OPTICS

PHYS 311

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SUPERPOSITION

OF

WAVES

Introduction

Our earlier look at the mathematics of wave motion concentrated on the description of a single wave. However, much of Optics involve the superposition of waves in one way or another. Basic processes of reflection and refraction can only be treated satisfactory in terms of overlapping of waves.

The Superposition Principle

recall that the individual components of a wave satisfy the Laplace equation

$$\nabla^2\psi = \frac{1}{\upsilon^2}\frac{\partial^2\psi}{\partial t^2}$$

A LINEAR EQUATION

If ψ_1 and ψ_2 are independently solutions of the wave equation, then the *linear combination* of them, $\psi = \psi_1 + \psi_2$ is also a solution

This is known as the **principle of superposition**

Superposition of EM wave can be expressed in terms of E or B

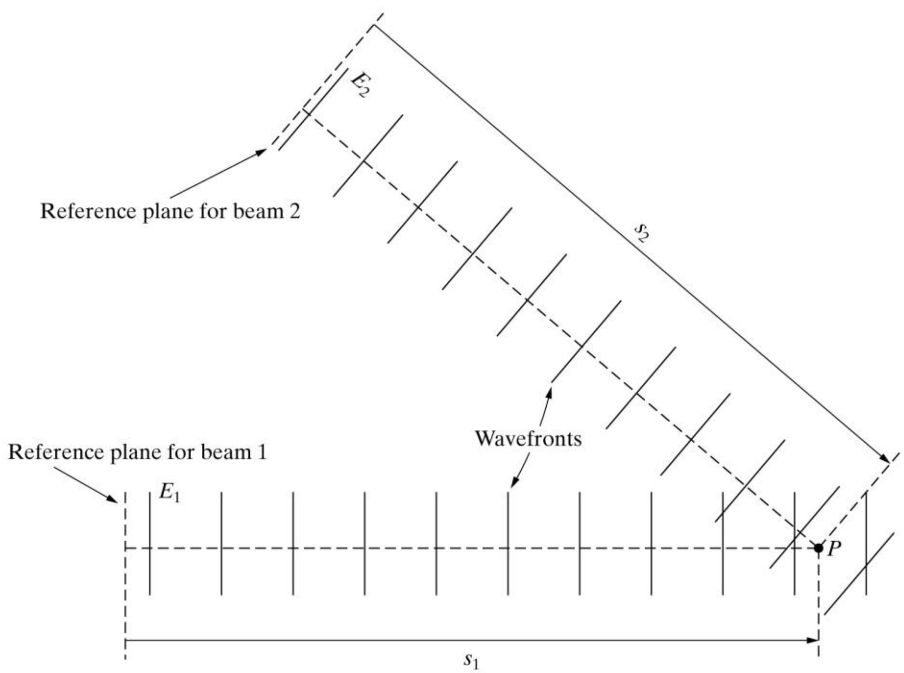
$$\vec{E} = \vec{E}_1 + \vec{E}_2 \quad , \quad \vec{B} = \vec{B}_1 + \vec{B}_2$$

Superposition of waves ← same ω

lets consider waves which have different amplitudes and phase but the same frequency

