Geometry of Curves and Surfaces

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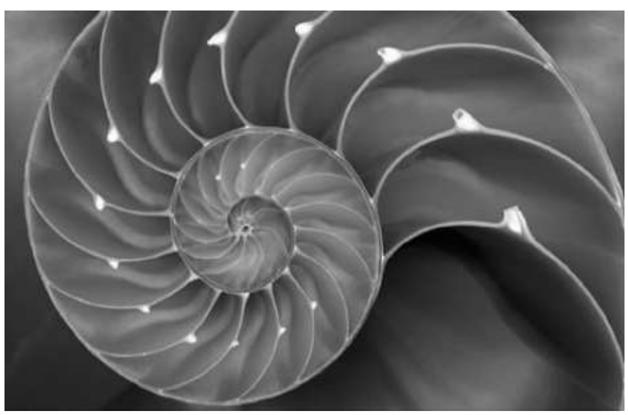
0 Why study geometry?

Curves and surfaces are all around us in the natural world, and in the built environment. The first step in *understanding* these structures is to find a mathematically natural way to *describe* them. That is the primary aim of this course, which will focus particularly on the concept of *curvature*.

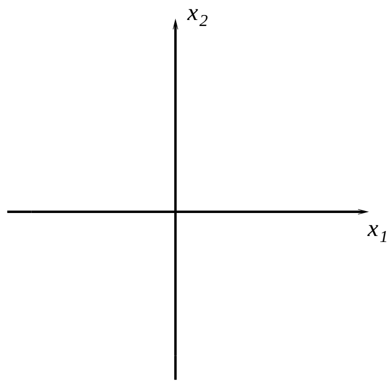


As a side effect, you will develop some very useful transferable skills. Key among these is the ability to translate a mathematical system (a set of equations, or formulae, or inequalities) into a visual picture. Your geometric intuition about the picture can then give you useful insight into the original mathematical system. This trick of visualizing mathematical systems can be very powerful and is, unfortunately, not strongly emphasized in the teaching of maths.

Example 0 How does the number of solutions of the pair of simultaneous equations

$$\begin{aligned} xy &= 1 \\ x^2 + y^2 &= a^2 \end{aligned}$$

depend on the constant $a > 0$?



So for $a < a_0$ (in fact, $a_0 = \sqrt{2}$), the system has 0 solutions, for $a = a_0$ it has 2 and for $a > a_0$ it has 4. One could easily verify this by solving the equations explicitly, but the point is that *visualizing* the system gave us a very quick (in fact, almost instantaneous) short cut.

□

The above example featured a pair of curves, each associated with an algebraic equation. In fact, throughout this course we will think of curves in a rather different way, not as a set of points satisfying an equation, but rather as the range of a suitable mapping. That is, we will deal primarily with *parametrized* curves.

1 Regularly Parametrized Curves

1.1 Basic definitions

Let $I \subseteq \mathbb{R}$ be an open interval,

$$\text{e.g.} \qquad (0,\pi), \qquad (-\infty,1), \qquad \mathbb{R} \quad \text{etc.}$$

Recall that a function $f : I \to \mathbb{R}$ is **smooth** if all its derivatives $f'(t)$, $f''(t)$, $f'''(t), \ldots$ exist for all t (shorthand: $\forall\, t \in I$).

E.g. polynomials, trigonometric functions (sin, cos etc.), exponentials, logarithms, hyperbolic trig functions (sinh, cosh etc.) are all smooth.

$f : \mathbb{R} \to \mathbb{R}$ such that $f(t) = t^{\frac{4}{3}}$ is **not** smooth. Check:

$f'(t) =$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\Rightarrow \quad f''(t) =$

so $f''(0)$ does not exist.

We can extend this definition of smoothness to maps $\gamma : I \to \mathbb{R}^n$ where $n \geq 2$. Such a map is a rule which assigns to each "time" t a vector

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)) \in \mathbb{R}^n.$$

We say that the map γ is smooth if every one of its component functions $\gamma_i : I \to \mathbb{R}$, $i = 1, 2, \ldots, n$ is smooth in the usual sense (all derivatives exist everywhere). We may think of γ as describing the trajectory of a point particle moving in $\mathbb{R}^n$. This leads us to:

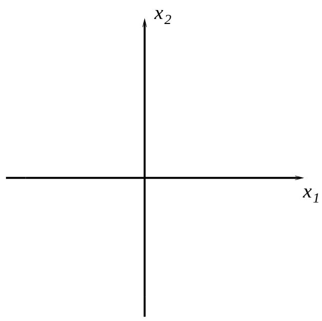
Definition 1 A **parametrized curve** (PC) in $\mathbb{R}^n$ is a smooth map $\gamma : I \to \mathbb{R}^n$. A time $t \in I$ is a **regular point** of γ if $\gamma'(t) \neq 0$. If $\gamma'(t) = 0$, then $t \in I$ is a **singular point** of γ. If every $t \in I$ is regular then γ is said to be a **regularly parametrized curve** (RPC). In other words, a PC is a RPC if and only if

$$\boxed{\text{there does not exist a time } t \in I \text{ such that } \gamma'(t) = 0 = (0, 0, \ldots, 0).}$$

The **image set** of a curve γ is the range of the mapping, that is,

$$\gamma(I) = \{\gamma(t) \in \mathbb{R}^n \mid t \in I\} \subset \mathbb{R}^n \qquad \square$$

Example 2 Consider the parabola $x_2 = x_1^2$. There are infinitely many PCs whose image set is this parabola.



Two examples:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (t, t^2)$$
$$\delta : \mathbb{R} \to \mathbb{R}^2, \qquad \delta(t) = (t^3, t^6).$$

γ is a **regularly** parametrized curve:

$$\gamma'(t) =$$

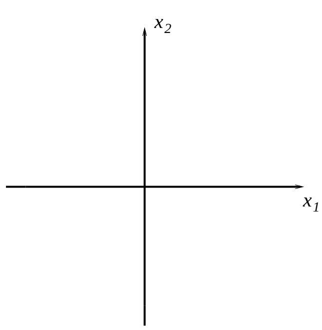
But δ is **not**:

$$\delta'(t) =$$

. $\square$

So Definition 1 concerns the *parametrization* of the curve, not just its image set $\gamma(I) \subset \mathbb{R}^n$. Why? A PC is a *smooth* map $\gamma : I \to \mathbb{R}^n$, but it does not necessarily represent a *"smooth"* curve! A **R**PC does, however.

Example 3 $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t^2, t^3)$ is a smooth map, hence a PC. But its image set is not "smooth" – it has a cusp.



Note that γ is not a **R**PC:

$$\gamma'(t) =$$

Note also that the nasty point in $\gamma(I)$ occurs precisely where $\gamma'(t) = 0$, that is, at the *singular point* of γ. $\square$

Definition 4 Given a PC $\gamma : I \to \mathbb{R}^n$, its **velocity** is $\gamma' : I \to \mathbb{R}^n$, its **acceleration** is $\gamma'' : I \to \mathbb{R}^n$ and its **speed** is $|\gamma'| : I \to [0, \infty)$. $\square$

Notes:

- $|v|$ denotes the Euclidean norm of a vector $v \in \mathbb{R}^n$, that is,

$$|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \geq 0.$$

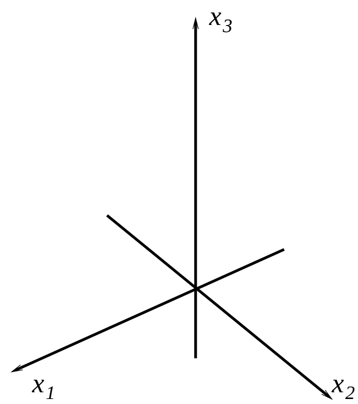
- It's important to distinguish between *vector* and *scalar* quantities.

  Velocity is a .... Acceleration is a .... Speed is a ....

- We can rephrase definition 1 as follows:

$$\boxed{\text{PC } \gamma \text{ is a } \underline{\mathbf{R}}\text{PC} \iff \text{its velocity (or its speed) never vanishes}}$$

Example 5 (straight line) Simple but important.



For any $v \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ one has the PC

$$\gamma : \mathbb{R} \to \mathbb{R}^n, \qquad \gamma(t) = x + vt.$$

Note that $\gamma'(t) = v$, constant, so γ is a RPC unless $v = 0$.
Note also that the *direction* of the straight line is determined solely by v. $\square$

A RPC has a well-defined tangent line at each $t_0 \in I$:

Definition 6 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. Then its **tangent line** at $t_0 \in I$ is the PC

$$\widehat{\gamma}_{t_0} : \mathbb{R} \to \mathbb{R}^n, \qquad \widehat{\gamma}_{t_0}(t) = \gamma(t_0) + t\gamma'(t_0).$$

Note that $\widehat{\gamma}'_{t_0}(t) = 0 + \gamma'(t_0) \neq 0$ since γ is a RPC. Hence every tangent line $\widehat{\gamma}_{t_0}$ is a RPC too. $\square$

Example 7 Consider the PC $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t^3 - t, t^2 - 1)$.

(A) Is it a RPC?

(B) Are any of its tangent lines vertical?

(A) Just check whether its velocity ever vanishes. Assume it does at time t. Then:

Hence γ is a RPC.

(B) Tangent line to γ at $t_0 \in \mathbb{R}$ is

$\widehat{\gamma}_{t_0}(t) =$

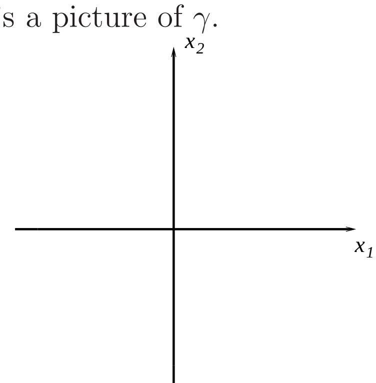
The direction of the tangent line is given by its (constant) velocity vector

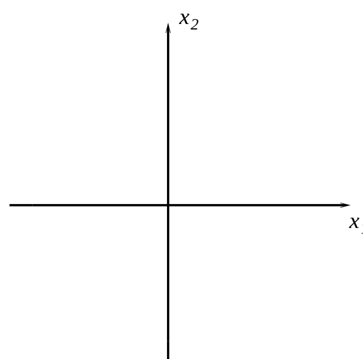
$$\widehat{\gamma}_{t_0}'(t) = (3t_0^2 - 1, 2t_0) = \gamma'(t_0).$$

The tangent line is *vertical* if the horizontal component (the x_1 component) of this vector is 0. Hence $\widehat{\gamma}_{t_0}$ is vertical if and only if

so $\widehat{\gamma}_{\frac{1}{\sqrt{3}}}$ and $\widehat{\gamma}_{-\frac{1}{\sqrt{3}}}$ are vertical.

Here's a picture of γ.





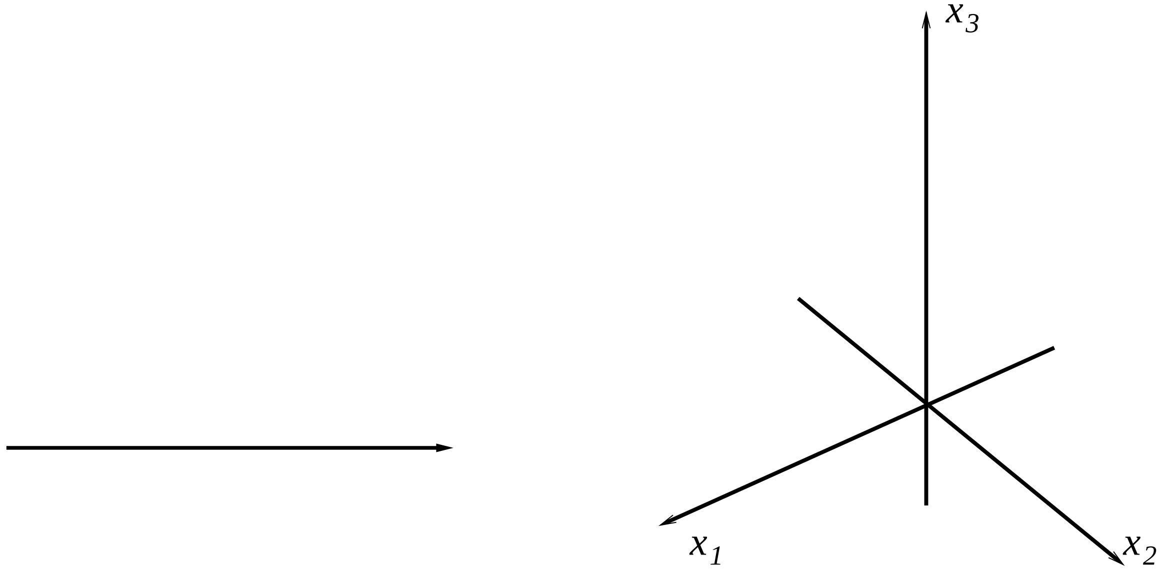
Note that the curve intersects itself exactly once. A self-intersection point is one where $\gamma(t_1) = \gamma(t_2)$ but $t_1 \neq t_2$, which implies, in this case,

$$\begin{aligned}
& t_1^3 - t_1 = t_2^3 - t_2, \qquad \text{and} \qquad t_1^2 - 1 = t_2^2 - 1 \\
\Rightarrow \quad & t_1^2 = t_2^2 \\
\Rightarrow \quad & t_2 = -t_1 \\
\Rightarrow \quad & t_1^3 - t_1 = -t_1^3 + t_1 = -(t_1^3 - t_1) \\
\Rightarrow \quad & 0 = t_1^3 - t_1 = t_1(t_1^2 - 1) \\
\Rightarrow \quad & t_1 = 0 \qquad \text{or} \qquad t_1 = \pm 1
\end{aligned}$$

If $t_1 = 0$ then $t_2 = -0 = t_1$, so this doesn't give a self-intersection point. Likewise $t_1 = 1$ and $t_1 = -1$ give the same self-intersection point, $\gamma(1) = \gamma(-1) = (0, 0)$.

Another question: what is the **length** of the segment of γ from $t = 0$ to $t = 1$? $\square$

1.2 Arc length



Given a RPC $\gamma : I \to \mathbb{R}^n$, what is the length of the curve segment from $t = t_0$ to $t = \widehat{t}$ say? Partition $[t_0, \widehat{t}]$ into N pieces $[t_{n-1}, t_n]$, $n = 1, 2, \ldots, N$ (with $t_N = \widehat{t}$) of equal length $\delta t = \frac{\widehat{t}-t_0}{N}$. If N is large, then δt is small. The length of the straight line segment from $\gamma(t_{n-1})$ to $\gamma(t_n)$ is

$$\delta s_n = |\gamma(t_n) - \gamma(t_{n-1})| = |\gamma(t_{n-1} + \delta t) - \gamma(t_{n-1})| \approx |\gamma'(t_{n-1})|\delta t.$$

So the total length of the piecewise straight line from $\gamma(t_0)$ to $\gamma(\widehat{t})$ is

$$s_N = \sum_{n=1}^{N} \delta s_n \approx \sum_{n=1}^{N} |\gamma'(t_{n-1})|\delta t.$$

In the limit $N \to \infty$, $\delta t \to 0$ and the piecewise straight line tends to the real curve γ. So the total length of the curve segment is

$$s = \lim_{N\to\infty} s_N = \int_{t_0}^{\widehat{t}} |\gamma'(t)|\, dt.$$

This motivates the following definition:

Definition 8 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. The **arc length** along γ from $t = t_0$ to $t = t_1$ is

$$s = \int_{t_0}^{t_1} |\gamma'(t)|\, dt. \qquad \square$$

More informally, $\boxed{\text{distance travelled} = \text{integral of speed.}}$

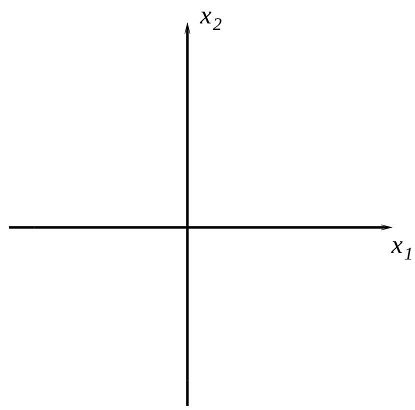
Example 9 $\gamma : (0, \infty) \to \mathbb{R}^2$, $\gamma(t) = (t, \frac{2}{3}t^{\frac{3}{2}})$.
Arc length from $t = 3$ to $t = 15$?

$$\begin{aligned} \gamma'(t) &= \\ |\gamma'(t)| &= \\ s &= \end{aligned}$$

$\square$

WARNING! This example was cooked up to be easy. It's usually impossible to compute s in practice.

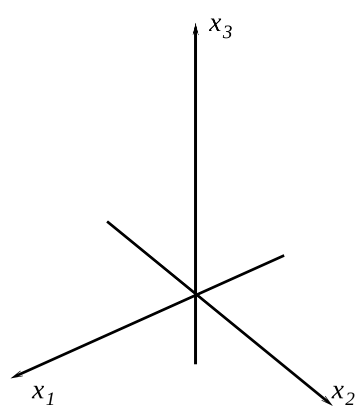
Example 7 (revisited) $\gamma(t) = (t^3 - t, t^2 - 1)$. Arc length from $t = 0$ to $t = 1$?



$$\begin{aligned} s &= \\ &= \\ &= \\ &= \\ &= \text{??? } (1.36 \quad \text{to 2 d.p.}) \end{aligned}$$

$\square$

Note that we can use Definition 8 even if $t_1 < t_0$:



$$\begin{aligned} s &= \int_{t_0}^{t_1} \underbrace{|\gamma'(t)|}_{\text{positive}} \, dt = -\int_{t_1}^{t_0} |\gamma'(t)| \, dt < 0 \\ &\Rightarrow \textbf{signed} \text{ arc length.} \end{aligned}$$

Once we've chosen a base point $t_0 \in I$, at every other time we can assign a unique signed arc length.

Definition 10 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. The **arc length function** based at $t_0 \in I$ is

$$\sigma_{t_0} : I \to \mathbb{R}, \qquad \sigma_{t_0}(t) = \int_{t_0}^{t} |\gamma'(u)|\, du \qquad \square$$

Example 9 (revisited) The arc length function based at $t_0 = 1$ for $\gamma(t) = (t, \frac{2}{3}t^{\frac{3}{2}})$ is

$$\begin{aligned} \sigma_1(t) &= \\ \\ &= \end{aligned}$$

Trick questions: (A) What is $\sigma_1(1)$?
(B) What is $\sigma_1'(t)$? $\square$

$\sigma_{t_0}(t)$ is very hard (usually impossible) to compute explicitly in practice. But the fact that it *exists* is crucial to the theory of curves. We need to understand its properties.

Remark 11 The arc length function $\sigma_{t_0} : I \to \mathbb{R}$ has the following properties:

(a) $\sigma_{t_0}(t_0) = \displaystyle\int_{t_0}^{t_0} |\gamma'(q)|\, dq = 0$.

