any curved motion** is due to some **FORCE** that provides the ***Centripetal acceleration*** , and for **the** *Uniform Circular Motion* this acceleration is**:** $\boxed{a = v^2 / R}$

Then, ***the Centripetal Force*** must be given by something like;

$$\boxed{F = m\, v^2 / R}$$

Kepler's Third Law Provides a key;

$$T^2 = k\, R^3$$

- But, period = $T = 2\pi\, R / v$ ⇨ $4\pi^2 R^2 / v^2 = k\, R^3$
- Therefore, $v^2 = 4\pi^2 / k\, R$
- Substituting this form for v^2 into Newton's 2nd Law gives:

$$\boxed{F = \frac{4\pi^2}{k}\frac{m}{R^2}}$$

$$F = \frac{4\pi^2}{k}\frac{m}{R^2}$$

- This is the **force** that the Sun must exert on a planet of mass m, orbital radius R, in order that the planet obeys Kepler's Laws in the circular motion.

- Consider Newton's 3rd Law, there must be an equal force also exert on the Sun by the planet, but in the opposite direction.

The only form of the law that is symmetric in the two masses is:

$$F = GM\frac{m}{R^2}$$

Where M and m are the mass of the sun and mass of the planet, R is the distance between them and G is a universal constant. The numerical value of G (in SI units) is

G = 6.67×10-11 *N.m2/kg2*

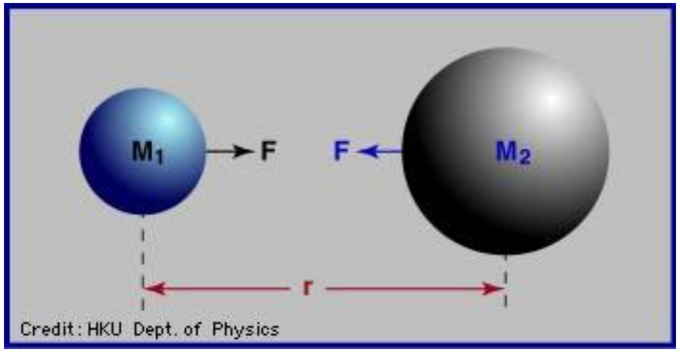
So, Kepler's constant k is equal to;

$$k = \frac{4\pi^2}{GM}$$

❍ Newton, then, generated this result for all bodies in his famous **law of gravitation** which may be stated as follows:

Every particle of matter in the universe attracts every other particle with force (Fg) that is directly proportional to the product of the masses (m1, m2) of the particles and inversely proportional to the square of the distance between them (r).

$$\boxed{F_g = \frac{Gm_1m_2}{r^2}}$$



❍ This law tells us that if the distance r is doubled, the force decreases by a factor of four, and so on.

❍ Even when the masses of the particles are different, the two interaction forces have equal magnitude.

gravitational potential energy

- The earth's gravitational force on a body of mass m at any point outside the earth is given by ; $F_g = Gm_Em/r^2$, where m_E is the mass of the earth and r is the distance of the body from the earth's center. Therefore, we can express the ***gravitational potential energy*** (U) in more general form as;

$$\boxed{U_g = -\frac{Gm_E m}{r}}$$