$$E_1 = E_{01}\,cos\left(ks_1 - \omega t + \varphi_1\right)$$

$$E_2 = E_{02}\,cos\left(ks_2 - \omega t + \varphi_2\right)$$



Superposition of waves ← same ω

$$E_1 = E_{01}\cos(ks_1 - \omega t + \varphi_1)$$

$$E_2 = E_{02}\cos(ks_2 - \omega t + \varphi_2)$$

For simplification…

$$\alpha_1 = ks_1 + \varphi_1$$

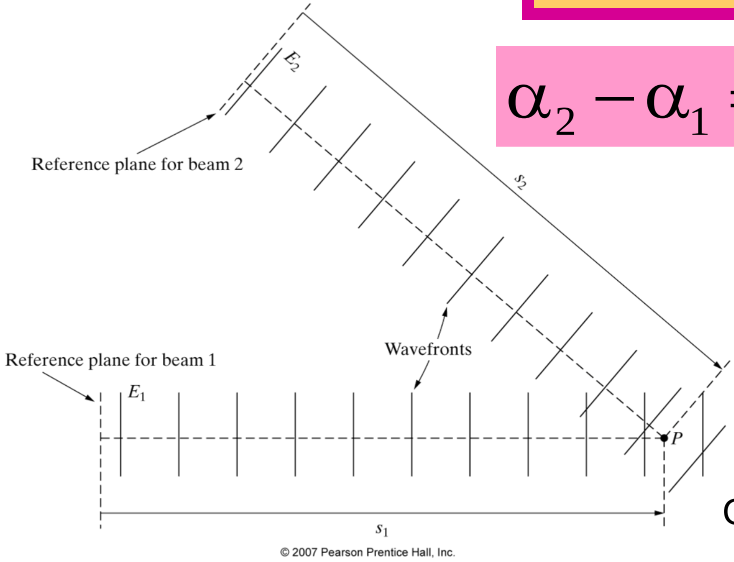
$$\alpha_2 = ks_2 + \varphi_2$$

$$E_1 = E_{01}\cos(\alpha_1 - \omega t)$$

$$E_2 = E_{02}\cos(\alpha_2 - \omega t)$$

The phase difference…

$$\alpha_2 - \alpha_1 = k(s_2 - s_1) + (\varphi_2 - \varphi_1)$$

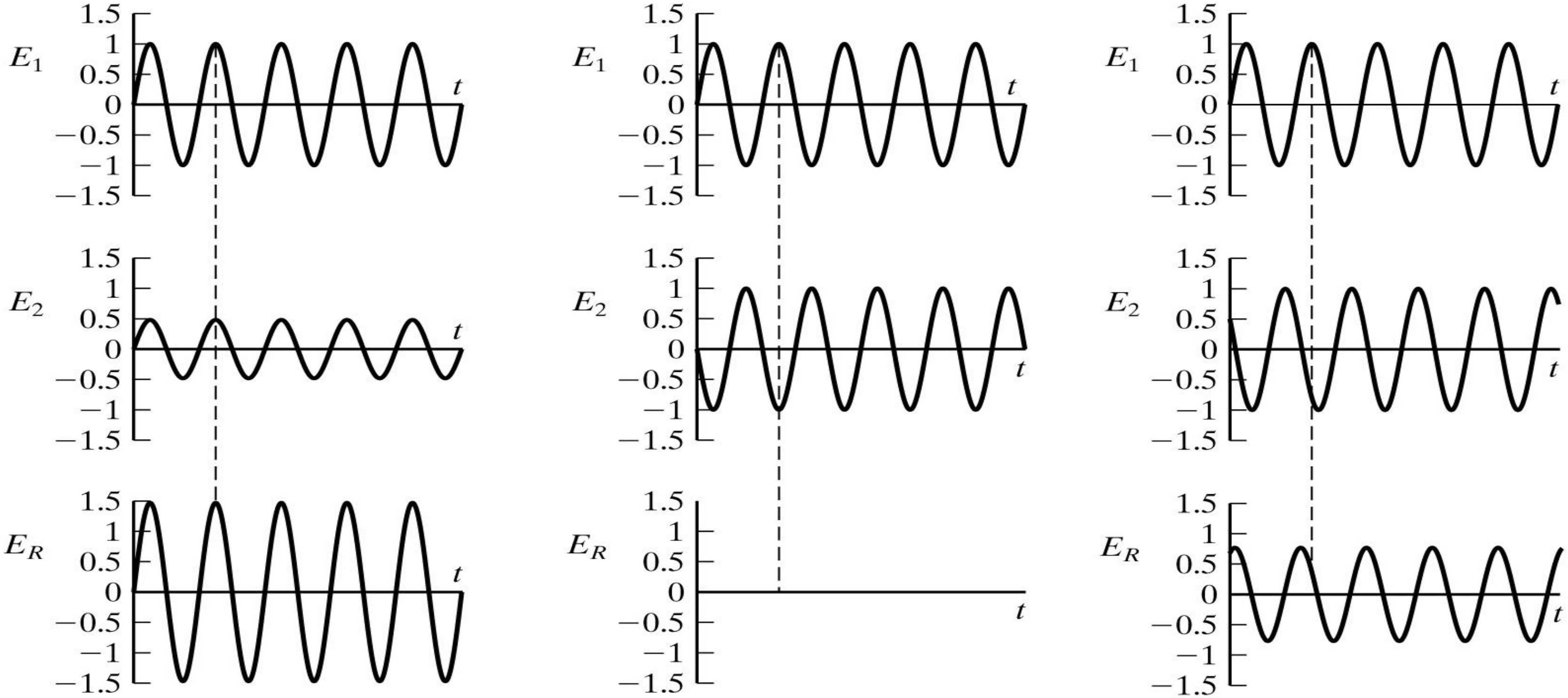


$$E_R = E_1 + E_2 = E_{01}\cos(\alpha_1 - \omega t) + E_{02}\cos(\alpha_2 - \omega t)$$

Superposition of waves ← same ω

$$E_R = E_1 + E_2 = E_{01}\,cos(\alpha_1 - \omega t) + E_{02}\,cos(\alpha_2 - \omega t)$$

Constructive | **Destructive** | **General**

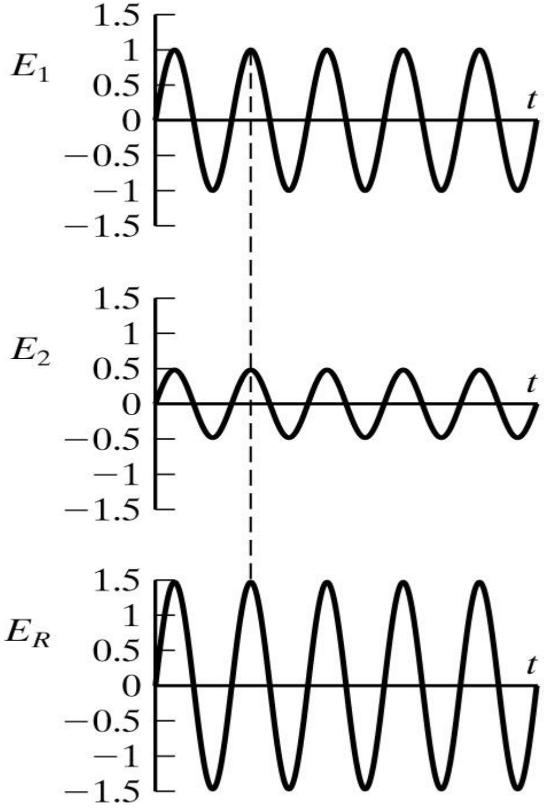


(a) (b) (c)

Superposition of waves ← same ω

Constructive

$$E_R = E_1 + E_2 = E_{01} \cos(\alpha_1 - \omega t) + E_{02} \cos(\alpha_2 - \omega t)$$