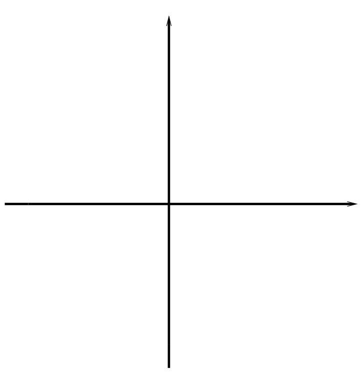
(b) σ_{t_0} is a *strictly increasing function*:

$$\sigma_{t_0}'(t) = \frac{d}{dt}\int_{t_0}^{t} |\gamma'(a)|\, da = |\gamma'(t)| > 0$$

for all $t \in I$ since γ is a **<u>R</u>**PC.

(c) It follows that σ_{t_0} is *one-to-one* by the Mean Value Theorem.

(d) Let $J \subseteq \mathbb{R}$ denote the *range* of σ_{t_0}. It's another (possibly unbounded) open interval. Given (c), there exists an *inverse function* to σ_{t_0}, let's call it $\tau_{t_0} : J \to I$, so that



$$\begin{aligned} \tau_{t_0}(\sigma_{t_0}(t)) &= t \qquad \text{for all } t \in I, \text{ and} \\ \sigma_{t_0}(\tau_{t_0}(s)) &= s \qquad \text{for all } s \in J \qquad (\clubsuit) \end{aligned}$$

$$\begin{array}{ccc} I & \xrightarrow{\sigma_{t_0}} & J \\ \text{time} & & \begin{array}{c}\text{arc}\\ \text{length}\end{array} \\ t & \underset{\tau_{t_0}}{\leftarrow} & s \end{array}$$

Trick question: what is $\tau_{t_0}(0)$?

(e) The inverse function τ_{t_0} is also strictly increasing. To see this, differentiate (♣) with respect to s using the chain rule:

$$\begin{aligned} \sigma'_{t_0}(\tau_{t_0}(s))\tau'_{t_0}(s) &= 1 \\ \Rightarrow \quad \tau'_{t_0}(s) &= \frac{1}{\sigma'_{t_0}(\tau_{t_0}(s))} = \frac{1}{|\gamma'(\tau_{t_0}(s))|} > 0 \end{aligned}$$

□

Example 12 $\gamma : \mathbb{R} \to \mathbb{R}^2$ $\gamma(t) = (a\cos t, a\sin t)$, where $a > 0$ is a constant.

$$\begin{aligned} \sigma_0(t) &= \\ \tau_0(s) &= \end{aligned}$$

Note both these functions are increasing functions from $\mathbb{R}$ to $\mathbb{R}$. □

Example 9 (revisited) $\gamma : (0,\infty) \to \mathbb{R}^2$ $\gamma(t) = (t, \frac{2}{3}t^{\frac{3}{2}})$. Recall

$$\sigma_1 : (0,\infty) \to \mathbb{R} \qquad \sigma_1(t) = \frac{2}{3}((1+t)^{\frac{3}{2}} - 2^{\frac{3}{2}}).$$

What is τ_1? Domain of τ_1 = range of σ_1. But since σ_1 is increasing, this is just the interval $J = (a, b)$, where

$$\begin{aligned} a &= \lim_{t\to 0} \sigma_1(t) = -\frac{2}{3}(2^{\frac{3}{2}} - 1) \quad \text{and} \\ b &= \lim_{t\to\infty} \sigma_1(t) = \infty. \end{aligned}$$

To find a formula for $\tau_1(s)$, we must solve $s = \sigma_1(t)$ to find t as a function of s:

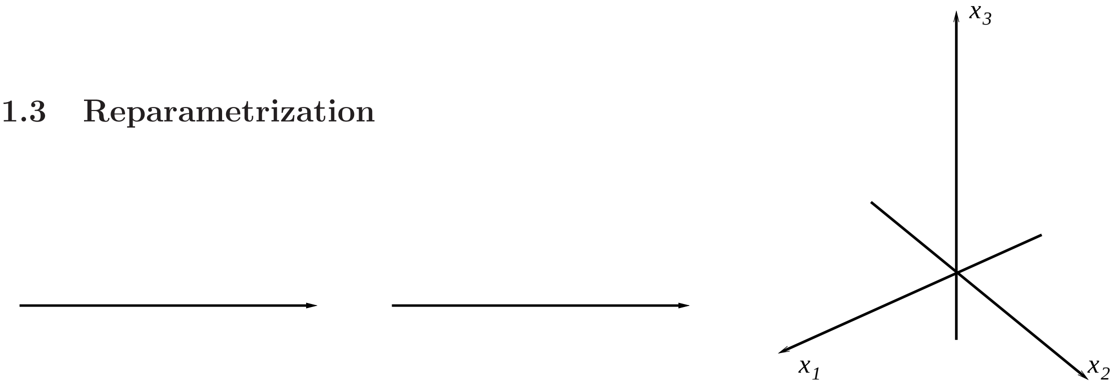
$$\begin{aligned} s &= \frac{2}{3}\left[(1+t)^{\frac{3}{2}} - 2^{\frac{3}{2}}\right] \\ \Rightarrow \quad \left(\frac{3}{2}s + 2^{\frac{3}{2}}\right)^{\frac{2}{3}} &= 1 + t \\ \Rightarrow \quad \tau_1(s) &= \left(\frac{3}{2}s + 2^{\frac{3}{2}}\right)^{\frac{2}{3}} - 1. \end{aligned}$$

[Check: $\tau_1(0) =$]

What is $\tau'_1(s)$?

$$\tau'_1(s) = \frac{1}{\sigma'_1(\tau_1(s))} = \frac{1}{|\gamma'(\tau_1(s))|} = \frac{1}{\sqrt{1+\tau_1(s)}} = \left(\frac{3}{2}s + 2^{\frac{3}{2}}\right)^{-\frac{1}{3}} \quad \square$$

1.3 Reparametrization



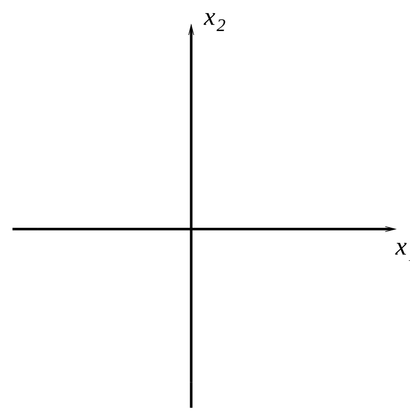
A reparametrization of a RPC $\gamma : I \to \mathbb{R}^n$ is a redefinition of "time":

$$\beta : J \to \mathbb{R}^n, \qquad \beta(u) = \gamma(h(u)) = (\gamma \circ h)(u)$$

Example 13 Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(t) = (t, e^t)$. Let $h : (0, \infty) \to \mathbb{R}$ such that $h(u) = \log u$. This gives the reparametrization

$$\beta : (0, \infty) \to \mathbb{R}^2,$$

$$\beta(u) = \gamma(h(u)) =$$



Note that the *image sets* of γ and β are *identical*: $\beta((0, \infty)) = \gamma(\mathbb{R})$. We've just changed the way we *label* the points on the curve.

Note also that both γ and β are **R**PCs in this case:

$$\gamma'(t) = \qquad \beta'(u) =$$

But we can't allow h to be any function $J \to \mathbb{R}$ if we want β to be a RPC. For example, let $h : \mathbb{R} \to \mathbb{R}$ such that $h(u) = \sin u$. Then

$$\beta(u) = \gamma(h(u)) =$$

$$\beta'(u) =$$

so $\beta'(\frac{\pi}{2}) = (0, 0)$, and β is not regular! This is an example of a bad redefinition of time. We want to exclude things like this from our formal definition of reparametrization. $\square$

Definition 14 A **reparametrization** of a PC $\gamma : I \to \mathbb{R}^n$ is a map $\beta : J \to \mathbb{R}^n$ defined by $\beta(u) = \gamma(h(u))$, where J is an open interval, and $h : J \to I$ is smooth, surjective and increasing (that is $h'(u) > 0$ for all $u \in J$). $\square$

Notes:

- We require h to be smooth so that β is smooth (by the Chain Rule), hence a PC.
- We require h to be surjective so that $\beta(J) = \gamma(I)$. In other words, this ensures that β covers all of γ, not just part of it.
- Since h is increasing, it is injective. Hence for each time $t \in I$ there is one (h surjective) and only one (h injective) corresponding new time $u \in J$.

Lemma 15 *Any reparametrization of a RPC is also a RPC.*

Proof: Let $\beta(u) = \gamma(h(u))$ where γ is a RPC. Then

$$\begin{aligned} \beta'(u) = \gamma'(h(u))h'(u) &= 0 \quad \Rightarrow & \\ \gamma'(h(u)) &= 0 & \text{(impossible since } \gamma \text{ is regular)} \\ \text{or} \qquad h'(u) &= 0 & \text{(impossible since } h \text{ is increasing).} \end{aligned}$$

Hence β is a RPC. $\square$

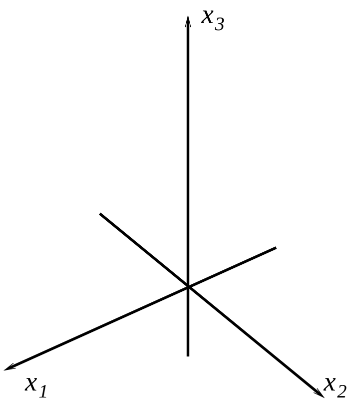
Note that reparametrization preserves the *direction* and *sense* of the velocity vector, but not its length:

$$\text{new velocity} = \beta'(u) = h'(u)\gamma'(h(u)) = \text{positive number} \times \text{old velocity}.$$

However:

Lemma 16 *Reparametrization preserves arc length.*

Proof: Let γ be a PC and s be the arc length along γ from $\gamma(t_0)$ to $\gamma(t_1)$. Let $\beta = \gamma \circ h$ be a reparametrization of γ where $h(u_0) = t_0$ and $h(u_1) = t_1$. We must show that s is also the arc length along β from $\beta(u_0)$ to $\beta(u_1)$. In fact



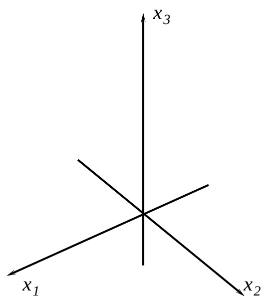
$$\begin{aligned} s &= \int_{t_0}^{t_1} |\gamma'(t)|\, dt \qquad \text{Substitution: } t = h(u),\ dt = h'(u)du \\ &= \int_{h^{-1}(t_0)}^{h^{-1}(t_1)} |\gamma'(h(u))|\, h'(u)du \\ &= \int_{u_0}^{u_1} |\gamma'(h(u))h'(u)|\, du \qquad \text{since } h'(u) > 0 \\ &= \int_{u_0}^{u_1} |\beta'(u)|\, du \end{aligned}$$

as required. $\square$

Definition 17 A **unit speed curve** (USC) is a smooth map $\gamma : I \to \mathbb{R}^n$ such that $|\gamma'(s)| = 1$ for all $s \in I$ $\square$

Notes:

- Clearly USC $\Rightarrow$ RPC.
- It's conventional to denote the "time" parameter of a unit speed curve by s rather than t, because the parameter *is* signed arc length (up to a constant):



$$\sigma = \int_{s_0}^{s_1} |\gamma'(s)| ds = \int_{s_0}^{s_1} 1 \, ds = s_1 - s_0. \quad (\clubsuit)$$

Unit speed curves may seem very special, but in fact they are, in a sense, completely universal:

Theorem 18 *Every RPC $\gamma : I \to \mathbb{R}^n$ has a unit speed reparametrization (USR) $\beta : J \to \mathbb{R}^n$. This USR is unique up to "time" translation. More precisely, if $\delta : K \to \mathbb{R}^n$ is another USR of γ, then there exists a constant $c \in \mathbb{R}$ such that*

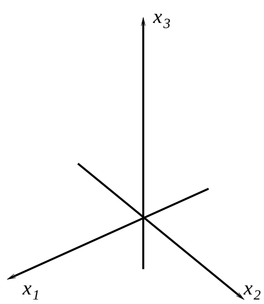
$$\beta(s) = \delta(s - c).$$

Proof:
(A) Existence:
Choose $t_0 \in I$ and let $\sigma_{t_0} : I \to J$ be the signed arc length function of γ based at t_0. Recall that there exists a smooth, well-defined inverse function $\tau_{t_0} : J \to I$ which is increasing (Remark 11(d) and (e)). So $h = \tau_{t_0}$ gives a reparametrization of γ, according to Definition 14: $\beta : J \to \mathbb{R}^n$, $\beta(s) = \gamma(\tau_{t_0}(s))$. But

$$\beta'(s) = \gamma'(\tau_{t_0}(s))\tau_{t_0}'(s) = \gamma'(\tau_{t_0}(s)) \frac{1}{|\gamma'(\tau_{t_0}(s))|}$$

by Remark 11(e). Hence $|\beta'(s)| = 1$ for all $s \in J$, and β is a USC as required.

(B) Uniqueness:
Let $\delta : K \to \mathbb{R}^n$ be another USR of γ, σ be the arc length along γ from $\gamma(t_0) = \beta(s_0) = \delta(\widetilde{s}_0)$ to $\gamma(t) = \beta(s) = \delta(\widetilde{s})$. By Lemma 16 and ($\clubsuit$),

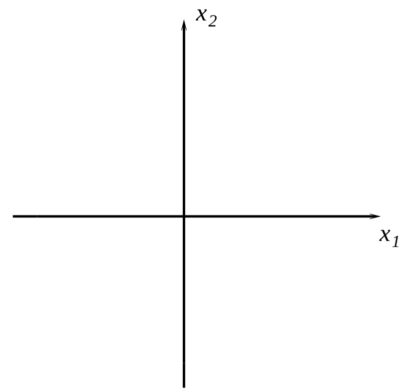


$$\begin{aligned} \sigma &= s - s_0 = \widetilde{s} - \widetilde{s}_0 \\ \Rightarrow \quad \widetilde{s} &= s - (s_0 - \widetilde{s}_0) \end{aligned}$$

Hence $\qquad \beta(s) = \delta(\widetilde{s}) = \delta(s - (s_0 - \widetilde{s}_0)). \qquad \square$

Example 19 Circle $\gamma(t) = a(\cos t, \sin t)$, or radius $a > 0$. To find a USR of γ we compute the arclength function and invert it, then use τ_0 to reparametrize γ:

$$\begin{aligned} \gamma'(t) &= \\ |\gamma'(t)| &= \\ \sigma_0(t) &= \\ \tau_0(s) &= \\ \beta(s) &= \gamma(\tau_0(s)) = \end{aligned}$$

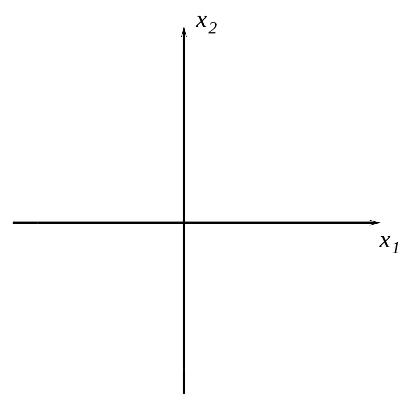


□

Although Theorem 18 ensures a USR exists, we may not be able to construct it explicitly:

Example 20 Ellipse $\gamma(t) = (a\cos t, \sin t)$, $a > 1$.

$$\begin{aligned} \gamma'(t) &= \\ |\gamma'(t)| &= \\ \sigma_0(t) &= \\ \tau_0(s) &= \\ \beta(s) &= \gamma(\tau_0(s)) = \end{aligned}$$



In this case we can't write down a USR explicitly. □

Summary

- A **regularly parametrized curve** (RPC) is a smooth map $\gamma : I \to \mathbb{R}^n$ (where I is an open interval) such that, for all $t \in I$, $\gamma'(t) \neq 0$.
- Given a RPC γ, its **velocity** is γ', its **speed** is $|\gamma'|$ and its **acceleration** is γ''.
- The **tangent line** to γ at $t_0 \in I$ is

$$\widehat{\gamma}_{t_0} : \mathbb{R} \to \mathbb{R}^n, \qquad \widehat{\gamma}_{t_0}(t) = \gamma(t_0) + \gamma'(t_0)t.$$

- The **arclength function** based at $t_0 \in I$ is

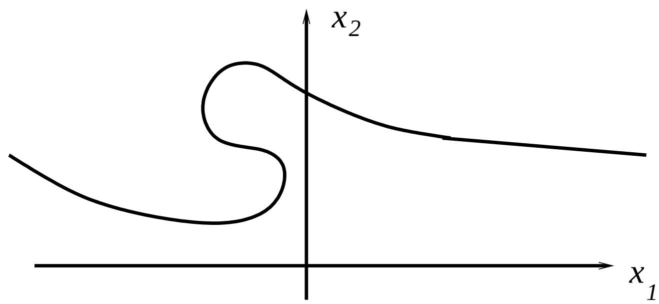
$$\sigma_{t_0}(t) = \int_{t_0}^{t} |\gamma'(u)| du.$$

  Geometrically, this is the arclength along γ from $\gamma(t_0)$ to $\gamma(t)$ if $t \geq t_0$ (and minus the arclength if $t < t_0$).
- A **reparamatrization** of a curve $\gamma : I \to \mathbb{R}^n$ is a curve $\gamma \circ h : J \to \mathbb{R}^n$ where $h : J \to I$ is smooth, increasing and surjective. If γ is a RPC, so is every reparametrization of γ.
- Arclength is unchanged by reparametrization.
- A **unit speed curve** (USC) is a curve with $|\gamma'(t)| = 1$ for all t.
- Every RPC has a reparametrization which is a USC. One can construct it, in principle, by reparametrizing with $h = \sigma_{t_0}^{-1}$.

2 Curvature of a parametrized curve

2.1 Basic definition

In this section we will develop a measure of the *curvature* of a RPC.



What distinguishes a region of high curvature from one of low curvature?

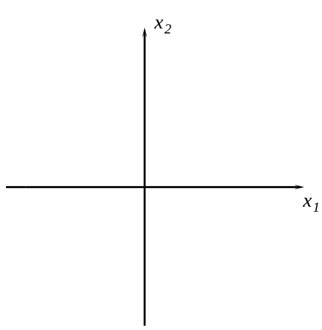
High curvature:	tangent lines change direction very rapidly
Low curvature:	tangent lines change direction only slowly
No curvature (straight line):	tangent lines don't change direction at all

So we require a measure of the rate of change of direction of the curve's tangent lines. Recall that the tangent line at $\gamma(t_0)$ of a RPC γ is

$$\widehat{\gamma}_{t_0}(u) = \gamma(t_0) \quad + \quad u\,\underbrace{\gamma'(t_0)}_{\substack{\text{determines}\\ \text{direction}}}$$

Perhaps we can use $\dfrac{d}{dt_0}(\gamma'(t_0)) = \gamma''(t_0)$ to measure the curvature of γ at $t = t_0$? Not quite: γ'' tells us about rate of change of *length* of γ' as well as rate of change of *direction*.

Example 21 $\gamma(t) = (\log t, 2\log t + 1)$ is a straight line, and hence is not curved.



However

$\gamma'(t) \quad =$

$\gamma''(t) \quad =$

which is not zero. □

But what if γ happens to be a unit speed curve, $\gamma(s)$? Then $|\gamma'(s)| = 1$ for all s, so $\gamma''(s)$ *does* tell us only about the rate of change of the *direction* of the tangent lines.

Definition 22 (a) Let $\gamma : I \to \mathbb{R}^n$ be a USC. Then the **curvature vector** of γ is $k : I \to \mathbb{R}^n$ where

$$k(s) = \gamma''(s).$$

Note that k is a *vector* quantity. We shall refer to the norm of k, $|k| : I \to [0, \infty)$ as the curvature of γ.

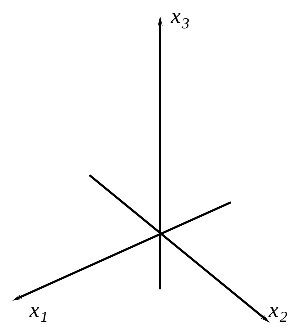
(b) If a RPC $\gamma : I \to \mathbb{R}^n$ is *not* a USC, then Theorem 18 says that it has a unit speed reparametrization $\beta : J \to \mathbb{R}^n$, $\beta = \gamma \circ h$. In that case, we define the curvature vector of γ at $t = h(s)$ to be the curvature vector of β at s, as in part (a), that is, $\beta''(s)$. In other words, $k : I \to \mathbb{R}^n$ such that

$$k = \beta'' \circ h^{-1}.$$

Note: To make sense, this definition should be independent of the choice of unit speed reparametrization β of γ. It is. Recall, by Theorem 18, that any pair of USRs of γ differ only by shifting the origin of the new time coordinate s. But such a shift has no effect on the second (or indeed the first) derivative of β. □

Example 23 Helix $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(s) = \dfrac{1}{\sqrt{2}}(\cos s, \sin s, s)$.

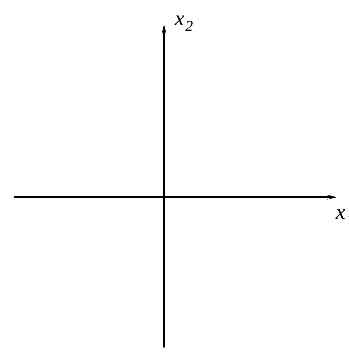
$$\begin{aligned} \gamma'(s) &= \\ |\gamma'(s)| &= \\ k(s) = \gamma''(s) &= \\ |k(s)| &= \end{aligned}$$



. □

Example 24 Circle $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (2\cos t, 2\sin t)$

$$\begin{aligned} \gamma'(t) &= \\ |\gamma'(t)| &= \end{aligned}$$



so *not* a USC. But we can find a USR of γ. How?

$$\begin{aligned} \text{Arclength function:} \quad & \sigma_0(t) = \\ \text{Inverse function:} \quad & \tau_0(s) = \\ \text{USR:} \quad & \beta(s) = \gamma(\tau_0(s)) = \end{aligned}$$

Now compute curvature vector of β:

$$\beta'(s) =$$

$$\beta''(s) =$$

Change back to old parametrization, $s = \sigma_0(t) =$

$$k(t) = \beta''(\sigma_0(t)) =$$

$\square$

In general this process is rather clumsy. In particular, it's usually impossible to write down β, the USR of γ, explicitly. Recall, for instance, Example 20, the ellipse $\gamma(t) = (a\cos t, \sin t)$.

So let's calculate k once and for all for a general RPC γ, to obtain a more user-friendly version of Definition 22. Two preliminary observations:

(A) Given two vectors $u, v \in \mathbb{R}^n$, we define their scalar product $u \cdot v \in \mathbb{R}$ by

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots u_n v_n.$$

In particular, the norm of v may be rewritten

$$|v| = \sqrt{v \cdot v}.$$

(B) Given two vector valued functions $u, v : I \to \mathbb{R}^n$, we have a product rule for differention,

$$\frac{d}{dt}(u(t) \cdot v(t)) = u'(t) \cdot v(t) + u(t) \cdot v'(t).$$

In particular,

$$\frac{d}{dt}|u(t)| = \frac{d}{dt}(u \cdot u)^{\frac{1}{2}} = \frac{u \cdot u' + u' \cdot u}{2(u \cdot u)^{\frac{1}{2}}} = \frac{u(t) \cdot u'(t)}{|u(t)|}.$$

Let $\gamma(t)$ be a RPC and $\beta(s) = \gamma(h(s))$ be a USR of γ. Then

$$\beta'(s) = \frac{d}{ds}(\gamma(t)) = \frac{d\gamma}{dt}\frac{dt}{ds} = \frac{\gamma'(t)}{ds/dt}.$$

But $|\beta'(s)| = 1$, so $ds/dt = |\gamma'(t)|$, and

$$\beta'(s) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Differentiating this equation w.r.t. s once again yields

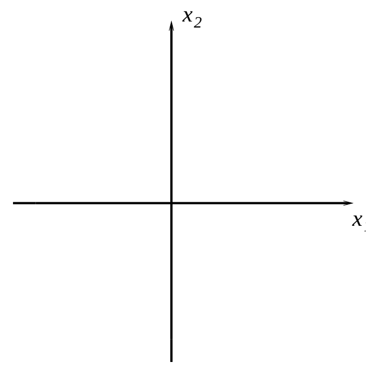
$$\begin{aligned}
\beta''(s) &= \frac{d}{ds}\left(\frac{\gamma'(t)}{|\gamma'(t)|}\right) = \frac{d}{dt}\left(\frac{\gamma'(t)}{|\gamma'(t)|}\right)\frac{dt}{ds} \\
&= \left\{\frac{\gamma''(t)}{|\gamma'(t)|} - \frac{\gamma'(t)}{|\gamma'(t)|^2}\frac{d}{dt}|\gamma'(t)|\right\}\frac{1}{|\gamma'(t)|} \\
&= \left\{\frac{\gamma''(t)}{|\gamma'(t)|} - \frac{\gamma'(t)}{|\gamma'(t)|^2}\left(\frac{\gamma'(t) \cdot \gamma''(t)}{|\gamma'(t)|}\right)\right\}\frac{1}{|\gamma'(t)|} \\
&= \frac{1}{|\gamma'(t)|^2}\left\{\gamma''(t) - \left(\frac{\gamma'(t) \cdot \gamma''(t)}{|\gamma'(t)|^2}\right)\gamma'(t)\right\},
\end{aligned}$$

where we have used observation (B) above. This leads us to

Definition 22 (*) The curvature vector of a RPC $\gamma : I \to \mathbb{R}^n$ is $k : I \to \mathbb{R}^n$, where

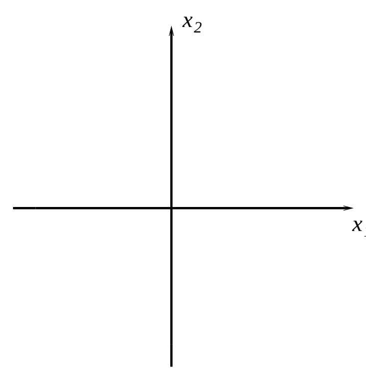
$$k(t) = \frac{1}{|\gamma'(t)|^2} \left\{ \gamma''(t) - \left(\frac{\gamma'(t) \cdot \gamma''(t)}{|\gamma'(t)|^2} \right) \gamma'(t) \right\}. \qquad \square$$

Example 25 Parabola $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t, t^2)$.



$$\begin{aligned}
\gamma'(t) &= \\
|\gamma'(t)|^2 &= \\
\gamma''(t) &= \\
\gamma'(t) \cdot \gamma''(t) &= \\
k(t) &=
\end{aligned} \qquad \square$$

Example 26 Ellipse $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (2\cos t, \sin t)$.



$$\begin{aligned}
\gamma'(t) &= \\
|\gamma'(t)|^2 &= \\
\gamma''(t) &= \\
\gamma'(t) \cdot \gamma''(t) &= \\
k(t) &= \\
|k(t)| &=
\end{aligned} \qquad \square$$

Definition 22(*) is much easier to use than definition 22, but it's not so memorable. Can we improve it?

2.2 The unit tangent vector and normal projection

We start with a simple observation about the curvature vector:

Fact 27 The curvature vector k of a RPC γ is always orthogonal to its velocity, $k(t) \cdot \gamma'(t) = 0$. This follows directly from Definition 22(*):

$$k(t)\cdot\gamma'(t) = \frac{1}{|\gamma'(t)|^2}\left\{\gamma''(t)\cdot\gamma'(t) - \left(\frac{\gamma'(t)\cdot\gamma''(t)}{|\gamma'(t)|^2}\right)\gamma'(t)\cdot\gamma'(t)\right\} = 0. \qquad \square$$

Looking back at examples 23, 24, 26 one sees several illustrations of this:

Helix	$\gamma(s) = \frac{1}{\sqrt{2}}(\cos s, \sin s, s)$	$\gamma'(s) = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1)$	$k(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, 0)$
Circle	$\gamma(t) = (2\cos t, 2\sin t)$	$\gamma'(t) = (-2\sin t, 2\cos t)$	$k(t) = \frac{1}{2}(-\cos t, -\sin t)$
Ellipse	$\gamma(t) = (2\cos t, \sin t)$	$\gamma'(t) = (-2\sin t, \cos t)$	$k(t) = \frac{-2}{(1+3\sin^2 t)^2}(\cos t, 2\sin t)$

We can use this fact to give a more memorable version of Definition 22(*). In preparation for this, we need:

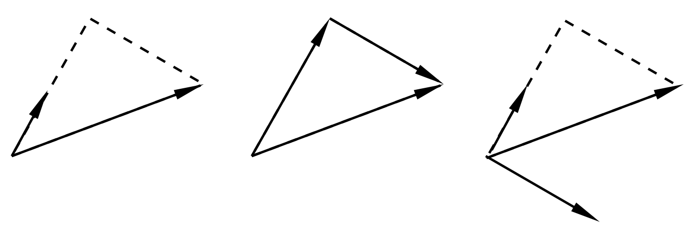
Definition 28 Let $\gamma : I \to \mathbb{R}^n$ be a RPC. Its **unit tangent vector** $u : I \to \mathbb{R}^n$ is

$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$

Note that u is well defined because γ is a **R**PC (so $|\gamma'(t)| > 0$ for all t). Note also that $|u(t)| = 1$ for all t by construction.

Given any other vector valued function $v : I \to \mathbb{R}^n$, we define its **normal projection**, $v_\perp : I \to \mathbb{R}^n$ by

$$v_\perp(t) = v(t) - [v(t)\cdot u(t)]u(t).$$



We can think of $v_\perp$ as that part of v left over after we have subtracted off the component of v in the direction of u. $\square$

Example 29 $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(t) = (\frac{t^2}{2}, \sin t, \cos t)$. What are u and $\gamma''_\perp$?

$$
\begin{aligned}
\gamma'(t) &= \\
|\gamma'(t)| &= \\
u(t) &= \\
\gamma''(t) &= \\
\gamma''(t) \cdot u(t) &= \\
\gamma''_\perp(t) &= \\
&= \\
&= \\
&= \qquad \square
\end{aligned}
$$

The normal projection of the acceleration vector $\gamma''_\perp : I \to \mathbb{R}^n$ is of particular interest, because it occurs in Definition 22(*):

$$
\begin{aligned}
k(t) &= \frac{1}{|\gamma'(t)|^2} \left\{ \gamma''(t) - \left(\frac{\gamma'(t)}{|\gamma'(t)|} \cdot \gamma''(t) \right) \frac{\gamma'(t)}{|\gamma'(t)|} \right\} \\
&= \frac{1}{|\gamma'(t)|^2} \{ \gamma''(t) - [u(t) \cdot \gamma''(t)] u(t) \} = \frac{\gamma''_\perp(t)}{|\gamma'(t)|^2}
\end{aligned}
$$

Definition 22 (**) The **curvature vector** of a RPC $\gamma : I \to \mathbb{R}^n$ is $k : I \to \mathbb{R}^n$,

$$
k(t) = \frac{\gamma''_\perp(t)}{|\gamma'(t)|^2} \qquad \square
$$

Example 29 (revisited) $\gamma(t) = (\frac{t^2}{2}, \sin t, \cos t)$ has curvature vector

$k(t) =$

$\square$

Summary

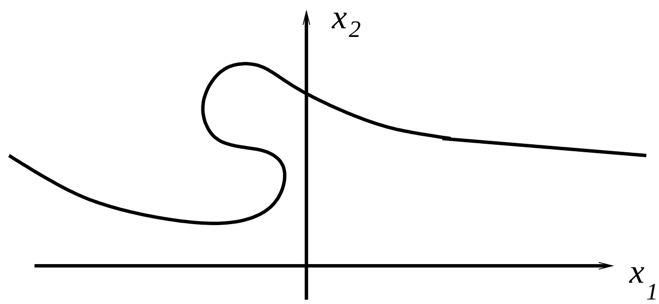
- The curvature vector of a RPC measures how fast the tangent lines to the curve change **direction**.
- If γ is a unit speed curve, the **curvature vector** is $k(s) = \gamma''(s)$.
- In general
$$k(t) = \frac{1}{|\gamma'(t)|^2} \left\{ \gamma''(t) - \frac{\gamma''(t) \cdot \gamma'(t)}{|\gamma'(t)|^2} \gamma'(t) \right\}.$$
- The **unit tangent vector** along a curve $\gamma : I \to \mathbb{R}^n$ is
$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}.$$
- The **normal projection** of $v : I \to \mathbb{R}^n$ is
$$v_\perp(t) = v(t) - [u(t) \cdot v(t)]u(t).$$
- An alternative formula for the curvature vector is
$$k(t) = \frac{\gamma''_\perp(t)}{|\gamma'(t)|^2}.$$

3 Planar curves

3.1 Signed curvature of a planar curve

The theory of curvature can be developed further in the special case of *planar curves*, that is, RPCs $\gamma : I \to \mathbb{R}^2$ in two dimensions. These are special because all the curvature information associated with γ may be encoded in a single real-valued function $\kappa : I \to \mathbb{R}$, called the *signed curvature*. How?

Recall that k in this case is a 2-vector, so it consists of a *pair* of real functions $k(t) = (k_1(t), k_2(t))$ say. However, Fact 27 states that k is always orthogonal to the unit tangent vector u. So in fact we already know the *direction* of k. The only extra information we need to provide is the length of k, and its *sense*: whether it points to the left or the right of u.



Definition 30 Let $\gamma : I \to \mathbb{R}^2$ be a RPC with unit tangent vector u (recall $u = \gamma'/|\gamma'|$). The **unit normal vector** of γ is $n : I \to \mathbb{R}^2$,

$$n(t) = (-u_2(t), u_1(t)).$$

The **signed curvature** of γ is $\kappa : I \to \mathbb{R}$,

$$\kappa(t) = k(t) \cdot n(t)$$

where $k : I \to \mathbb{R}^2$ is the curvature vector of γ, as in Definition 22(**). □

Notes:

(a) n is orthogonal to u ($n(t)\cdot u(t) = 0$) and has unit length ($|n|^2 = u_2^2 + u_1^2 = |u|^2 = 1$) by construction. In fact, n is the vector obtained by rotating u 90° anticlockwise.

(b) Since both n and k are orthogonal to u, they must be parallel. In fact we can re-interpret the above definition as follows: given that $k(t)$ and $n(t)$ are parallel, we define $\kappa(t)$ to be the constant of proportionality,

$$k(t) = \kappa(t) n(t).$$

It follows that $|\kappa(t)| = |k(t)|$. However, κ contains more information than the (unsigned) curvature $|k|$ — its sign tells us the “sense” of k.