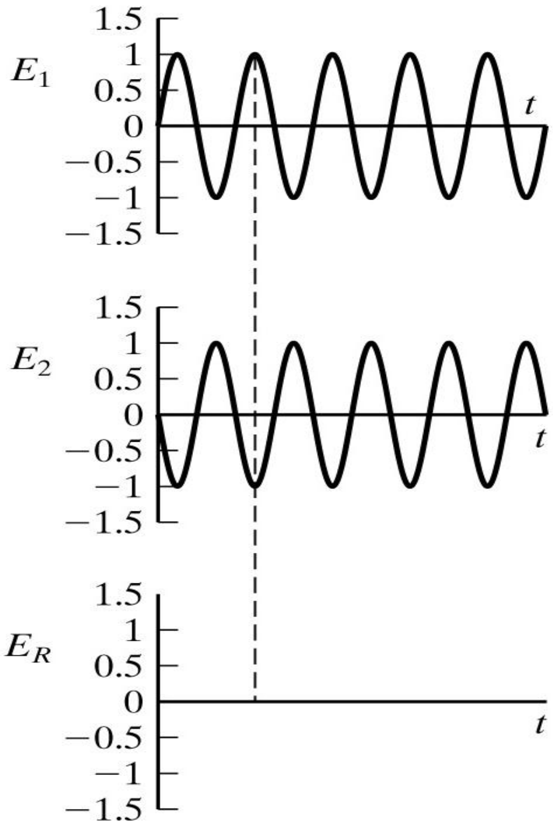
(a)

$$\begin{aligned} E_R &= E_1 + E_2 \\ &= E_{01} \cos(\alpha_1 - \omega t) + E_{02} \cos(\alpha_1 + 2m\pi - \omega t) \\ &= (E_{01} + E_{02}) \cos(\alpha_1 - \omega t) \end{aligned}$$

Superposition of waves ← same ω

Destructive

$$E_R = E_1 + E_2 = E_{01}\,cos(\alpha_1 - \omega t) + E_{02}\,cos(\alpha_2 - \omega t)$$



(b)

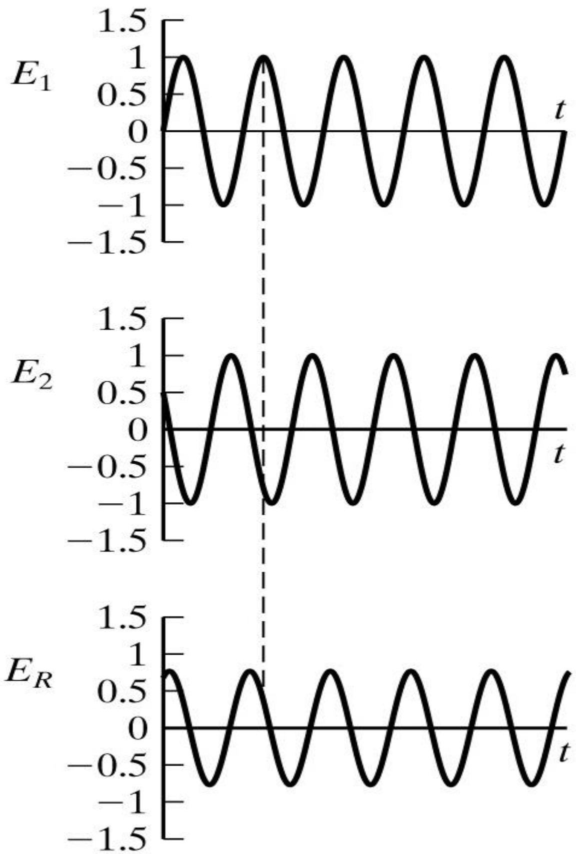
$$\begin{aligned} E_R &= E_1 + E_2 \\ &= E_{01}\,cos(\alpha_1 - \omega t) + E_{02}\,cos(\alpha_1 + (2m+1)\pi - \omega t) \\ &= (E_{01} - E_{02})\,cos(\alpha_1 - \omega t) \end{aligned}$$

Superposition of waves ← same ω

General

$$E_R = E_1 + E_2 = E_{01}\cos(\alpha_1 - \omega t) + E_{02}\cos(\alpha_2 - \omega t)$$

For simplicity, to find the resultant field, we use the complex form and phasor diagram..



(c)

$$E_R = E_1 + E_2$$
$$= Re(E_{01}e^{i(\alpha_1 - \omega t)} + E_{02}e^{i(\alpha_2 - \omega t)})$$
$$= Re(e^{-i\omega t}(E_{01}e^{i\alpha_1} + E_{02}e^{i\alpha_2}))$$

$$E_0 e^{i\alpha} = E_{01}e^{i\alpha_1} + E_{02}e^{i\alpha_2}$$

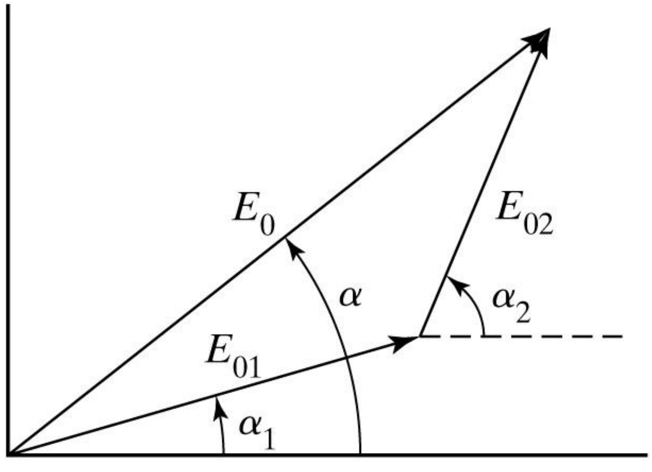
$$E_R = Re(E_0 e^{i(\alpha - \omega t)})$$
$$= E_0 \cos(\alpha - \omega t)$$