(c) Recall that $k = \gamma''_\perp / |\gamma'|^2$, so

$$\kappa(t) = \frac{\gamma''_\perp(t) \cdot n(t)}{|\gamma'(t)|^2}.$$

However,

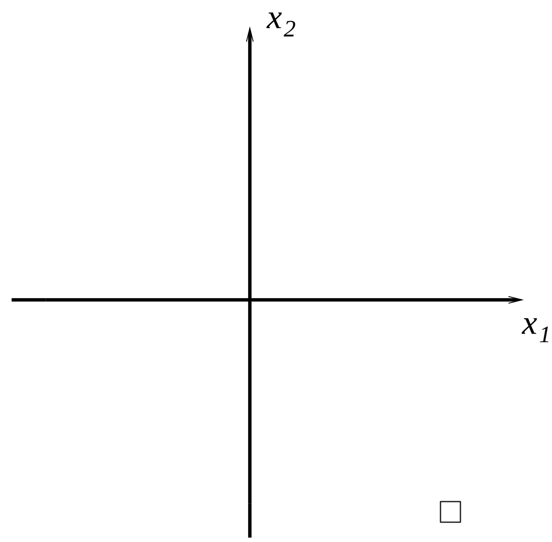
$$\gamma_\perp'' \cdot n = (\gamma'' - (\gamma'' \cdot u)u) \cdot n = \gamma'' \cdot n,$$

so we can give a slightly more convenient definition of κ:

Definition 30 (*) Let $\gamma : I \to \mathbb{R}^2$ be a RPC, $u = \gamma'/|\gamma'|$ be its unit tangent vector and $n = (-u_2, u_1)$ be its unit normal. Then its **signed curvature** $\kappa : I \to \mathbb{R}$ is

$$\kappa(t) = \frac{\gamma''(t) \cdot n(t)}{|\gamma'(t)|^2}. \qquad \square$$

Example 31 A sinusoidal curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\sin t, t)$

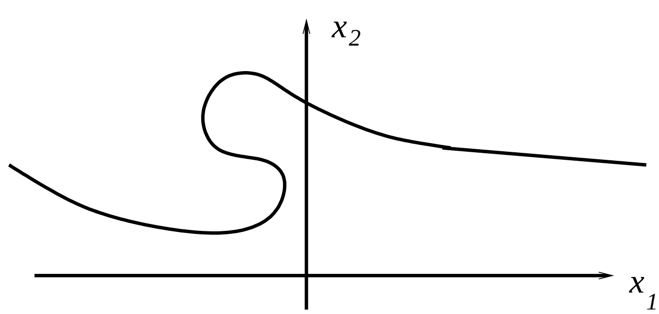


$$\begin{aligned} \gamma'(t) &= \\ |\gamma'(t)| &= \\ u(t) &= \\ n(t) &= \\ \gamma''(t) &= \\ \kappa(t) &= \end{aligned}$$

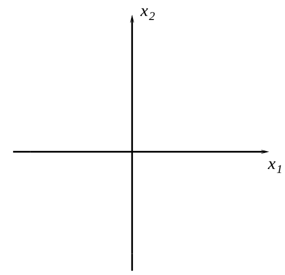
. $\square$

Note that

$\kappa > 0$	$\Rightarrow$	k points in same sense as n	$\Rightarrow$	curve is turning **left**
$\kappa < 0$	$\Rightarrow$	k points in opposite sense to n	$\Rightarrow$	curve is turning **right**



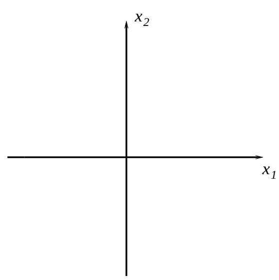
Example 32 Looking at the parabola $\gamma(t) = (t, t^2)$ it's immediately clear that $\kappa(t)$ is always ____________. Let's check:



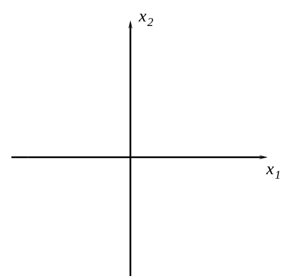
$$\begin{aligned} \gamma'(t) &= \\ u(t) &= \\ n(t) &= \\ \gamma''(t) &= \\ \kappa(t) &= \end{aligned}$$

Note that this observation depends crucially on the *orientation* of the curve, that is, the direction in which it is traversed. For example, for both of the parabolae below, the signed curvature is always ____________:

$\gamma(t) = (-t, t^2)$



$\gamma(t) = (t^2, t)$



. $\square$

Points on a curve where the signed curvature changes sign have a special name:

Definition 33 Let $\gamma : I \to \mathbb{R}^2$ be a RPC and $\kappa : I \to \mathbb{R}$ be its signed curvature. If there exists a time $t_* \in I$ such that $\kappa(t_*) = 0$ and κ changes sign at $t = t_*$, then $\gamma(t_*)$ is an **inflexion point** of γ. $\square$

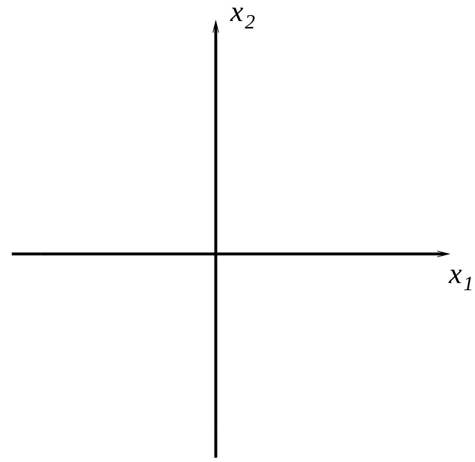
Example 31 (revisited) What are the inflexion points of the sinusoidal curve $\gamma(t) = (\sin t, t)$? Recall that its signed curvature function is

$$\kappa(t) =$$

So $\kappa(t) = 0$ if and only if $t = N\pi$, where $N \in \mathbb{Z}$. Further, the *sign* of $\kappa(t)$ changes at each such time. Hence for all $N \in \mathbb{Z}$,

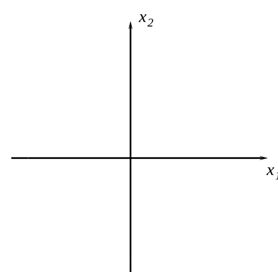
$$\gamma(N\pi) = (0, N\pi)$$

is an inflexion point.



WARNING!
$\kappa(t_*) = 0$ does NOT imply that $\gamma(t_*)$ is an inflexion point!
κ must CHANGE SIGN at $t = t_*$ too!

Counterexample 34 The quartic curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t, t^4)$ has $\kappa(0) = 0$. However, it clearly has no inflexion points ($\kappa(t) \geq 0$, since the curve never turns right),



$$\begin{aligned}
\gamma'(t) &= \\
\gamma''(t) &= \\
\gamma''(0) &= \\
\kappa(0) &= \frac{\gamma''(0) \cdot n(0)}{|\gamma'(0)|^2} = 0
\end{aligned}$$

Exercise: show that

$$\kappa(t) = \frac{12t^2}{(1 + 16t^6)^{\frac{3}{2}}}. \quad \square$$

3.2 Planar curves of prescribed curvature

In this section we will consider only planar unit speed curves (PUSCs) $\gamma(s)$ ($|\gamma'(s)| = 1$ for all s). Note this entails no loss of generality by Theorem 18.

So far, given a PUSC $\gamma(s)$ we can construct its signed curvature $\kappa(s)$. Can we go the other way? That is, given a function $\kappa(s)$, can we reconstruct the PUSC $\gamma(s)$ whose curvature is κ? Yes, provided we also specify an initial position $\gamma(0)$ and tangent vector $\gamma'(0)$.

How? Note $|\gamma'(s)| = 1$ so each velocity vector is determined by just its *direction*. That is, there exists smooth $\theta : I \to \mathbb{R}$ such that

$$\gamma'(s) = (\cos\theta(s), \sin\theta(s)) \qquad (A).$$

Note that $u(s) = \gamma'(s)$ and hence the unit normal vector is

$$n(s) = (-\sin\theta(s), \cos\theta(s)).$$

Now, for a USC, curvature $k(s) = \gamma''(s) = (-\sin\theta(s), \cos\theta(s))\theta'(s)$, and hence the signed curvature is

$$\kappa(s) = n(s) \cdot k(s) = \theta'(s) \qquad (B).$$

We may collect (A), (B) into a coupled system of 3 nonlinear ordinary differential equations (ODEs):

$$(*) \begin{cases} \dfrac{d\theta}{ds} = \kappa(s) & (1) \\[2ex] \dfrac{d\gamma_1}{ds} = \cos\theta & (2) \\[2ex] \dfrac{d\gamma_2}{ds} = \sin\theta & (3) \end{cases}$$

So any PUSC of curvature κ is a solution of system (*), and *vice versa*. Given a *prescribed* $\kappa(s)$, we can solve (*) for the curve $\gamma(s)$.

Theorem 35 *Given any smooth $\kappa : I \to \mathbb{R}$, $0 \in I$, and constants $\theta_0 \in \mathbb{R}$ and $\gamma_0 \in \mathbb{R}^2$, there exists a unique USC $\gamma : I \to \mathbb{R}^2$ with $\gamma(0) = \gamma_0$, $\gamma'(0) = (\cos\theta_0, \sin\theta_0)$ and signed curvature κ.*

Proof: Must prove existence of a unique global solution $(\theta(s), \gamma_1(s), \gamma_2(s))$ of the initial value problem [IVP] $(\theta(0), \gamma_1(0), \gamma_2(0)) = (\theta_0, a, b)$ for system (*) [where $\gamma_0 = (a, b)$].

In fact, (*) is separable. First consider IVP (1):

$$\frac{d\theta}{ds} = \kappa(s), \qquad \theta(0) = \theta_0.$$

This has solution

$$\theta(s) = f(s) := \theta_0 + \int_0^s \kappa(\alpha)d\alpha.$$

[Note that $f'(s) = \kappa(s)$ by the Fundamental Theorem of Calculus, and $f(0) = \theta_0$, so $\theta = f$ is a solution with the right initial data.]
Unique? Yes, by the Mean Value Theorem (MVT).
[Assume not unique. Then there exists another solution $\theta(s) = g(s)$ with $g(0) = \theta_0$ but $g \neq f$, that is, there exists $s_0 \in I$ such that $g(s_0) \neq f(s_0)$. Consider the function $F(s) = f(s) - g(s)$. Clearly $F(0) = 0$, $F(s_0) \neq 0$ and F is differentiable on I. Hence, by MVT, there exists s_* between 0 and s_0 such that

$$F'(s_*) = \frac{F(s_0) - F(0)}{s_0 - 0} = \frac{F(s_0)}{s_0} \neq 0.$$

But f and g both solve (1), so $F'(s) = \kappa(s) - \kappa(s) = 0$ for all s, a contradiction.]

Now substitute this unique solution ($\theta = f$) into IVP (2):

$$\frac{d\gamma_1}{ds} = \cos f(s) \qquad\qquad \gamma_1(0) = a.$$

This has unique solution

$$\gamma_1(s) = a + \int_0^s \cos f(\alpha)\, d\alpha$$

by an identical argument. Similarly, substituting $\theta = f$ into IVP (3),

$$\frac{d\gamma_2}{ds} = \sin f(s) \qquad\qquad \gamma_2(0) = b,$$

one has the unique solution

$$\gamma_2(s) = b + \int_0^s \sin f(\alpha)\, d\alpha. \qquad\qquad \square$$

Note that Theorem 34 doesn't just prove existence and uniqueness of the curve $\gamma(s)$, it also gives a *formula* for it:

$$\boxed{\gamma(s) = \gamma_0 + \left(\int_0^s \cos\theta(\alpha)\, d\alpha, \int_0^s \sin\theta(\alpha)\, d\alpha\right), \quad \text{where} \quad \theta(\alpha) = \theta_0 + \int_0^\alpha \kappa(\beta)\, d\beta. \qquad (C)}$$

Example 36 Curves of constant curvature: $\kappa(s) = \kappa_0 \neq 0$, constant. Let's always choose $\gamma(0) = (0,0)$, $\theta(0) = 0$ henceforth. Then formula (C) implies

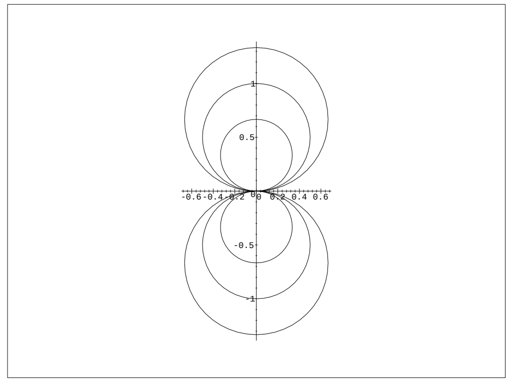
$$\begin{aligned}
\theta(\alpha) &= \\
\Rightarrow \gamma_1(s) &= \\
\gamma_2(s) &= \\
\Rightarrow \gamma(s) &=
\end{aligned}$$

This curve is a circle of radius $1/|\kappa_0|$ centred on $(0, 1/\kappa_0)$. Question: what happens when $\kappa_0 = 0$? We can (a) go back and use formula (C) with $\kappa(\beta) \equiv 0$ or (b) take the limit as $\kappa_0 \to 0$ of the solution above (and appeal to continuity properties of solutions of ODEs).

Either way we find that

$$\kappa_0 = 0 \quad \Rightarrow \quad \gamma(s) = (s, 0),$$

that is, the solution degenerates to a horizontal straight line.



Example 37 $\kappa(s) = \dfrac{1}{1+s^2}$. Note $\kappa(s) > 0$ for all s, so curve always turns left and has no inflexion points. Applying formula (C) again:

$$\begin{aligned} \theta(\alpha) &= \\ \Rightarrow \gamma_1(s) &= \\ \gamma_2(s) &= \\ \Rightarrow \gamma(s) &= \end{aligned}$$

In fact, using the reparametrization $s = \tau(t) = \sinh t$, this becomes

$$\widetilde{\gamma}(t) = (\gamma \circ \tau)(t) = (t, \cosh t - 1),$$

the graph of the cosh function shifted down one unit.$\square$

It's actually quite difficult to cook up curvature functions $\kappa(s)$ for which the integrals in formula (C) are explicitly calculable. Even a seemingly simple choice such as $\kappa(s) = s$ turns out to be intractable:

$$\begin{aligned} \theta(\alpha) &= \int_0^\alpha \beta \, d\beta = \frac{1}{2}\alpha^2 \\ \gamma_1(s) &= \int_0^s \cos \frac{\alpha^2}{2} \, d\alpha = ??? \\ \gamma_2(s) &= \int_0^s \sin \frac{\alpha^2}{2} \, d\alpha = ??? \end{aligned}$$

What can we say about the geometry of this curve?

Reminder: A function $f : I \to \mathbb{R}$ is

even if $f(-t) = f(t)$ for all $t \in I$
odd if $f(-t) = -f(t)$ for all $t \in I$.

Proposition 38 *Let $\gamma(s)$ be the USC of curvature $\kappa(s)$ with $\gamma(0) = 0$, $\gamma'(0) = (1, 0)$.*

(a) If κ is even, γ is symmetric under reflexion in the x_2 axis.

(b) If κ is odd, γ is symmetric under rotation by 180 degrees about $(0, 0)$.

Proof: From formula (C) one sees that

$$\theta(-\alpha) =$$

where $\xi := -\beta$. Hence κ even implies θ odd, while κ odd implies θ even. Similarly,

$$\gamma_1(-s) =$$

so if κ is odd or even (meaning θ is even or odd) then γ_1 is odd. Further,

$$\gamma_2(-s) =$$

so if κ is even (θ odd) γ_2 is even while if κ is odd (θ even) γ_2 is odd. Summarizing:

$$\begin{aligned} \kappa \text{ even} \quad &\Rightarrow \quad \gamma(-s) \equiv \\ \kappa \text{ odd} \quad &\Rightarrow \quad \gamma(-s) \equiv \end{aligned}$$

and hence γ has the symmetry claimed. $\square$

Applying Proposition 38 to $\kappa(s) = s$, an odd function, we see that the corresponding curve γ must have rotational symmetry about the origin. Also, γ has one and only one inflexion point: $\gamma(0) = (0, 0)$. For $s > 0$, $\kappa > 0$ meaning the curve always turns leftwards, and as s grows this turning gets tighter and tighter ($|\kappa|$ is unbounded). The behaviour for $s < 0$ is determined by that for $s > 0$ by the symmetry property.

To get an idea of the specific shape of the curve γ, we can solve system (*) approximately using an ODE solver package, in Maple for example. The following Maple code is adapted from *Differential Geometry and its Applications* by J. Oprea. Given a function κ and an interval $I = (s_1, s_2)$ it computes the curve $\gamma : I \to \mathbb{R}^2$ of curvature κ ($\gamma(0) = 0$, $\gamma'(0) = (1, 0)$) and then plots it.

```
recreate:=proc(kappa,s1,s2)
local sys,gamma1,gamma2,theta,IC,soln:
with(plots):
sys:={diff(theta(s),s)=kappa(s),
      diff(gamma1(s),s)=cos(theta(s)),
      diff(gamma2(s),s)=sin(theta(s))}:
IC:={theta(0)=0,gamma1(0)=0,gamma2(0)=0}:
soln:=dsolve(sys union IC,{theta(s),gamma1(s),gamma2(s)},type=numeric):
odeplot(soln,[gamma1(s),gamma2(s)],s1..s2,numpoints=400,scaling=constrained);
end:
```

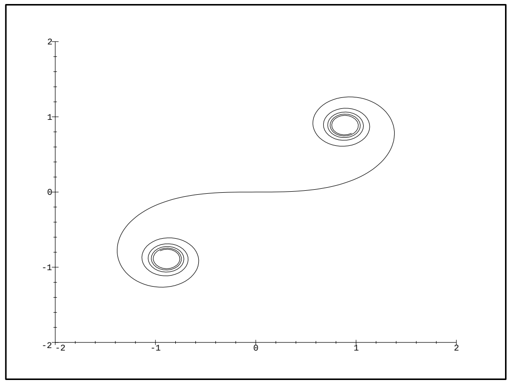
To apply this to our case ($\kappa(s) = s$) one defines

```
kappa:=s->s;
```

and then executes (for example)

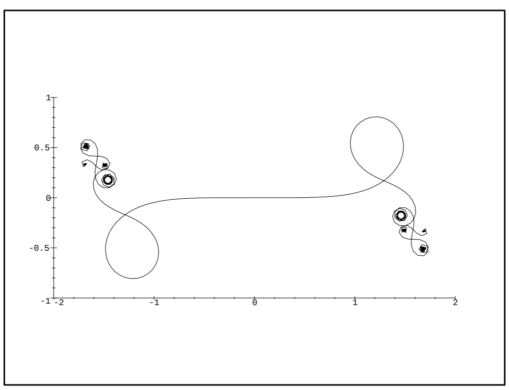
```
recreate(kappa,-8,8);
```

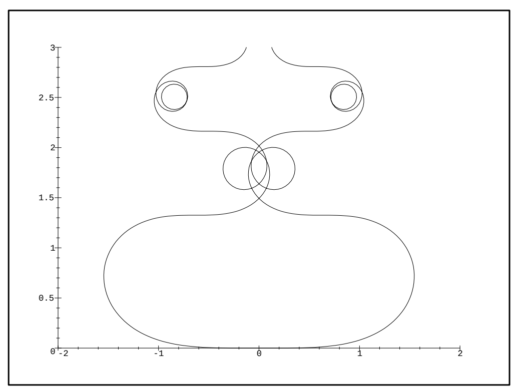
The result is:



Note the curve has the predicted turning behaviour and symmetry. It is now straightforward to turn the program loose on just about any curvature function. The results can be quite entertaining.

Example 39 Let $\kappa(s) = s^2 \sin s$ (execute `kappa:s->s^2*sin(s);`). Note that κ is again odd, so the corresponding curve must have rotational symmetry about the origin. Note also that $\gamma(s)$ has infinitely many inflexion points, since κ changes sign at every $s = m\pi$, where $m \in \mathbb{Z}$. Executing `recreate(kappa,-15,15);` one obtains (below left):





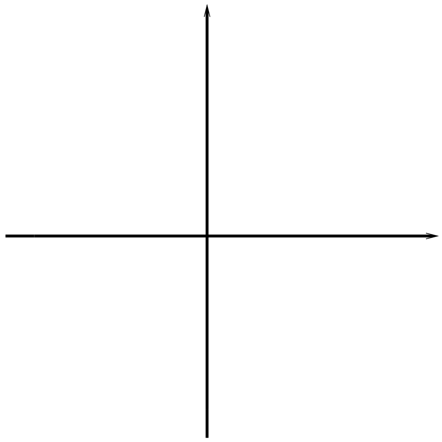
Compare with $\kappa(s) = s \sin s$, an even function:

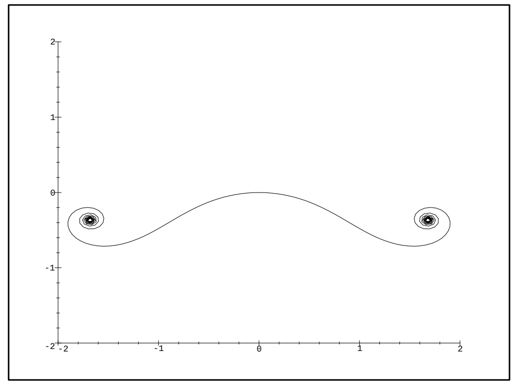
```
kappa:=s->s*sin(s);
recreate(kappa,-10,10);
```

The corresponding curve is depicted above, right. Note the reflexion symmetry in the x_2 axis. $\square$

Example 40 Let $\kappa(s) = s^2 - 1$ (execute `kappa:s->s^2-1;`). Note that κ is __________, so the corresponding curve must have _______________ symmetry. Note also that $\gamma(s)$

has exactly _____ inflexion points, since κ changes sign at $s =$_______. Executing
`recreate(kappa,6,6);`
one obtains (below right):





It's not hard to identify the inflexion points on the curve above.