We use **phasors**

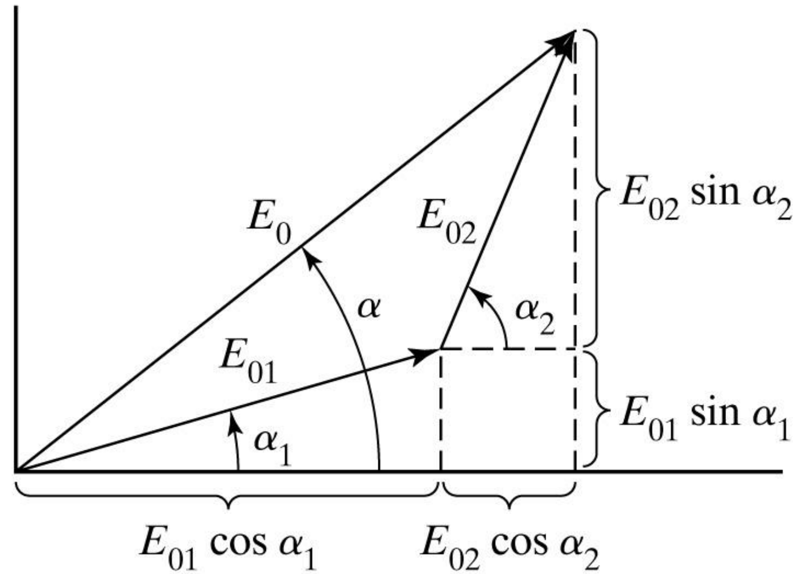
Superposition of waves ← same ω

General

Phasor diagram



(a)



(b)

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02}\cos(\alpha_1 - \alpha_2)$$

$$\tan\alpha = \frac{E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2}{E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2}$$

$$E_0\cos\alpha = E_{01}\cos\alpha_1 + E_{02}\cos\alpha_2$$

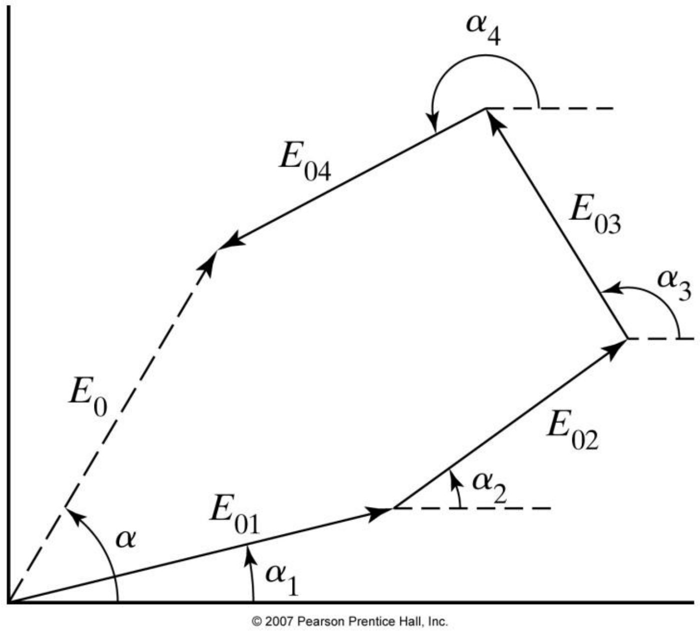
$$E_0\sin\alpha = E_{01}\sin\alpha_1 + E_{02}\sin\alpha_2$$

Superposition of waves ← same ω

General

Phasor diagram – more than two waves

$$\tan\alpha = \frac{\sum_{i=1}^{N} E_{0i}\sin\alpha_i}{\sum_{i=1}^{N} E_{0i}\cos\alpha_i}$$



$$E_0^2 = \left(\sum_{i=1}^{N} E_{0i}\sin\alpha_i\right)^2 + \left(\sum_{i=1}^{N} E_{0i}\cos\alpha_i\right)^2$$

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\cos(\alpha_j - \alpha_i)$$

RANDOM AND COHERENT SOURCES

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\cos(\alpha_j - \alpha_i)$$

Random phase

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 = NE_{01}^2$$

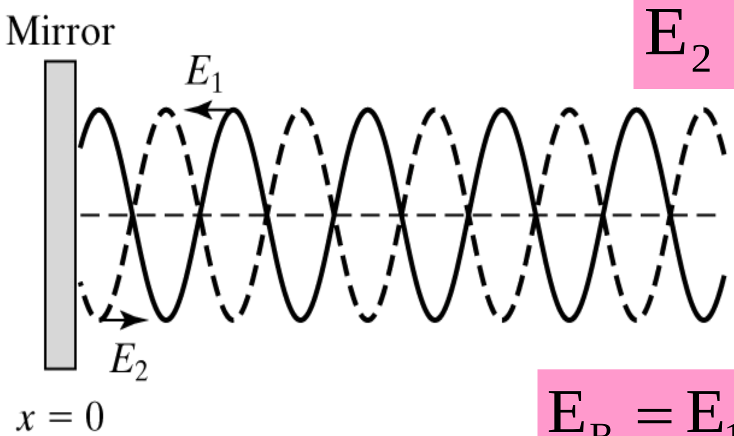
Coherent phase

$$E_0^2 = \sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>i}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}$$

$$E_0^2 = \left(\sum_{i=1}^{N} E_{0i}\right)^2 = (NE_{01})^2 = N^2E_{01}^2$$

STANDING WAVES

Two waves with equal amplitudes traveling in the directions +x and –x add as:



$$E_2 = E_0\, sin(\omega t - kx - \varphi_R)$$ **← To the right**

$$E_1 = E_0\, sin(\omega t + kx)$$ **← To the left**

$$E_R = E_1 + E_2 = E_0 [sin(\omega t + kx) + sin(\omega t - kx + \varphi_R)$$

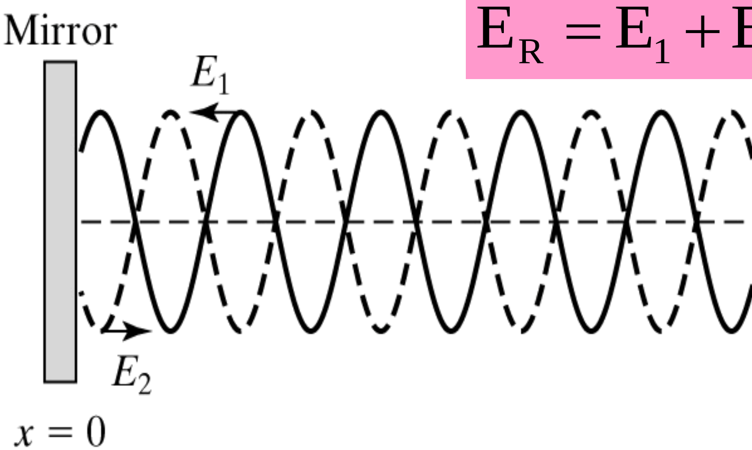
$$\beta_+ = \omega t + kx \quad , \quad \beta_- = \omega t - kx - \varphi_R$$

$$sin\beta_+ + sin\beta_- \equiv 2\, sin\frac{1}{2}(\beta_+ + \beta_-)cos\frac{1}{2}(\beta_+ - \beta_-)$$

$$E_R = 2E_0\, cos\left(kx + \frac{\varphi_R}{2}\right) sin\left(\omega t - \frac{\varphi_R}{2}\right)$$

STANDING WAVES

There is a π phase shift upon reflection

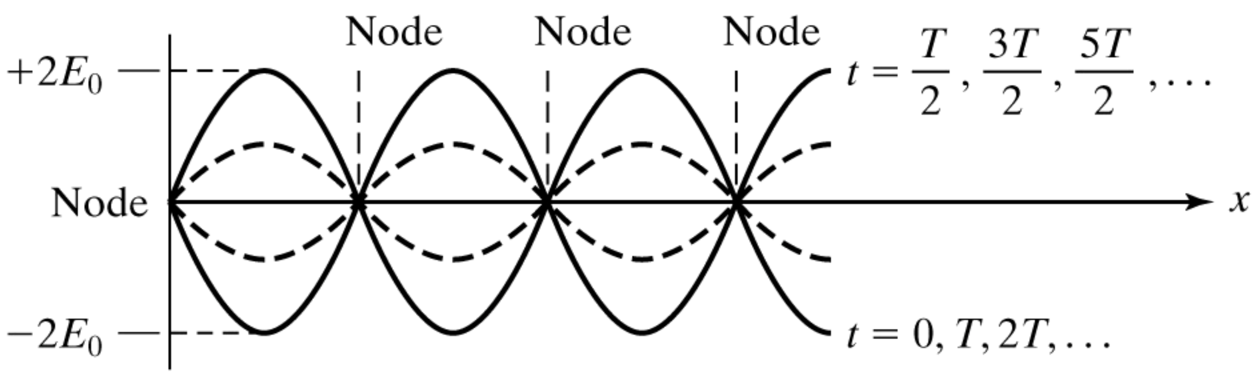


$$E_R = E_1 + E_2 = E_0 [sin(\omega t + kx) + sin(\omega t - kx + \varphi_R)$$

$$\frac{\varphi_R}{2} = \frac{\pi}{2}$$

$$E_R = 2E_0 \cos\left(kx + \frac{\varphi_R}{2}\right) \sin\left(\omega t - \frac{\varphi_R}{2}\right)$$

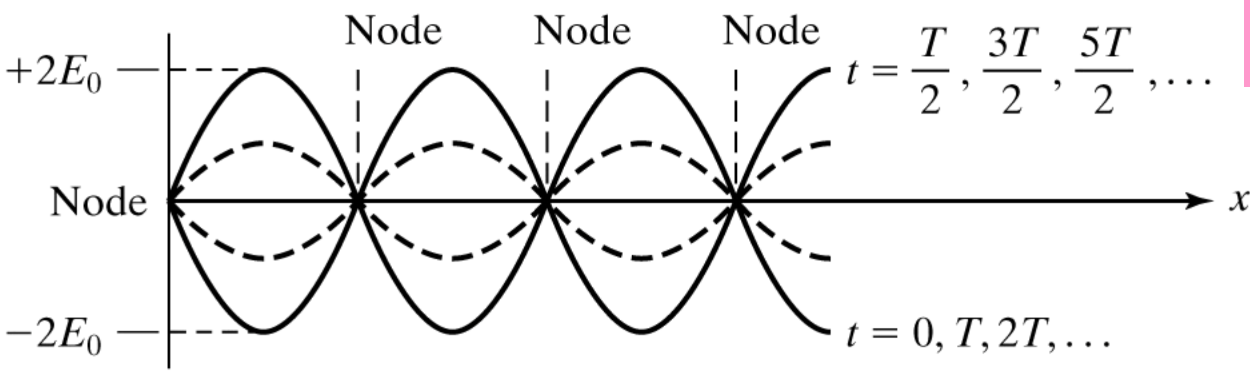
$$E_R = (2E_0 \sin kx) \cos \omega t$$



Space dependent amplitude

$$E_R = A(x) \cos \omega t$$

STANDING WAVES



$$E_R = A(x)\cos\omega t$$

$$A(x) = 2E_0 \sin kx$$

There exist values of x for which A(x) = o

$$\sin kx = 0 \quad , \quad kx = \frac{2\pi x}{\lambda} = m\pi \quad , \quad m = 0, \pm 1, \pm 2, \ldots$$