Trick question: What's the arc length along the curve from one inflexion point to the next?
Answer:______________ $\square$

Go forth and experiment

You are encouraged to experiment in Maple with the procedure `recreate`. The value of this is that it will help develop your intuition about the geometric relationship between the function κ and the corresponding curve γ. Of course, in the final exam for this module, Maple will not be available to you. Hopefully the intuition you've developed will. In particular, relying heavily on `recreate` to complete Problem Sheet 2 would be a mistake. You have been warned...

3.3 The evolute of a planar curve

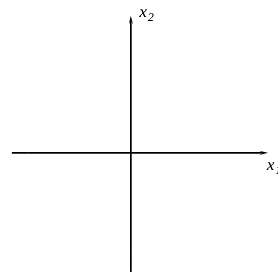
Given one planar curve, there are a number of geometrically interesting ways to generate new curves from it. One of these is the **evolute** of the curve. Throughout this section, $\gamma : I \to \mathbb{R}^2$ will denote a planar RPC, not necessarily unit speed, $u : I \to \mathbb{R}^2$ will denote its unit tangent vector, $n : I \to \mathbb{R}^2$ its unit normal vector and $\kappa : I \to \mathbb{R}$ its signed curvature. First we need:

Definition 41 Given $t_0 \in I$, the **centre of curvature** of γ at $t = t_0$ is

$$c(t_0) = \gamma(t_0) + \frac{1}{\kappa(t_0)} n(t_0).$$

Note this definition only makes sense if $\kappa(t_0) \neq 0$. $\square$

Example 42 We saw in example 32 that the parabola $\gamma(t) = (t, t^2)$ has, at $t = 0$,



$$n(0) =$$

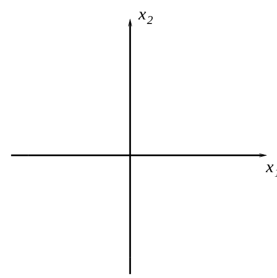
$$\kappa(0) =$$

$$\Rightarrow c(0) = \qquad .$$

$\square$

We can give a nice geometric interpretation of the centre of curvature in terms of the behaviour of the **normal lines** to the curve γ.

Theorem 43 *For $t_0 \in I$ fixed and $t \in I$ variable, consider the normal lines to γ through $\gamma(t_0)$ and $\gamma(t)$. If $\kappa(t_0) \neq 0$ and $|t - t_0|$ is sufficiently small then these normals intersect at some point $\alpha(t) \in \mathbb{R}^2$. Then*



$$\lim_{t \to t_0} \alpha(t) = c(t_0),$$

the centre of curvature of γ at t_0. .

To prove this, we'll need a useful lemma:

Lemma 44 *For all $t \in I$,*

$$(a) \quad n'(t) = -\kappa(t)\gamma'(t), \qquad (b) \quad u'(t) = \kappa(t)|\gamma'(t)|n(t).$$

Proof: Since $[u(t), n(t)]$ is an *orthonormal* pair of vectors, they form a (t dependent) basis for $\mathbb{R}^2$. So we can always express any $\mathbb{R}^2$ valued function as a linear combination of $u(t)$ and $n(t)$. Applying this idea to $n'(t)$, we see that there must exist smooth functions $\lambda : I \to \mathbb{R}$ and $\mu : I \to \mathbb{R}$ such that

$$n'(t) = \lambda(t)u(t) + \mu(t)n(t).$$

Taking the scalar product of both sides of this equation with $u(t)$ gives

$$u \cdot n' = \lambda \underbrace{u \cdot u}_{1} + \mu \underbrace{u \cdot n}_{0} = \lambda \quad \text{(orthonormality)}$$

Now

$$u \cdot n = 0 \quad \Rightarrow \quad u' \cdot n + u \cdot n' = 0 \quad \Rightarrow \quad u \cdot n' = -n \cdot u'.$$

Hence

$$\lambda = -n \cdot \frac{d}{dt}\left(\frac{\gamma'}{|\gamma'|}\right) = -n \cdot \left(\frac{\gamma''}{|\gamma'|} - \gamma' \frac{d|\gamma'|^{-1}}{dt}\right) = -\frac{n \cdot \gamma''}{|\gamma'|} = -\kappa|\gamma'|$$

Similarly,

$$n \cdot n' = \lambda \underbrace{n \cdot u}_{0} + \mu \underbrace{n \cdot n}_{1} = \mu \quad \text{(orthonormality)}$$

and

$$n \cdot n = 1 \quad \Rightarrow \quad n' \cdot n + n \cdot n' = 0 \quad \Rightarrow \quad n \cdot n' = 0$$

so it follows that $\mu(t) = 0$. Hence,

$$n'(t) = -\kappa(t)|\gamma'(t)|u(t) = -\kappa(t)\gamma'(t)$$

which proves part (a).

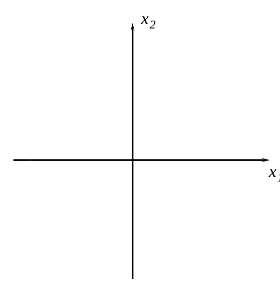
Part (b) follows from similar reasoning, so we shall be more brief.

$$u' = (u \cdot u')u + (n \cdot u')n = 0u - (u \cdot n')n = \kappa|\gamma'|n$$

by part (a). $\square$

We may now give a

Proof of Theorem 43: We may give the normals through $\gamma(t)$, $\gamma(t_0)$ the (unit speed) parametrizations (with parameters s and s_0 respectively)



$$\gamma(t) + sn(t), \qquad \gamma(t_0) + s_0 n(t_0).$$

Hence, their intersection point, which will depend on t, is

$$\alpha(t) = \gamma(t) + s(t)n(t) = \gamma(t_0) + s_0(t)n(t_0), \qquad \clubsuit$$

where $s(t)$ and $s_0(t)$ are two unknown functions of t. Differentiating ($\clubsuit$) with respect to t and taking the limit $t \to t_0$, we find that

$$\gamma'(t_0) + s'(t_0)n(t_0) + s(t_0)n'(t_0) = s_0'(t_0)n(t_0). \qquad \spadesuit$$

Taking the scalar product of ($\spadesuit$) with $u(t_0)$ yields

$$\begin{aligned} |\gamma'(t_0)| + 0 + s(t_0)n'(t_0) \cdot u(t_0) &= 0 \\ \Rightarrow \quad |\gamma'(t_0)| - s(t_0)\kappa(t_0)\gamma'(t_0) \cdot u(t_0) &= 0 \\ \Rightarrow \quad s(t_0) &= \frac{1}{\kappa(t_0)} \end{aligned}$$

by Lemma 44. Hence

$$\lim_{t \to t_0} \alpha(t) = \gamma(t_0) + s(t_0)n(t_0) = \gamma(t_0) + \frac{1}{\kappa(t_0)}n(t_0)$$

as was to be proved. $\square$

Definition 45 The **evolute** of the planar curve γ is $E_\gamma : I \to \mathbb{R}^2$, defined such that

$$E_\gamma(t) = \gamma(t) + \frac{1}{\kappa(t)}n(t),$$

in other words, it is the curve of centres of curvature of the curve γ. $\square$

Notes:

- E_γ is only well defined (on the whole of I) provided γ has nonvanishing signed curvature κ. In particular, γ must have no inflexion points, or else E_γ "escapes to infinity."
- Even if $\kappa(t) \neq 0$ for all $t \in I$, the evolute may not be a **R**PC, as we will now show.

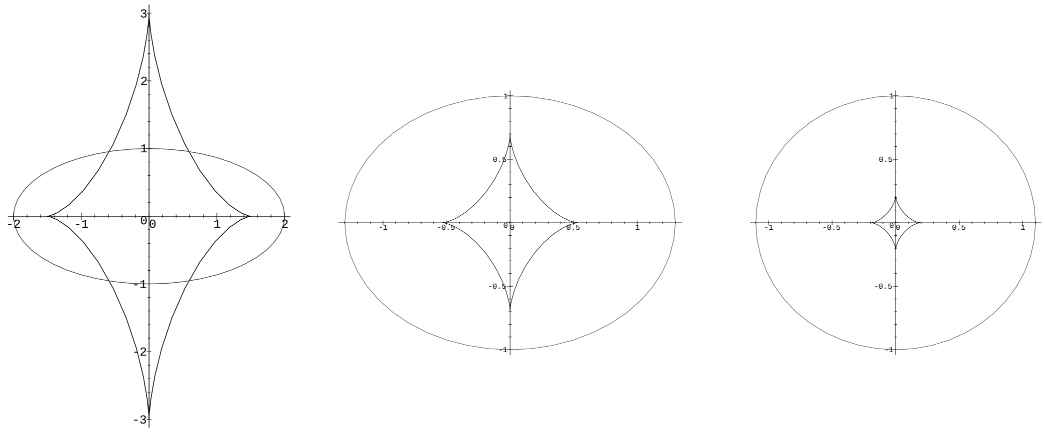
$$\begin{aligned} E'_\gamma(t) &= \gamma'(t) - \frac{\kappa'(t)}{\kappa(t)^2}n(t) + \frac{1}{\kappa(t)}n'(t) \\ &= \gamma'(t) - \frac{\kappa'(t)}{\kappa(t)^2}n(t) - \frac{1}{\kappa(t)}\kappa(t)\gamma'(t) \\ &= -\frac{\kappa'(t)}{\kappa(t)^2}n(t) \qquad \diamondsuit \end{aligned}$$

by Lemma 44. So $E'_\gamma(t) = (0,0)$ if and only if $\kappa'(t) = 0$. Hence, if κ has critical points, then E_γ is not regular. At such points, E_γ may exhibit "cusps".

Example 46 (ellipse) $\gamma(t) = (a\cos t, \sin t)$

$$\begin{aligned} \gamma'(t) &= \\ n(t) &= \\ \gamma''(t) &= \\ \kappa(t) &= \\ E_\gamma(t) &= \\ &= \\ &= \end{aligned}$$

$a = 2$ $\qquad$ $a = 1.3$ $\qquad$ $a = 1.1$



In the limit $a \to 1$, the ellipse degenerates to a circle, and its evolute collapses to a single point $(0, 0)$. $\square$

We can relate the geometric quantities associated with E_γ to those of γ.

Theorem 47 *Let u^E, n^E, $\sigma^E_{t_0}$ denote the unit tangent, unit normal and arclength function of E_γ respectively. If $\kappa'(t) < 0$ for all $t \in I$, then*

$$(a) \quad u^E(t) = n(t), \qquad (b) \quad n^E(t) = -u(t), \qquad (c) \quad \sigma^E_{t_0}(t) = \frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)}.$$

Proof: (a) From ($\diamondsuit$), we know that

$$E'_\gamma(t) = -\frac{\kappa'(t)}{\kappa(t)^2} n(t)$$

so if $\kappa'(t) < 0$ then

$$|E'_\gamma(t)| = -\frac{\kappa'(t)}{\kappa(t)^2}.$$

Hence $u^E = E'_\gamma / |E'_\gamma| = n$.
(b) $n^E = (-u^E_2, u^E_1) = (-n_2, n_1) = (-u_1, -u_2)$ by part (a).
(c) Arclength is the integral of speed, which by part (a) is

$$\sigma^E_{t_0}(t) = -\int_{t_0}^{t} \frac{\kappa'(q)}{\kappa(q)^2} dq = \int_{t_0}^{t} \left(\frac{1}{\kappa}\right)'(q) dq = \frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)}$$

which was to be proved. $\square$

3.4 Involutes and parallels of a planar curve

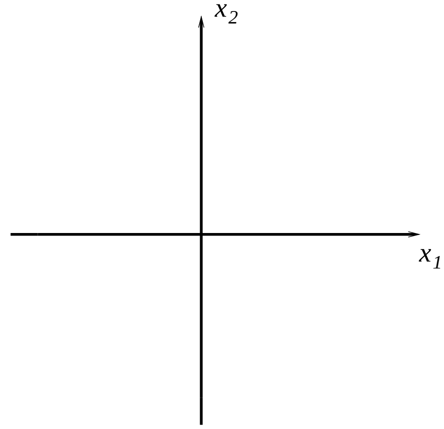
Once again, let $\gamma : I \to \mathbb{R}^2$ be a RPC, and let u, n, κ denote its unit tangent, unit normal and signed curvature respectively. In this section we describe a different way to generate new planar curves from γ which is in some sense the inverse of taking the evolute.

Imagine we have a piece of string wrapped along the curve from $\gamma(t_0)$ to $\gamma(t_1)$, and that the end $\gamma(t_1)$ is fixed. Imagine now that we peel the free end of the string away from γ, always keeping the string taut. Then the released section of string is a straight line *tangent* to γ at some point $\gamma(t)$, $t_0 < t < t_1$. The length of the released section is the arclength along γ from t_0 to t, that is

$$\sigma_{t_0}(t) = \int_{t_0}^{t} |\gamma'(q)| dq.$$

So when the contact point between string and curve is $\gamma(t)$, the free end of the string is located at

$$\gamma(t) - \sigma_{t_0}(t)u(t).$$



Definition 48 The **involute** of γ starting at $t_0 \in I$ is $I_\gamma : I \to \mathbb{R}^2$, defined by

$$I_\gamma(t) = \gamma(t) - \sigma_{t_0}(t)u(t).$$

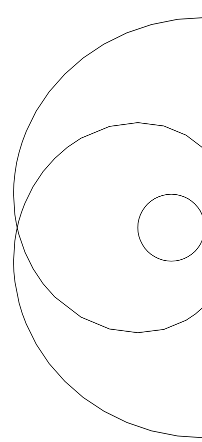
Note that if $\gamma(t)$ is a USC then $\sigma_{t_0}(t) = t - t_0$ so

$$I_\gamma(t) = \gamma(t) - (t - t_0)\gamma'(t).$$

[Question: why don't we denote the "time" parameter s in this case? If γ is a USC, does it follow that I_γ is a USC too?] □

Example 49 A circle $\gamma(t) = (\cos t, \sin t)$. This is a USC, so the involute of γ based at $t = 0$ is

$$\begin{aligned} I_\gamma(t) &= \\ &= \\ &= \end{aligned}$$



. □

Note that I_γ has a cusp at $t = 0$ in this case. Accident? No: I_γ is **never** a RPC.

$$\begin{aligned} I_\gamma(t) &= \gamma(t) - \sigma_{t_0}(t)u(t) \\ \Rightarrow \quad I'_\gamma(t) &= \\ &= \\ &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\spadesuit) \end{aligned}$$

where we have used Lemma 44. Hence $I'_\gamma(t) = (0,0)$ if and only if $\sigma_{t_0}(t) = 0$ or $\kappa(t) = 0$ (since γ is a RPC, $|\gamma'(t)|$ is never 0). So $I'_\gamma(t) = (0,0)$ when $t = t_0$ and whenever $\kappa(t) = 0$ (for example, where the contact point is an inflexion point of γ).

A useful analogy:

$$\begin{array}{ccc} & \text{function } f(x) & \\ \text{derivative} \swarrow & & \searrow \text{ integral} \\ f'(x) & & \int_{x_0}^x f(q)dq \end{array}$$

$$\begin{array}{ccc} & \text{curve } \gamma & \\ \text{evolute} \swarrow & & \searrow \text{ involute} \\ E_\gamma & & I_\gamma \end{array}$$

Fundamental Theorem of Calculus: derivative of integral is the original function f.

Theorem 50 *Let I_γ be an involute of γ. Then the evolute E_I of I_γ is γ.*

Proof: The strategy is simple: construct the evolute of I_γ, the involute of γ starting at t_0. To do this, we will need the signed curvature κ^I and the unit normal n^I of I_γ.

By Theorem 18 we may assume without loss of generality that γ is a USC. Then

$$I'_\gamma(t) = -(t - t_0)\kappa(t)n(t) = -(t - t_0)k(t) \qquad (\clubsuit)$$

by equation ($\spadesuit$). Hence I_γ has unit tangent

$$u^I = \frac{I'_\gamma}{|I'_\gamma|} = -\frac{(t - t_0)\kappa}{|t - t_0||\kappa|}n$$

and so

$$n^I = (-u^I_2, u^I_1) = \frac{(t - t_0)\kappa}{|t - t_0||\kappa|}u.$$

Differentiating ($\clubsuit$) gives

$$I''_\gamma = -k - (t - t_0)k',$$

whence the signed curvature of I_γ is

$$\begin{aligned} \kappa^I &= \frac{I''_\gamma \cdot n^I}{|I'_\gamma|^2} = -\frac{(t - t_0)\kappa}{|t - t_0|^3|\kappa|^3}[k + (t - t_0)k'] \cdot u \\ &= -\frac{\kappa}{|t - t_0||\kappa|^3}k' \cdot u \end{aligned}$$

since $k \cdot u = 0$ by Fact 27. But

$$k \cdot u = 0 \quad \Rightarrow \quad k' \cdot u + k \cdot u' = 0 \quad \Rightarrow \quad k' \cdot u = -k \cdot u' = -|k|^2 = -\kappa^2$$

since γ is a USC (so $u = \gamma'$ and $u' = \gamma'' = k$). Hence

$$\kappa^I = \frac{\kappa}{|\kappa|} \frac{1}{|t - t_0|}.$$

So

$$\begin{aligned} E_I(t) &= I_\gamma(t) + \frac{1}{\kappa^I(t)} n^I(t) \\ &= \gamma - (t - t_0)u + \frac{|t - t_0||\kappa|}{\kappa} \frac{(t - t_0)\kappa}{|t - t_0||\kappa|} u = \gamma(t) \end{aligned}$$

as was to be proved. □

So taking the evolute "undoes" the involute, just as differentiation "undoes" integration. What about the converse? If we first differentiate $f(x)$, then take a definite integral of f', we don't necessarily get the same function $f(x)$ back again – it could be shifted by a constant c. For example:

$$\begin{aligned} & f(x) = \cos x \qquad \Rightarrow \qquad f'(x) = -\sin x \\ \Rightarrow \quad & \int_0^x f'(q)dq = -\int_0^x \sin q \, dq = \cos x - 1 = f(x) - 1 \end{aligned}$$

Thinking of the graphs of the functions, we could say that integrating the derivative in general gives a shifted, or "parallel", function. An analogous statement holds for curves too, i.e. if we take an involute of the evolute of γ, the result is a curve "parallel" to γ, in the following precise sense:

Definition 51 Given a RPC $\gamma : I \to \mathbb{R}^2$ and a constant $\lambda \in \mathbb{R}$, the curve $\gamma_\lambda : I \to \mathbb{R}^2$ defined by

$$\gamma_\lambda(t) = \gamma(t) + \lambda n(t)$$

is a **parallel curve** to γ. □

Note the similarity between γ_λ and the evolute of γ,

$$E_\gamma(t) = \gamma(t) + \kappa(t)^{-1} n(t).$$

Under what circumstances is γ_λ regular?

Lemma 52 *The parallel γ_λ to γ is a RPC if and only if $\kappa(t) \neq 1/\lambda$ for all $t \in I$. In other words, λ must lie outside the range of the function $1/\kappa$.*

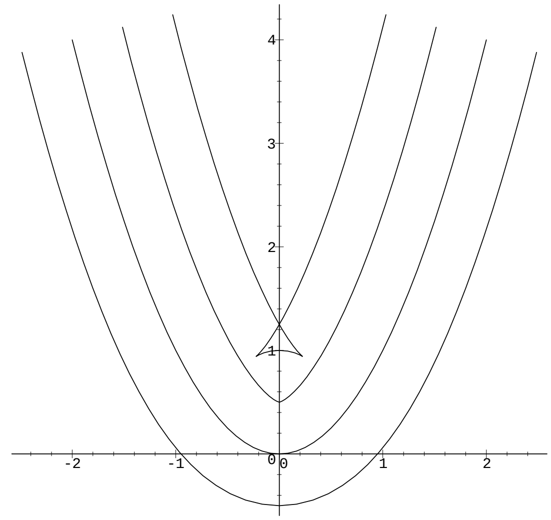
Proof:

$$\gamma'_\lambda(t) = \gamma'(t) + \lambda n'(t) = \gamma'(t) - \lambda \kappa(t) \gamma'(t)$$

by Lemma 44. Hence $\gamma'_\lambda(t) = (0, 0)$ if and only if $\kappa(t) = 1/\lambda$. □

Example 53 Let's construct the general parallel curve to the parabola $\gamma(t) = (t, t^2)$:

$$\begin{aligned} \gamma'(t) &= \\ n(t) &= \\ \gamma_\lambda(t) &= \\ &= \end{aligned}$$



Which values of the constant λ give *regular* parallels? Example 32 $\Rightarrow$

$$\kappa(t) = \frac{2}{(1+4t^2)^{\frac{3}{2}}}.$$

Hence $1/\kappa(t)$ has range $[\frac{1}{2}, \infty)$, so γ_λ is regular if and only if $\lambda < \frac{1}{2}$.

$\square$

Theorem 54 *Let γ have evolute E_γ. Then every involute I_E of E_γ is a parallel curve to γ.*

Proof: We simply construct the involute of E_γ starting at $t_0 \in I$. To do this we will need the arclength function $\sigma^E_{t_0}(t)$ and the unit tangent $u^E(t)$ of E_γ. We shall assume that $\kappa'(t) < 0$ (a similar argument works for the case $\kappa'(t) > 0$). Recall that in this case,

$$\sigma_{t_0}(t) = \frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)} \quad \text{and} \quad u^E(t) = n(t)$$

by Theorem 47, parts (c) and (a). Hence,

$$\begin{aligned} I_E(t) &= E_\gamma(t) - \sigma^E(t) u^E(t) \\ &= \gamma(t) + \frac{1}{\kappa(t)} n(t) - \left(\frac{1}{\kappa(t)} - \frac{1}{\kappa(t_0)} \right) n(t) \\ &= \gamma(t) - \frac{1}{\kappa(t_0)} n(t). \end{aligned}$$

But this is just γ_λ where $\lambda = 1/\kappa(t_0)$. $\square$

Summary

- For a planar curve $\gamma : I \to \mathbb{R}^2$ we can define the **unit normal vector**

$$n(t) = (-u_2(t), u_1(t)),$$

  where u is the unit tangent vector.

- Since the curvature vector is parallel to n, there is a scalar function $\kappa : I \to \mathbb{R}$ called the **signed curvature**, such that

$$k(t) = \kappa(t)n(t).$$

- A convenient formula for $\kappa(t)$ is

$$\kappa(t) = \frac{\gamma''(t) \cdot n(t)}{|\gamma'(t)|^2}.$$

- If $\kappa(t) > 0$, the curve is turning to the **left**. If $\kappa(t) < 0$, the curve is turning to the **right**.

- Given a function $\kappa(s)$, there is a planar USC $\gamma(s)$ whose signed curvature is $\kappa(s)$. This curve is unique up to rigid motions. Symmetries of κ imply symmetries of γ.

- The curve obtained from γ by tracing out the locus of its centres of curvature is called the **evolute** of γ. Explicitly

$$E_\gamma(t) = \gamma(t) + \frac{1}{\kappa(t)}n(t).$$

- The **involute** of γ based at $t_0 \in I$ is

$$I_\gamma(t) = \gamma(t) - \sigma_{t_0}(t)u(t).$$

- A **parallel** to γ is a curve

$$\gamma_\lambda(t) = \gamma(t) + \lambda n(t)$$

  where $\lambda \in \mathbb{R}$ is a constant.

- The evolute of an involute of γ is γ. Every involute of the evolute of γ is a parallel to γ.

- The regularity properties of evolutes and parallels can be analyzed in terms of the curvature properties of γ.

4 Curves in $\mathbb{R}^3$ and the Frenet frame

4.1 The Frenet frame

Planar curves ($\gamma : I \to \mathbb{R}^2$) are special because given one vector in $\mathbb{R}^2$, the unit tangent vector $u = \gamma'/|\gamma'|$) say, one can uniquely determine an orthogonal one by rotating $90°$ anticlockwise, the unit normal n in this case.
Curves in $\mathbb{R}^3$ ($\gamma : I \to \mathbb{R}^3$) are also special: given an *ordered pair* of vectors in $\mathbb{R}^3$, e.g. $u = \gamma'/|\gamma'|$, $k = \gamma''_\perp/|\gamma'|^2$, one can uniquely determine a third orthogonal to both by using the **vector product** ($u \times k$).

Reminder 55 Given vectors $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in \mathbb{R}^3$, their **vector product** is

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

It has the following properties:

(a) $v \times u = -u \times v$

(b) For all $\lambda, \mu \in \mathbb{R}$, $(\lambda u + \mu v) \times w = \lambda(u \times w) + \mu(v \times w)$

(c) If u is parallel to v (i.e. $u = \lambda v$) then $u \times v = 0$ [follows from (a), (b)]

(d) $|u \times v|^2 = |u|^2|v|^2 - (u \cdot v)^2$.

(e) $u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \times u)$

(f) $u \times v$ is orthogonal to both u and v [follows from (e)] □

Definition 56 Let $\gamma : I \to \mathbb{R}^3$ be a RPC whose *curvature never vanishes* (i.e. for all $t \in I$, $|k(t)| \neq 0$). Then in addition to the **unit tangent vector**

$$u(t) := \frac{\gamma'(t)}{|\gamma'(t)|}$$

one defines the **principal unit normal vector**

$$n(t) := \frac{k(t)}{|k(t)|}$$

and the **binormal vector**

$$b(t) := u(t) \times n(t).$$

The ordered triplet $[u(t), n(t), b(t)]$ is called the **Frenet frame** of the curve. γ □

Note: we call $n(t)$ the **principal** unit normal to distinguish it from the infinitely many other unit vectors lying in the plane orthogonal to $u(t)$. This is only possible if $k(t) \neq 0$.

Lemma 57 *The Frenet frame is orthonormal (the vectors u, n, b are mutually orthogonal and each have unit length).*

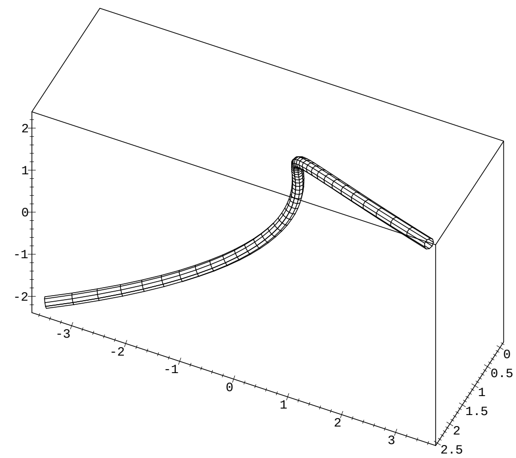
Proof: $|u(t)| = 1$ and $|n(t)| = 1$ for all t by definition. Also $n(t)$ is parallel to $k(t)$ which is orthogonal to $u(t)$ by Fact 27. It remains to show that (i) $|b(t)| = 1$ and (ii) $b(t)$ is orthogonal to $u(t)$ and $n(t)$. But (i) follows from Reminder 55(d),

$$|b|^2 = |u \times n|^2 = |u|^2|n|^2 - (u \cdot n)^2 = 1 \times 1 - 0^2 = 1,$$

and (ii) follows directly from Reminder 55(f). $\square$

So given any regularly parametrized curve of nonvanishing curvature (RPCNVC) $\gamma : I \to \mathbb{R}^3$, the Frenet frame $[u, n, b]$ forms an orthonormal basis for the vector space $\mathbb{R}^3$.

Example 58 Construct the Frenet frame at $t = 0$ for the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(t) = (\frac{t^2}{2}, \frac{t^3}{3}, t)$. Note: this is only possible provided $k(0) \neq 0$!



$$\begin{aligned}
\gamma'(t) &= \\
\gamma''(t) &= \\
\gamma'(0) &= \\
\gamma''(0) &= \\
\gamma''_\perp(0) &= \\
k(0) &= \frac{\gamma''_\perp(0)}{|\gamma'(0)|^2} =
\end{aligned}$$

which is nonzero, so the Frenet frame is well-defined. So

$$\begin{aligned}
u(0) &= \frac{\gamma'(0)}{|\gamma'(0)|} = \\
n(0) &= \frac{k(0)}{|k(0)|} = \\
b(0) &= u(0) \times n(0) \\
&=
\end{aligned}$$

$\square$

4.2 Torsion of a unit speed curve in $\mathbb{R}^3$

Our aim is to describe the geometry of curves in $\mathbb{R}^3$ by analysing the time dependence of the Frenet frame. This process simplifies greatly if the curve under consideration is a unit speed curve, so henceforth, we will consider only USCs. Note that this entails no loss of generality by Theorem 18: every RPC has a unit speed reparametrization, unique up to time translation.

The construction of the Frenet frame for a USC $\gamma : I \to \mathbb{R}^3$ simplifies somewhat because γ' is already a unit vector, and the curvature $k = \gamma''$. Hence,

$$u(s) = \gamma'(s), \qquad n(s) = \frac{u'(s)}{|u'(s)|}, \qquad b(s) = u(s) \times n(s).$$

It is conventional in the context of curves in $\mathbb{R}^3$ to denote the **scalar** curvature by $\kappa(s)$ rather than $|k(s)|$. This should not be confused with the **signed** curvature of a curve in $\mathbb{R}^2$. Here $\kappa(s)$ just means the length of the curvature vector $k(s)$, which of course is never negative. Noting that $k(s) = u'(s)$ for a USC, we may re-write the definition of the principal unit normal as

$$\boxed{u'(s) = \kappa(s)n(s)} \qquad (1)$$

Since the Frenet frame $[u(s), n(s), b(s)]$ spans $\mathbb{R}^3$ we should be able to find similar formulae for $n'(s)$ and $b'(s)$. The coefficient functions we extract should tell us about the geometry of γ, just as κ does. We start with $b'(s)$. Since it's a 3-vector, there must exist smooth functions $\lambda, \mu, \nu : I \to \mathbb{R}$ such that

$$b'(s) = \lambda(s)u(s) + \mu(s)n(s) + \nu(s)b(s)$$

Taking the scalar product of both sides with $b(s)$ gives:

$$b \cdot b' = \lambda \underbrace{b \cdot u}_{0} + \mu \underbrace{b \cdot n}_{0} + \nu \underbrace{b \cdot b}_{1} = \nu \quad \text{(orthonormality)}$$

But

$$b \cdot b = 1 \quad \Rightarrow \quad b' \cdot b + b \cdot b' = 0 \quad \Rightarrow \quad b' \cdot b = 0$$

and hence $\nu = 0$. Similarly,

$$u \cdot b' = \lambda \underbrace{u \cdot u}_{1} + \mu \underbrace{u \cdot n}_{0} + \nu \underbrace{u \cdot b}_{0} = \lambda \quad \text{(orthonormality)}$$

But

$$\begin{aligned} u \cdot b = 0 \quad \Rightarrow \quad u' \cdot b + u \cdot b' = 0 \quad \Rightarrow \quad u \cdot b' &= -u' \cdot b \\ &= -\kappa\, n \cdot b \qquad \text{(by eqn. (1))} \\ &= 0. \end{aligned}$$

Hence $\lambda = 0$ also. It follows that

$$b'(s) = \mu(s)n(s)$$

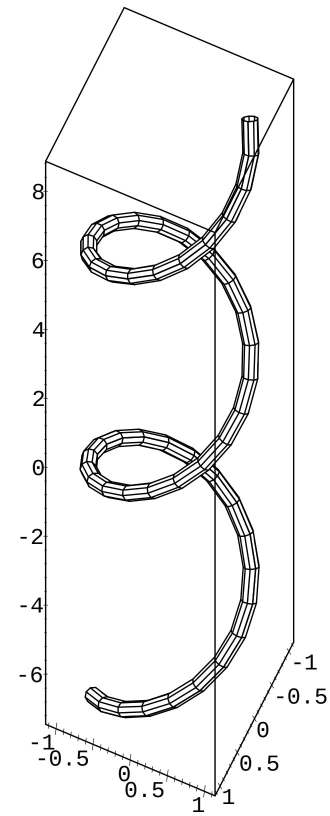
for some function μ. We call $-\mu(s)$ the *torsion* of the curve.

Definition 59 Let $\gamma : I \to \mathbb{R}^3$ be a USCNVC. The **torsion** of the curve is that function $\tau : I \to \mathbb{R}$ defined by the equation

$$\boxed{b'(s) = -\tau(s)n(s)} \qquad (2)$$

Alternatively, $\tau(s) = -b'(s) \cdot n(s)$. □

Example 60 Construct the Frenet frame and torsion for the helix $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(s) = (\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}})$.



First check it's a USCNVC:

$$\begin{aligned} \gamma'(s) &= \\ |\gamma'(s)| &= \\ k(s) &= \gamma''(s) = \end{aligned}$$

OK. Now compute $[u, n, b]$:

$$\begin{aligned} u(s) &= \gamma'(s) = \\ n(s) &= \frac{u'(s)}{|u'(s)|} = \\ b(s) &= u(s) \times n(s) = \end{aligned}$$

To find $\tau(s)$, we compute $b'(s)$ and compare with $n(s)$:

$$\begin{aligned} b'(s) &= \\ \tau(s) &= \end{aligned}$$

□

What is the geometric meaning of torsion? Recall that the curvature $\kappa = |u'|$ measures the rate of change of the direction of the tangent line to the curve. Torsion has a similar interpretation, but in terms of *planes* rather than lines.

At the point $\gamma(s_0)$ we define the **osculating plane** of the curve to be that plane through $\gamma(s_0)$ spanned by the orthonormal pair $[u(s_0), n(s_0)]$, or equivalently, by the velocity $\gamma'(s_0)$ and acceleration $\gamma''(s_0)$ of the curve. The orientation of this plane is uniquely determined by any vector normal to it, for example the binormal vector $b(s_0) = u(s_0) \times n(s_0)$. Consider the Taylor expansion of the curve $\gamma(s)$ based at the time s_0:

$$\gamma(s) = \underbrace{\gamma(s_0) + \gamma'(s_0)(s - s_0) + \frac{1}{2}\gamma''(s_0)(s - s_0)^2}_{\text{stays in osculating plane}} + \frac{1}{6}\gamma'''(s_0)(s - s_0)^3 + \cdots$$

The failure of $\gamma(s)$ to stay in the osculating plane is controlled (locally) by $\gamma'''(s_0)$. The osculating plane divides $\mathbb{R}^3$ in half: we can classify any vector v **not** in the plane as **positive** if $v \cdot b(s_0) > 0$ and **negative** if $v \cdot b(s_0) > 0$. Thinking of the osculating plane as horizontal, with $b(s_0)$ pointing "up", positive vectors point on the up side of the plane, negative vectors on the downside. So if $\gamma'''(s_0) \cdot b(s_0) > 0$, the curve punctures the osculating plane upwards as it passes through $\gamma(s_0)$, while if $\gamma'''(s_0) \cdot b(s_0) < 0$, it punctures the osculating plane downwards. What does this have to do with torsion?

Proposition 61 *Let $\gamma : I \to \mathbb{R}^3$ be a USCNVC. Then $\tau(s) = (\gamma'''(s) \cdot b(s))/\kappa(s)$.*

Proof: Recall $b = u \times n = \gamma' \times (\gamma''/|\gamma''|)$. Hence

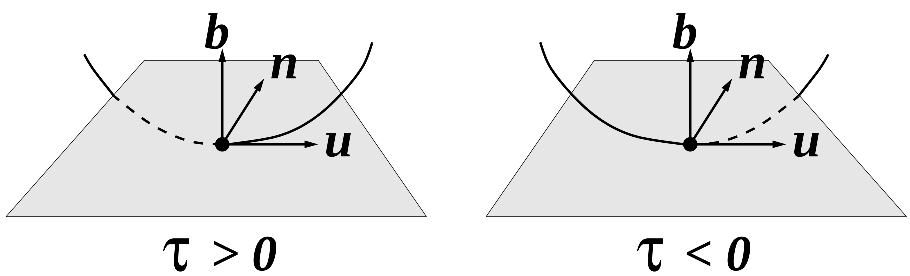
$$\begin{aligned} b' &= \gamma'' \times \left(\frac{\gamma''}{|\gamma''|}\right) + \gamma' \times \left[\frac{\gamma'''}{|\gamma''|} - \frac{\gamma''' \cdot \gamma''}{|\gamma''|^2}\gamma''\right] \\ &= 0 + u \times \left[\frac{\gamma'''}{\kappa} - (\gamma''' \cdot n)n\right]. \end{aligned}$$

But $b' = -\tau n$, so

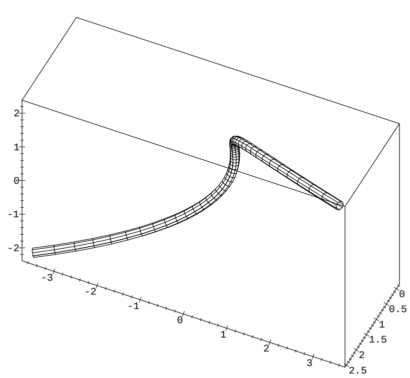
$$\tau = -n \cdot b' = -n \cdot \left[u \times \frac{\gamma'''}{\kappa}\right] = \frac{\gamma'''}{\kappa} \cdot (u \times n) = \frac{\gamma'''}{\kappa} \cdot b.$$

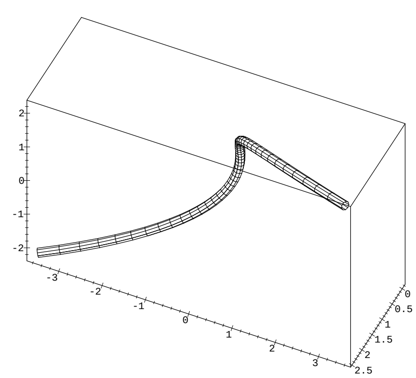
$\square$

So if $\tau(s_0) > 0$, the curve punctures its osculating plane at $\gamma(s_0)$ upwards, and if $\tau(s_0) > 0$ it punctures downwards.