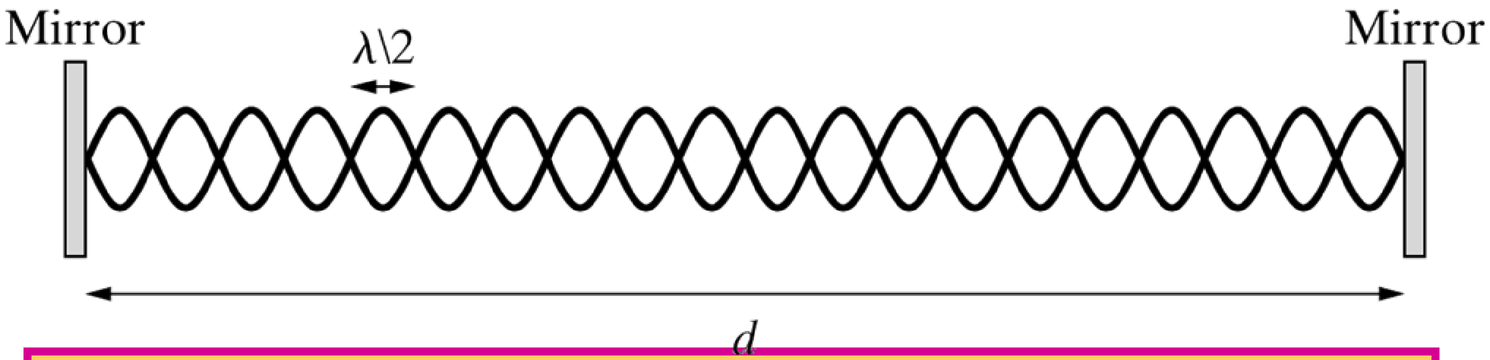
$$x = m\left(\frac{\lambda}{2}\right) = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots$$

These are called nodes

$$\omega t = 2\pi\nu t = \left(\frac{2\pi}{T}\right)t = m\pi \quad \Rightarrow t = m\left(\frac{T}{2}\right) = 0, \frac{T}{2}, T, \frac{3T}{2}, \ldots$$

Anti-nodes occur at time..

STANDING WAVES



The normal modes of a standing wave

$$d = m\left(\frac{\lambda_m}{2}\right)$$ **,m is a nonzero integer**

The wavelength and frequency of the standing waves modes

$$\lambda_m = \left(\frac{2d}{m}\right) \quad , \quad \nu_m = \frac{c}{\lambda_m} = m\left(\frac{c}{2d}\right)$$

THE BEAT PHENOMENON

Two waves with equal amplitudes with DIFFERENT frequencies but with the same speed (non-dispersive medium)

$$E_1 = E_0 \cos(k_1 x - \omega_1 t)$$

$$E_2 = E_0 \cos(k_2 x - \omega_2 t)$$

$$E_R = E_1 + E_2 = E_0 [\cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)]$$

$$\alpha = k_1 x - \omega_1 t \quad , \quad \beta = k_2 x - \omega_2 t$$

$$\cos\alpha + \cos\beta \equiv 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$$

$$E_R = 2E_0 \cos\left[\frac{(k_1+k_2)}{2}x - \frac{(\omega_1+\omega_2)}{2}t\right]\cos\left[\frac{(k_1-k_2)}{2}x - \frac{(\omega_1-\omega_2)}{2}t\right]$$

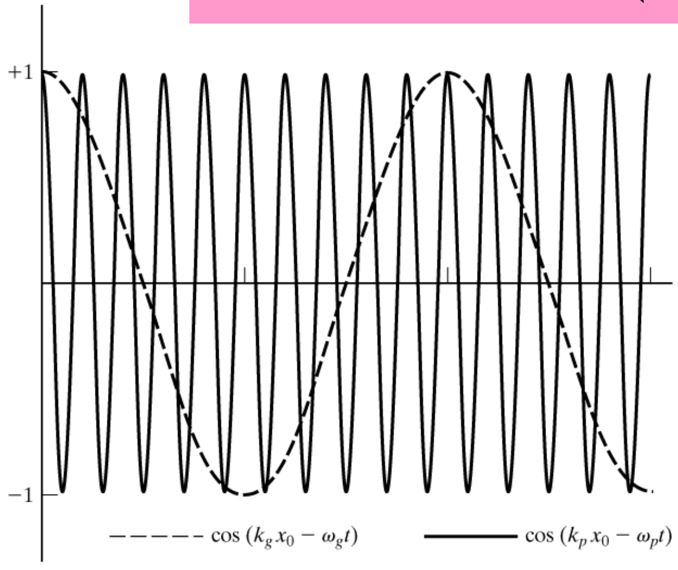
THE BEAT PHENOMENON

$$E_R = 2E_0 \cos\left[\frac{(k_1+k_2)}{2}x - \frac{(\omega_1+\omega_2)}{2}t\right]\cos\left[\frac{(k_1-k_2)}{2}x - \frac{(\omega_1-\omega_2)}{2}t\right]$$