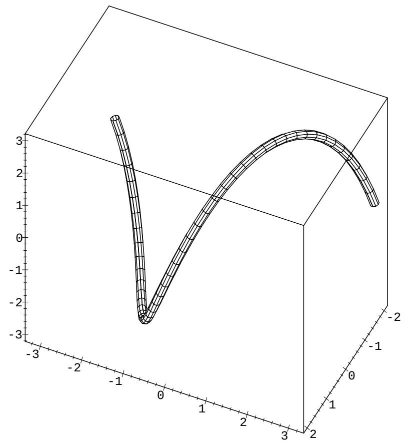


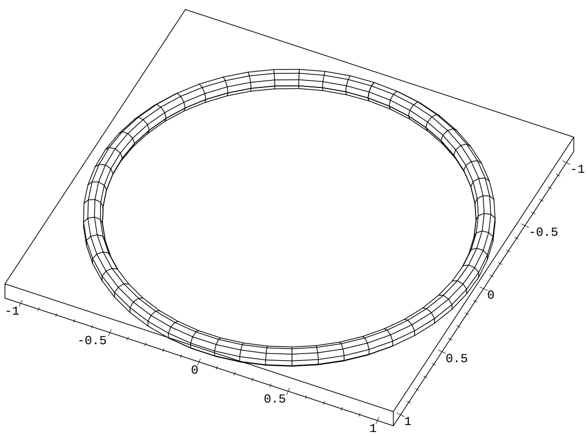
Let's test our understanding on a few examples. In each case, we want to determine whether the torsion of the curve at the marked point is positive, negative or zero.





Note: the sign of τ does *not* depend on the *orientation* of the curve, that is, the direction in which it is traversed.





4.3 The Frenet formulae

Recall that the curvature κ and torsion τ of a USCNVC in $\mathbb{R}^3$ can be extracted by decomposing $u'(s)$ and $b'(s)$ relative to the Frenet frame $[u(s), n(s), b(s)]$:

$$\boxed{u'(s) = \kappa(s)n(s)} \text{ — (1)} \qquad\qquad \boxed{b'(s) = -\tau(s)n(s)} \text{ — (2)}$$

What about $n'(s)$? Does this give us yet another scalar quantity analogous to κ and τ? In fact, it does not, as we shall now show.

As before, since $n' : I \to \mathbb{R}^3$ is a smooth, vector valued function, there must exist smooth scalar functions $\lambda, \mu, \nu : I \to \mathbb{R}$ such that

$$n'(s) = \lambda(s)u(s) + \mu(s)n(s) + \nu(s)b(s),$$

and, since $[u, n, b]$ is orthonormal (Lemma 57), we may extract the coefficients by taking scalar products:

$$\lambda(s) = u(s) \cdot n'(s), \qquad \mu(s) = n(s) \cdot n'(s), \qquad \nu(s) = b(s) \cdot n'(s).$$

- Now $u(s) \cdot n(s) = 0$ for all s, which upon differentiating with respect to s yields

  $$0 = u'(s) \cdot n(s) + u(s) \cdot n'(s) = \kappa(s)n(s) \cdot n(s) + u(s) \cdot n'(s) = \kappa(s) + \lambda(s),$$

  using equation (1). Hence $\lambda(s) = -\kappa(s)$.

- Similarly, $n(s) \cdot n(s) = 1$ for all s, and differentiating yields

  $$0 = n'(s) \cdot n(s) + n(s) \cdot n'(s) = 2\mu(s).$$

  Hence $\mu(s) = 0$.

- Finally, $b(s) \cdot n(s) = 0$ implies

  $$0 = b'(s) \cdot n(s) + b(s) \cdot n'(s) = -\tau(s)n(s) \cdot n(s) + b(s) \cdot n'(s) = -\tau(s) + \nu(s),$$

  using equation (2). Hence $\nu(s) = \tau(s)$.

Assembling the pieces, we have the formula

$$\boxed{n'(s) = -\kappa(s)u(s) + \tau(s)b(s)} \text{ — (3)}$$

Taking (1),(2),(3) together, we have just proved:

Theorem 62 (The Frenet formulae) *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed curve of nonvanishing curvature. Then its Frenet frame satisfies the formulae*

$$\begin{array}{rcccc} u'(s) & = & & \kappa(s)n(s) & \\ n'(s) & = & -\kappa(s)u(s) & & +\tau(s)b(s) \\ b'(s) & = & & -\tau(s)n(s) & \end{array}$$

□

We will use the Frenet formulae to prove two fundamental results:

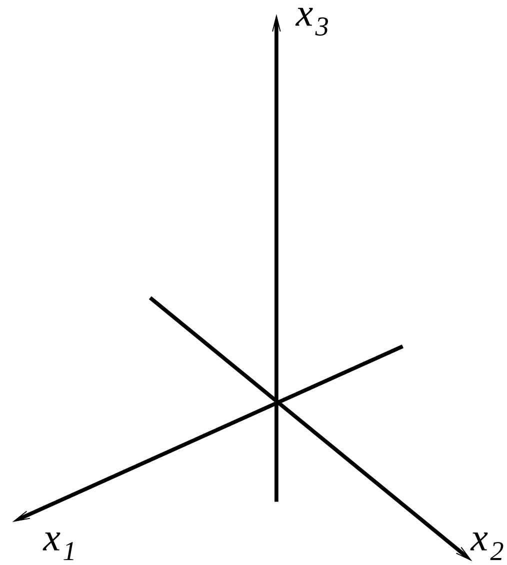
- That a USCNVC in $\mathbb{R}^3$ is *planar* if and only if its torsion is zero.
- That a USCNVC in $\mathbb{R}^3$ is uniquely determined by its scalar curvature and torsion (almost).

Definition 63 A **plane** $P \subset \mathbb{R}^3$ is the set of points $x \in \mathbb{R}^3$ satisfying the equation

$$B \cdot x = \nu,$$

where $B \in \mathbb{R}^3$ and $\nu \in \mathbb{R}$ are constants, and B is a unit vector ($|B| = 1$). Geometrically, B determines the orientation of the plane P, while $|\nu|$ is its distance from the origin, $(0, 0, 0)$. [Note that the pairs (B, ν) and $(-B, -\nu)$ describe the same plane.]

A curve $\gamma : I \to \mathbb{R}^3$ is **planar** if its image is contained in some plane P, that is, if there exist constants $B \in \mathbb{R}^3$, $|B| = 1$ and $\nu \in \mathbb{R}$, such that



$$B \cdot \gamma(s) = \nu$$

for all $s \in I$ $\square$

Example 64 The USCNVC $\gamma(s) = \frac{1}{2}(1 + 2\sin s, \sqrt{3} + \cos s, 1 - \sqrt{3}\cos s)$ is a **planar curve**. $\square$

How on earth do I know this? One way to settle the issue is to use:

Theorem 65 *Let $\gamma : I \to \mathbb{R}^3$ be a USCNVC. Then γ is planar if and only if $\tau(s) = 0$ for all $s \in I$.*

Proof: We must prove:

(A) γ planar $\Rightarrow$ $\tau = 0$
(B) $\tau = 0$ $\Rightarrow$ γ planar

(A) Assume γ is planar. Then there exist constants $B \in \mathbb{R}^3$ and $\nu \in \mathbb{R}$, as in definition 63 such that, for all $s \in I$,

$$\begin{aligned} \gamma(s) \cdot B &= \nu \\ \text{Diff. w.r.t. } s: \qquad u(s) \cdot B &= 0 \\ \text{Diff. w.r.t. } s \text{ again:} \qquad k(s) \cdot B &= 0 \\ \Rightarrow \quad \kappa(s)n(s) \cdot B &= 0 \\ \Rightarrow \quad n(s) \cdot B &= 0 \end{aligned}$$

since γ has nonvanishing curvature. So B is a unit vector, and is orthogonal to both $u(s)$ and $n(s)$ for all s. Compare this with $b(s) = u(s) \times n(s)$. This has precisely the same properties. Furthermore, in $\mathbb{R}^3$, the unit vector orthogonal to both $u(s)$ and $n(s)$ is *unique* up to sign. Hence

$$b(s) = \pm B,$$

a *constant* vector. Hence $b'(s) = 0$ and so $\tau(s) = 0$ for all $s \in I$ by the last Frenet formula.

(B) Assume $\tau \equiv 0$. Then $b' \equiv 0$ and hence $b(s) = b_0$, a constant unit vector (by the Mean Value Theorem). But then

$$\frac{d}{ds}(\gamma(s) \cdot b_0) = u(s) \cdot b_0 = u(s) \cdot b(s) = 0.$$

Hence $\gamma(s) \cdot b_0 = a$, some constant (by the Mean Value Theorem again). So γ lies in the plane determined by the unit vector $B = b_0$ and the constant $\nu = a$. $\square$
[Note: we only used the first and last Frenet formulae in this proof.]

Example 64 (revisited) Let's use theorem 65 to show that

$$\gamma(s) = \frac{1}{2}(1 + 2\sin s, \sqrt{3} + \cos s, 1 - \sqrt{3}\cos s)$$

is planar, and to construct the plane in which it lies. First, check it's a USCNVC:

$$\begin{aligned} \gamma'(s) &= \\ |\gamma'(s)| &= \\ k(s) = \gamma''(s) &= \\ \kappa(s) = |k(s)| &= \end{aligned}$$

Now construct its Frenet frame, compute $b'(s)$ and extract τ:

$$\begin{aligned} u(s) = \gamma'(s) &= \\ n(s) = \frac{k(s)}{|k(s)|} &= \\ b(s) = u(s) \times n(s) &= \end{aligned}$$

$$\begin{aligned} b'(s) &= \\ \tau(s) &= \end{aligned}$$

and hence γ is planar. Which plane?

$$B = b(s) \ =$$

$$\nu = \gamma(s) \cdot b(s) = \gamma(0) \cdot b(0) \ =$$

So the plane P containing γ has equation

$$(\quad , \quad , \quad) \cdot (x_1, x_2, x_3) = \quad , \quad \Rightarrow \qquad \square$$

So what kind of curve is this?

- It's planar.
- It has constant curvature.

Is it, perhaps, a **circle**? The answer is **yes**, but we will need to develop another piece of theory to demonstrate this.

Recall that a USC in $\mathbb{R}^2$ is uniquely determined by its signed curvature κ and its initial position γ_0 and initial velocity $(\cos\theta_0, \sin\theta_0)$ (Theorem 35). It turns out that a similar result holds for a USCNVC in $\mathbb{R}^3$, except now we have to specify the scalar curvature κ, the torsion τ, the initial position γ_0, the initial velocity u_0 **and** the initial normal vector n_0 (or equivalently, the initial osculating plane).

Theorem 66 *Given smooth functions $\kappa : I \to (0, \infty)$, $\tau : I \to \mathbb{R}$ $(0 \in I)$ and constants $\gamma_0, u_0, n_0 \in \mathbb{R}^3$, $|u_0| = |n_0| = 1$, $u_0 \cdot n_0 = 0$, there exists a unique unit speed curve $\gamma : I \to \mathbb{R}^3$ with*
(a) $\gamma(0) = \gamma_0$ (b) $\gamma'(0) = u_0$ (c) $n(0) = n_0$ (d) curvature κ (e) torsion τ.

Partial proof: Existence – too hard (see MATH 3181). Uniqueness: Let $\gamma, \widetilde{\gamma}$ be any pair of curves satisfying (a)–(e). We will show that $\gamma(s) = \widetilde{\gamma}(s)$ for all $s \in I$. Since both curves have identical torsion and (nonvanishing) curvature, their Frenet frames, call them $[u, n, b]$ and $[\widetilde{u}, \widetilde{n}, \widetilde{b}]$ respectively, both satisfy the Frenet formulae for κ and τ. It follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds}\left\{|u-\widetilde{u}|^2 + |n-\widetilde{n}|^2 + |b-\widetilde{b}|^2\right\} \\
&\quad = (u-\widetilde{u}) \cdot (u'-\widetilde{u}') + (n-\widetilde{n}) \cdot (n'-\widetilde{n}') + (b-\widetilde{b}) \cdot (b'-\widetilde{b}') \\
&\quad = -[u \cdot \widetilde{u}' + u' \cdot \widetilde{u} + n \cdot \widetilde{n}' + n' \cdot \widetilde{n} + b \cdot \widetilde{b}' + b' \cdot \widetilde{b}] \\
&\quad = -[u \cdot \kappa\widetilde{n} + \kappa n \cdot \widetilde{u} + n \cdot (-\kappa\widetilde{u} + \tau\widetilde{b}) + (-\kappa u + \tau b) \cdot \widetilde{n} + b \cdot (-\tau\widetilde{n}) - \tau n \cdot \widetilde{b}] \\
&\quad = 0
\end{aligned}$$

by the Frenet formulae. Hence

$$|u-\widetilde{u}|^2 + |n-\widetilde{n}|^2 + |b-\widetilde{b}|^2 = C \tag{1}$$

some constant. Substituting $s = 0$ in (1) and using properties (b) and (c), one sees that $C = 0$. Since all terms on the left hand side of (1) are non-negative, it follows that each is identically zero. Hence $|u(s) - \widetilde{u}(s)| = 0$ for all s. It follows that for all s,

$$\begin{aligned} \frac{d}{ds}(\gamma(s) - \widetilde{\gamma}(s)) &= 0 \\ \Rightarrow \gamma(s) - \widetilde{\gamma}(s) &= v \end{aligned}$$

a constant vector, which by property (a) must be 0. Hence $\gamma \equiv \widetilde{\gamma}$. $\square$

A less formal way of stating this result is that, up to rigid motions (translations and rotations of $\mathbb{R}^3$), a USCNVC is uniquely determined by $\kappa(s)$ and $\tau(s)$.

So, since a circle has constant scalar curvature (Example 36) and zero torsion (it's planar), Example 64 **must** be a circle, by Theorem 66.

Summary

- Given a RPC $\gamma : I \to \mathbb{R}^3$ of nonvanishing curvature we define its **unit tangent vector** $u(t)$, **principal unit normal** $n(t)$ and **binormal** $b(t)$ by

$$u(t) = \frac{\gamma'(t)}{|\gamma'(t)|}, \qquad n(t) = \frac{k(t)}{|k(t)|}, \qquad b(t) = u(t) \times n(t).$$

  This triple of vectors forms an orthonormal basis for $\mathbb{R}^3$ called the **Frenet frame**.

- The curvature $|k(t)|$ is usually denoted $\kappa(t)$. Note $\kappa \geq 0$.

- For a **unit speed** curve $\gamma : I \to \mathbb{R}^3$ of nonvanishing curvature we define the **torsion** $\tau : I \to \mathbb{R}$ by

$$b'(s) = -\tau(s)n(s).$$

- A USCNVC is **planar** if and only if $\tau \equiv 0$.

- The rate of change of the Frenet frame as one travels along a USCNVC is determined by the torsion and curvature according to the **Frenet formulae**

$$\begin{array}{rcccc} u'(s) & = & & \kappa(s)n(s) & \\ n'(s) & = & -\kappa(s)u(s) & & +\tau(s)b(s) \\ b'(s) & = & & -\tau(s)n(s) & \end{array}$$

- A USCNVC is uniquely determined (up to rigid motions) by its curvature and torsion.

5 Regularly parametrized surfaces

5.1 The basic definition

Our approach to studying the geometry of **curves** in $\mathbb{R}^n$ was to think of them as smooth maps $\gamma : I \to \mathbb{R}^n$. Recall that not every such map gives a nice, smooth, regular curve. We needed to impose a constraint on the derivative of γ, namely $\gamma'(t)$ should never vanish, in order to guarantee the curve was well behaved.

We now want to develop a similar method for studying **surfaces**, roughly speaking, nice smooth **two-dimensional** sets in $\mathbb{R}^3$. Actually, the techniques we will study generalize quite easily to work for $(n-1)$-dimensional surfaces in $\mathbb{R}^n$ for any n, but we will stick to two-dimensional surfaces in $\mathbb{R}^3$ because these are easiest to visualize. Again, we will think of surfaces as smooth maps $M : U \to \mathbb{R}^3$, where U is (a subset of) $\mathbb{R}^2$, so a surface will be a $\mathbb{R}^3$-valued function of **two** variables, $M(x_1, x_2)$. Once again, not every such map gives a nice smooth surface, and we need to impose constraints on the behaviour of $M(x_1, x_2)$ to guarantee its image set is well behaved. Just as for curves, the constraints amount to the requirement that the map $M : U \to \mathbb{R}^3$ be **regular**, but to make this notion precise requires a little more work.

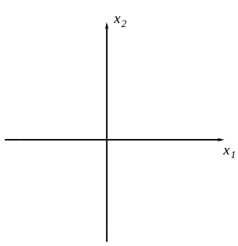
First, we identify the class of **domains** U we will be using. Our main concern is that we can do calculus with M, so we require that the domain has no boundary points (defining limits at boundary points is tricky).

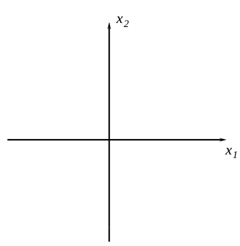
Definition 67 For each $x \in \mathbb{R}^2$ and $r > 0$ let $B_r(x) = \{y \in \mathbb{R}^2 : |y - x| < r\}$ be the **disk of radius r centred at x.** Then a subset $U \subseteq \mathbb{R}^2$ is **open** if for all $x \in U$ there exists $\delta > 0$ such that $B_\delta(x) \subseteq U$. $\square$

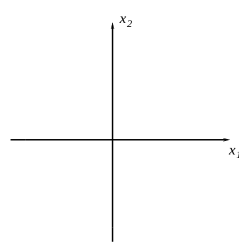
The idea is that no point in U lies at the edge of U, because every point can be surrounded by some disk in U. This is a generalization of the idea of an open interval in $\mathbb{R}$ (one which has no endpoints).