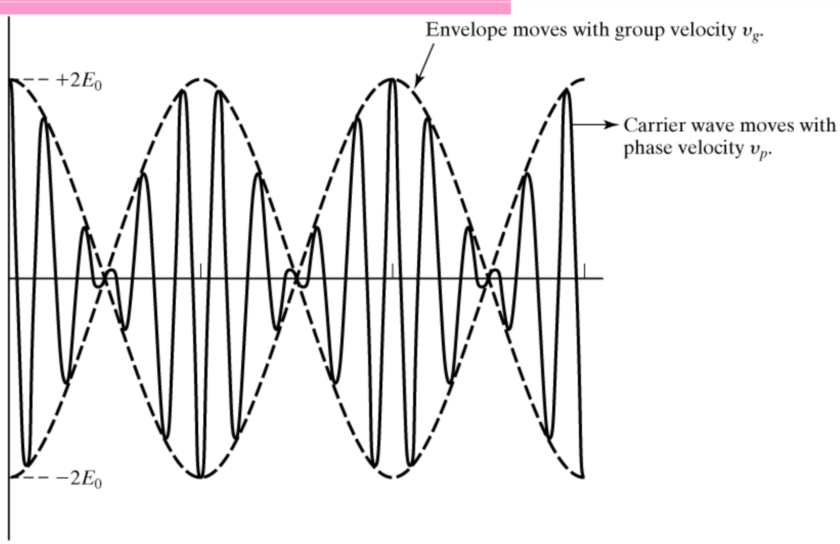
$$k_p = \frac{(k_1+k_2)}{2} \quad , \quad \omega_p = \frac{(\omega_1+\omega_2)}{2}$$

$$k_g = \frac{(k_1-k_2)}{2} \quad , \quad \omega_g = \frac{(\omega_1-\omega_2)}{2}$$

$$E_R = 2E_0 \cos(k_p x - \omega_p t)\cos(k_g x - \omega_g t)$$

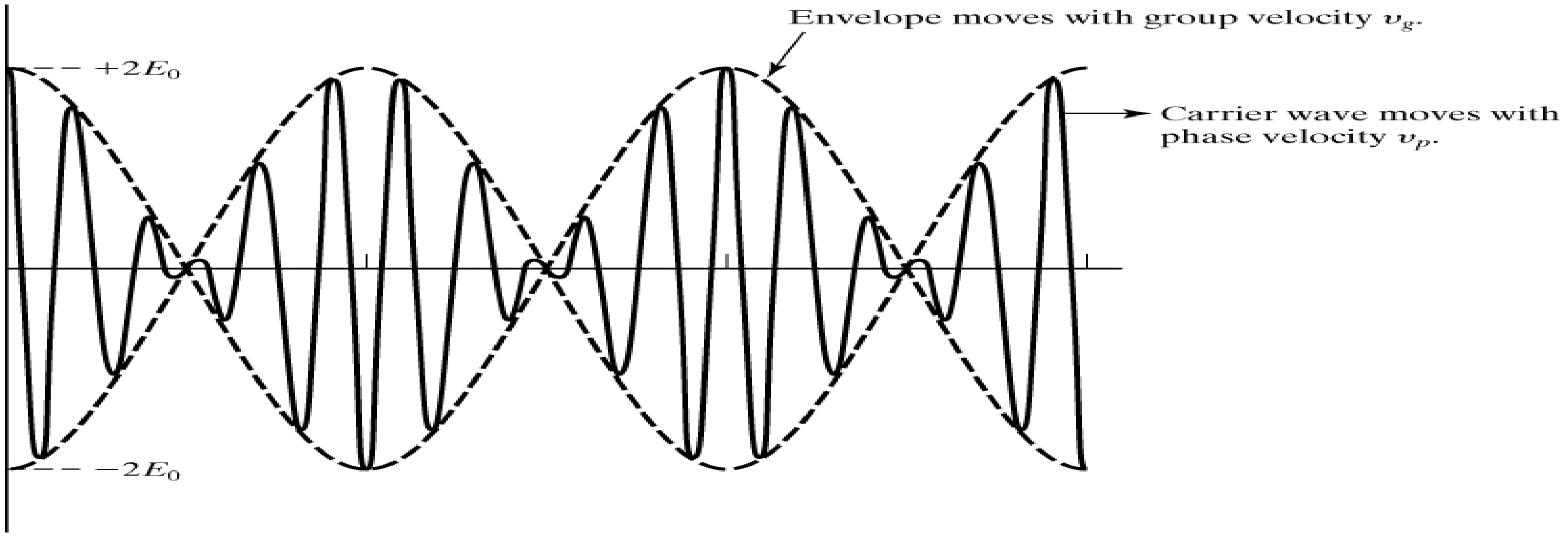


(a)



(b)

THE BEAT PHENOMENON



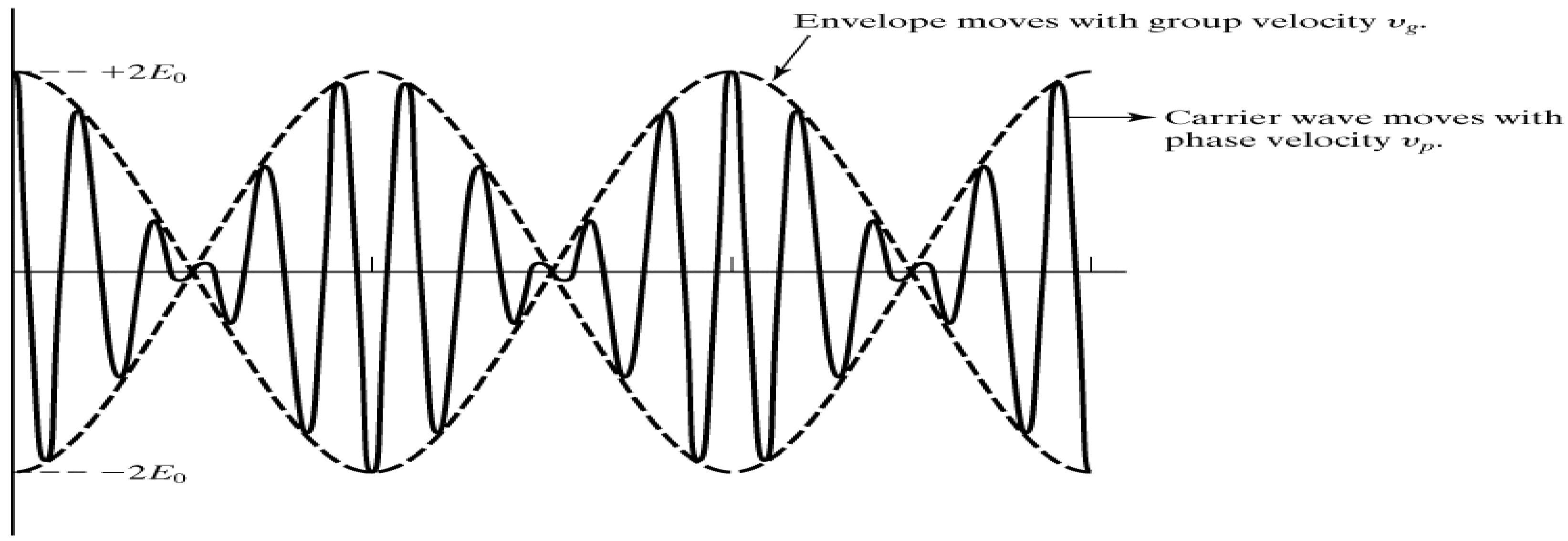
(b)

$$\omega_b = 2\omega_g = 2\left(\frac{\omega_1 - \omega_2}{2}\right) = \omega_1 - \omega_2$$

PHASE AND GROUP VELOCITIES

Any pulse of light can be viewed as a superposition of harmonic waves of different frequencies.

The duration of the pulse is inversely proportional to the range of frequencies.



(b)

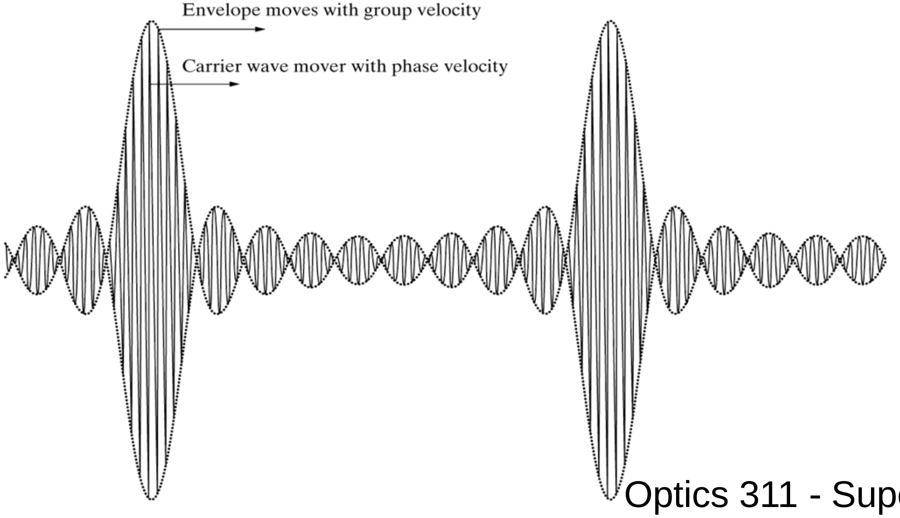
PHASE AND GROUP VELOCITIES

EM waves of different frequencies travel with different speeds through a given medium

dispersion

Phase velocity: The velocity of the harmonic waves that constitute the signal.

Group velocity: The velocity at which the positions of maximal constructive interference propagate.



PHASE AND GROUP VELOCITIES

$$\upsilon_p = \frac{\omega_p}{k_p} = \frac{\omega_1 + \omega_2}{k_1 + k_2} \cong \frac{\omega}{k}$$

$$\omega_1 \cong \omega_2 = \omega \quad , \quad k_1 \cong k_2 = k$$

$$\upsilon = \nu\lambda = \frac{\omega}{k}$$

$$\upsilon_g = \frac{\omega_g}{k_g} = \frac{\omega_1 - \omega_2}{k_1 - k_2} \cong \frac{d\omega}{dk}$$

The relation between phase and group velocities

$$\upsilon_g = \frac{d\omega}{dk} = \frac{d}{dk}(k\upsilon_p)$$

$$\upsilon_g = \upsilon_p + k\left(\frac{d\upsilon_p}{dk}\right)$$

In non-dispersive medium

$$\frac{d\upsilon_p}{dk} = 0$$

$$\upsilon_p = \upsilon_g = c$$

In dispersive medium

$$\frac{d\upsilon_p}{dk} = \frac{d}{dk}\left(\frac{c}{n}\right) = \frac{-c}{n^2}\left(\frac{dn}{dk}\right)$$

$$\frac{dn}{d\lambda} < 0$$

$$\upsilon_g < \upsilon_p$$

$$\upsilon_g = \upsilon_p\left[1 - \frac{k}{n}\left(\frac{dn}{dk}\right)\right]$$

$$\upsilon_g = \upsilon_p\left[1 + \frac{\lambda}{n}\left(\frac{dn}{d\lambda}\right)\right]$$