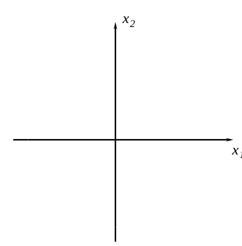
Example 68 The sets

$$(-1,1) \times \mathbb{R}, \quad (0,\pi) \times (0,1), \quad \{x \in \mathbb{R}^2 : 1 < |x| < 2\}, \quad \{x \in \mathbb{R}^2 : x_1 > -1\}$$



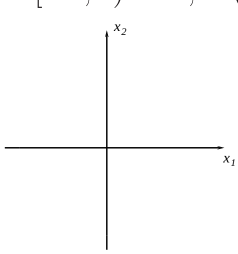


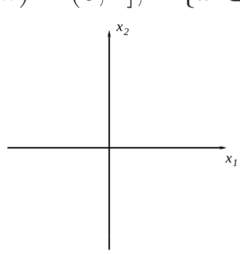


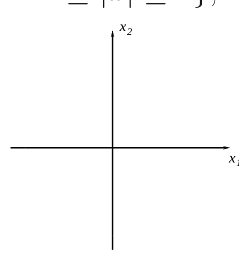


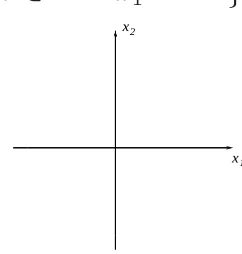
are all open sets, while

$$[-1,1) \times \mathbb{R}, \quad (0,\pi) \times (0,1], \quad \{x \in \mathbb{R}^2 : 1 \leq |x| \leq 2\}, \quad \{x \in \mathbb{R}^2 : x_1 = -1\}$$









are not. $\square$

Definition 69 Given a smooth map $M : U \to \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$ is open, let the **coordinate basis vectors** be $\varepsilon_1, \varepsilon_2 : U \to \mathbb{R}^3$,

$$\varepsilon_1 = \frac{\partial M}{\partial x_1}, \qquad \varepsilon_2 = \frac{\partial M}{\partial x_2}.$$

The point $(x_1, x_2) \in U$ is a **regular point** of M if $\{\varepsilon_1(x_1, x_2), \varepsilon_2(x_1, x_2)\}$ is linearly independent. The map M is **regular** if every $(x_1, x_2) \in U$ is a regular point. If M is not regular, we say it is **singular**, and refer to those values of x where $\varepsilon_1 \times \varepsilon_2 = (0, 0, 0)$ as the **singular points** of the map. $\square$

Since $\{\varepsilon_1(x_1, x_2), \varepsilon_2(x_1, x_2)\}$ contains only **two** vectors, it is linearly independent if and only if the vectors are not **parallel**. Two vectors in $\mathbb{R}^3$ are parallel if and only if their vector product vanishes, so another (more convenient) way to state Definition 69 is

Definition 69 (*) $M : U \to \mathbb{R}^3$ is **regular** if for all $x \in U$,

$$\varepsilon_1(x) \times \varepsilon_2(x) \neq (0, 0, 0).$$

Definition 70 A **regularly parametrized surface** (RPS) is a one-to-one, regular map $M : U \to \mathbb{R}^3$. $\square$

We will often denote the **image set** of the map by the same symbol, M, rather than $M(U)$, and refer to it as a RPS, in the same way that we often used the symbol γ to denote both a RPC and its image.

Example 71 (A sphere (almost)) Let $U = (0, \pi) \times (0, 2\pi)$. Note this is an open subset of $\mathbb{R}^2$ (it is a rectangle without boundary). Consider the mapping

$$M : U \to \mathbb{R}^3, \qquad M(x_1, x_2) = (\sin x_1 \cos x_2, \sin x_2 \sin x_2, \cos x_1)$$

I claim this is a RPS. Let's check this:

(A) is M one-to-one?
(B) is M regular?

(A)

(B) $\varepsilon_1(x_1, x_2) =$

$\varepsilon_2(x_1, x_2) =$

$\varepsilon_1 \times \varepsilon_2 =$

What does this RPS look like? Imagine we fix the value of parameter x_1 at some constant value, $\theta \in (0, \pi)$ say, and allow x_2 to take all values in $(0, 2\pi)$. We can then think of x_2 as a "time" coordinate so that, associated to the constant θ we have a **curve**

$$\beta_\theta : (0, 2\pi) \to \mathbb{R}^3, \qquad \beta_\theta(t) = M(\theta, t) = (\sin\theta\cos t, \sin\theta\sin t, \cos\theta)$$

This is a circle of radius $\sin\theta \in (0, 1)$ in the horizontal plane $y_3 = \cos\theta \in (-1, 1)$. Actually, it isn't quite a whole circle, as the point $(\sin\theta, 0, \cos\theta)$ is missing (t can't take the value 0 or 2π). Note that, by the very definition of partial differentiation,

$$\beta_\theta'(t) = \frac{d}{dt}M(\theta, t)\bigg|_{\theta \text{ const}} = \frac{\partial M}{\partial x_2}\bigg|_{(\theta,t)} = \varepsilon_2(\theta, t)$$

which gives a geometric interpretation of ε_2: it's the tangent vector along the curve defined by holding x_1 fixed but allowing x_2 to vary.

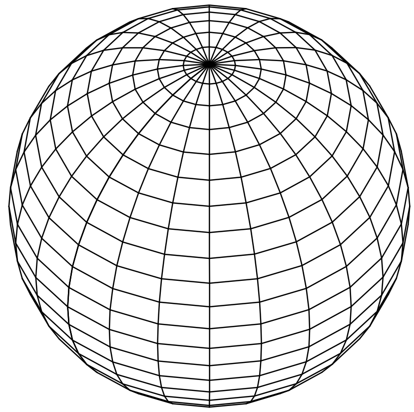
So, for each $\theta \in (0, \pi)$ we have a curve β_θ which is a circle of radius $\sin\theta$ in the plane $y_3 = \cos\theta$. As we allow θ to vary in $(0, \pi)$, the plane containg the circle shifts vertically from $y_3 = 1$ ($\theta = 0$) to $y_3 = -1$ ($\theta = \pi$) while at the same time growing from radius 0 ($\theta = 0$) to radius 1 ($\theta = \frac{\pi}{2}$) then shrinking back down to radius 0 again ($\theta = \pi$). The surface itself is the union of all these curves. In fact, for all θ and t,

$|M(\theta, t)|^2 =$

so every point on M lies distance 1 from $(0, 0, 0)$. Hence M is the surface of a sphere of radius 1. Actually, it isn't quite the whole sphere since it misses out one point on each circle so, in total, the whole semicircular line segment

$$\ell = \{(\sin\theta, 0, \cos\theta) \; : \; 0 \leq \theta \leq \pi\}$$

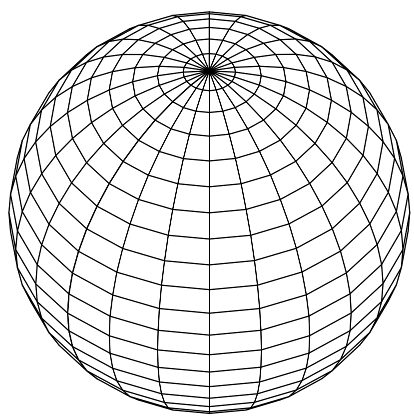
is missing.



Note that we could equally well have reconstructed the surface by considering the curves generated by fixing x_2 to a constant value, ϕ say, and allowing x_1 to take all values t in $(0, \pi)$:

$$\alpha_\phi : (0, \pi) \to \mathbb{R}^3, \qquad \alpha_\phi(t) = M(t, \phi) = (\sin t \cos \phi, \sin t \sin \phi, \cos t)$$

Each of these is a semicircle of radius 1 starting at $(0, 0, 1)$ and ending at $(0, 0, -1)$. As ϕ covers $(0, 2\pi)$ the semicircle α_ϕ rotates about the y_3 axis, sweeping out the unit sphere. As before, the semicircle ℓ (which would be α_0) is missing.



Remark 72 This idea applies to any RPS $M : U \to \mathbb{R}^3$. If we fix $x_2 = a$, some constant, we get a curve $\alpha_a(t) = M(t, a)$ along which x_1 varies. Changing the fixed value a generates a family of curves which sweep out (perhaps only part of) the surface M. Similarly if we fix $x_1 = b$, some constant, we get a curve $\beta_b(t)(t) = M(b, t)$ along which x_2 varies. Changing the fixed value b generates a family of curves which sweep out (perhaps only part of) the surface M. Furthermore

$$\varepsilon_1(t, a) = \alpha_a'(t), \qquad \varepsilon_2(b, t) = \beta_b'(t)$$

which gives a geometric interpretation of the coordinate basis vectors.

Example 73 (A paraboloid) Is the map $M : \mathbb{R}^2 \to \mathbb{R}^3$,

$$M(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$$

a RPS? Must check two things:

(A) is M one-to-one?
(B) is M regular?

(A)

(B) $\varepsilon_1(x_1, x_2) =$

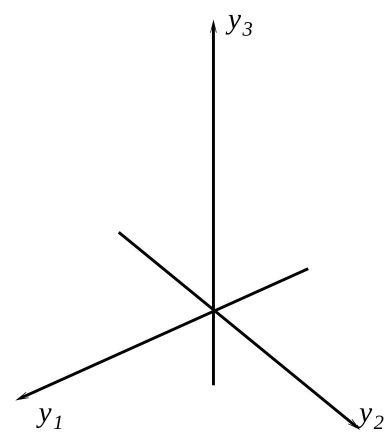
$\varepsilon_2(x_1, x_2) =$

$\varepsilon_1 \times \varepsilon_2 =$

So M is a RPS. Note that it is not always necessary to calculate all 3 components of $\varepsilon_1(x) \times \varepsilon_2(x)$ to show that M is regular at x. If we show that any one of the components is never 0, that suffices. In general, to show that a map $M : U \to \mathbb{R}^3$ is regular, we have to show that the 3 components of $\varepsilon_1 \times \varepsilon_2$ never **simultaneously** vanish.

Think of the map M as attaching a pair of coordinates (x_1, x_2) to each point y on the surface. For example,

- $y = (1, 1, 2) \in M$ has coords $x = (1, 1)$
- $y = (-1, 2, 5) \in M$ has coords $x = (\;,\;)$
- What about $y = (0, 1, -3)$?



□

This is why definition 70 requires the map $M : U \to \mathbb{R}^2$ to be one-to-one: for every $y \in M$ there is one **and only one** point $x \in U$ such that $M(x) = y$. This point $x = (x_1, x_2)$ defines the **local coordinates** of y. So we have a way of refering to points on the surface using 2 numbers (the components of x), rather than 3 (the components of y itself). We've already seen something like this: given a parametrized *curve* $\gamma : I \to \mathbb{R}^n$, we may refer to any point y on the curve either by its components $(y_1, y_2, \ldots, y_n)$, or more succinctly, by specifying the time $t \in I$ at which $\gamma(t) = y$.

Remark 74 In Example 71 our RPS covered only a part of the unit sphere

$$S = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \; : \; y_1^2 + y_2^2 + y_3^3 = 1\}.$$

This will be a common situation: often we want to "parametrize" a surface which can't be covered all in one go. Usually, a RPS which covers almost all of the surface is good enough. In Example 73 we had a RPS which covered all of the paraboloid

$$P = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \; : \; y_1^2 + y_2^2 - y_3 = 0\}.$$

in one go. What if we seek a RPS whose image set is the **cylinder**

$$C = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \; : \; y_1^2 + y_2^2 = 1\}?$$

Can we cover the whole of C is one go? It may surprise you to hear that the answer is **yes**! One example of such a RPS is $M : U \to \mathbb{R}^3$,

$$M(x_1, x_2) = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \log \sqrt{x_1^2 + x_2^2} \right)$$

where $U = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; (x_1, x_2) \neq (0, 0)\}$ the punctured plane.
Exercise: verify this, that is, show that M is a RPS and that its image set is the whole of C.

Example 75 If we are content with a RPS whose image set is almost all of the cylinder C, things are much easier. We can take $M : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$,

$$M(x_1, x_2) = (\cos x_1, \sin x_1, x_2)$$

for example. Let's check:

- Is M regular?

  $\varepsilon_1(x_1, x_2) \; =$

  $\varepsilon_2(x_1, x_2) \; =$

  $\varepsilon_1 \times \varepsilon_2 \; =$

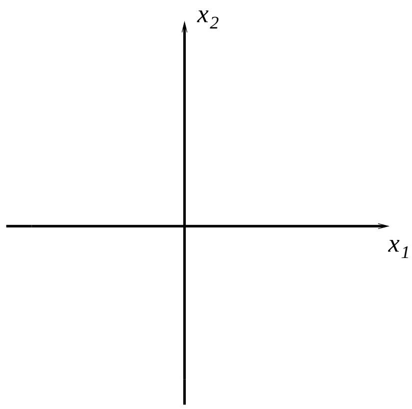
  which vanishes only if $\sin x_1 = 0$ and $\cos x_1 = 0$, which is impossible. So M is regular.

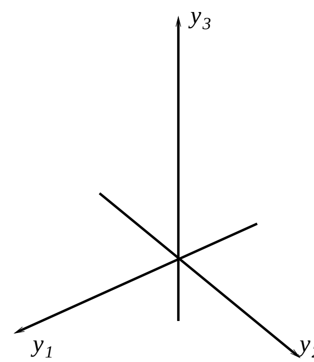
- Is it one-to-one? Assume $M(x_1, x_2) = M(\overline{x}_1, \overline{x}_2)$. Then

$$(\cos x_1, \sin x_1, x_2) = (\cos \overline{x}_1, \sin \overline{x}_1, \overline{x}_2)$$

  so $x_2 = \overline{x}_2$ and $x_1 - \overline{x}_1$ is an integer multiple of 2π. But x_1 and $\overline{x_1}$ are in $(0, 2\pi)$, so their difference must be 0. Hence $M(x_1, x_2) = M(\overline{x}_1, \overline{x}_2)$ implies $(x_1, x_2) = (\overline{x}_1, \overline{x}_2)$, that is, M is one-to-one.

The resulting surface is almost all of C. What points are missing?





This RPS omits the vertical line through $(1, 0, 0)$. We can always include this line, at the expense of omitting a different one, by changing the domain of M. For example, choosing $U = (-\pi, \pi) \times \mathbb{R}$ omits the vertical line through $(-1, 0, 0)$ instead.

We can give a geometric interpretation to the local coordinates (x_1, x_2) for this RPS. Given a point $y = M(x_1, x_2)$ on C we see that x_2 is the height of y above the plane $y_3 = 0$, and x_1 is an angular coordinate which measures the angular position of (y_1, y_2) in the plane (relative to the positive y_1 axis).

$\square$

5.2 The tangent and normal spaces

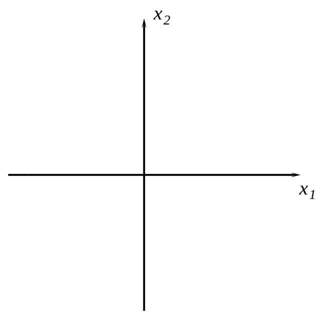
Recall that every point on a regularly parametrized curve has a well defined *tangent line*. The generalization of this to regularly parametrized surfaces is called the *tangent space*:

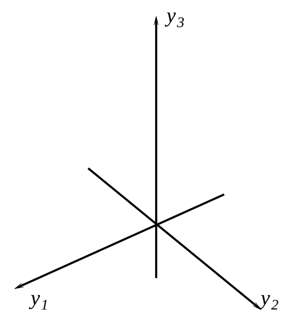
Definition 76 Let $y \in M$ be a point on a surface $M : U \to \mathbb{R}^3$. Then a **curve through** y **in** M is a smooth map $\alpha : I \to M$ $(0 \in I)$ with $\alpha(0) = y$. The **tangent space** to M at $y \in M$ is

$$T_yM = \{v \in \mathbb{R}^3 : \text{there exists a curve } \alpha \text{ through } y \text{ with } \alpha'(0) = v\}.$$

Any $v \in T_yM$ is called a **tangent vector** to M at y. $\square$

So the tangent space at y is the set of all possible velocity vectors $\alpha'(0) \in \mathbb{R}^3$ of curves α passing through y. Note that the curve α is *not* assumed to be regular.





Example 77 (paraboloid) Let $M : \mathbb{R}^2 \to \mathbb{R}^3$, $M(x_1, x_2) = (x_1, x_2, x_1^2 + x_2^2)$ and $y = (2, 1, 5) = M(2, 1)$. The easiest way to define a curve through y in M is to write down its *coordinate expression* $\widehat{\alpha} : I \to \mathbb{R}^2$. In order to be a curve *through* y, this must satisfy

$$\widehat{\alpha}(0) = M^{-1}(2, 1, 5) = (2, 1).$$

For example,

$$\begin{aligned}
\widehat{\alpha}(t) = (2, 1 + t) \quad &\Rightarrow \quad \alpha(t) = M(\widehat{\alpha}(t)) \;\; = \\
\widehat{\beta}(t) = (2 + t, 1 - t) \quad &\Rightarrow \quad \beta(t) = M(\widehat{\beta}(t)) \;\; = \\
\widehat{\gamma}(t) = (2 + t^2, 1 + \cos t) \quad &\Rightarrow \quad \gamma(t) = M(\widehat{\alpha}(t)) \;\; =
\end{aligned}$$

all give curves through y in M. The corresponding *tangent vectors* are:

$$\begin{aligned}
\alpha'(0) &= \\
\beta'(0) &= \\
\gamma'(0) &=
\end{aligned}$$

[Note that γ is *not* a RPC]. $\square$

A handy way to calculate with tangent vectors is to express them in terms of the coordinate basis vectors $\varepsilon_1, \varepsilon_2$ at $x = \bar{x}$. If α is a curve through $y = M(\bar{x})$, then

$$\begin{aligned} \alpha(t) &= M(\widehat{\alpha}(t)) = M(\widehat{\alpha}_1(t), \widehat{\alpha}_2(t)) \\ \Rightarrow \quad \alpha'(0) &= \left.\frac{\partial M}{\partial x_1}\right|_{\widehat{\alpha}(0)} \widehat{\alpha}_1'(0) + \left.\frac{\partial M}{\partial x_2}\right|_{\widehat{\alpha}(0)} \widehat{\alpha}_2'(0) = \widehat{\alpha}_1'(0)\varepsilon_1(\bar{x}) + \widehat{\alpha}_2'(0)\varepsilon_2(\bar{x}) \end{aligned}$$

So every tangent vector can be expressed as a linear combination of $\varepsilon_1(\bar{x}), \varepsilon_2(\bar{x})$. Furthermore, every linear combination of $\varepsilon_1(\bar{x}), \varepsilon_2(\bar{x})$ is a tangent vector, as we shall now prove. Let $v = a\varepsilon_1(\bar{x}) + b\varepsilon_2(\bar{x})$ where $a, b \in \mathbb{R}$. I claim that $v \in T_yM$, that is, that there exists a curve through y whose velocity vector at time $t = 0$ is v. One such curve is

$$\alpha(t) = M(\bar{x}_1 + at, \bar{x}_2 + bt)$$

Check:

$$\begin{aligned} \alpha'(0) &= \\ \\ &= \end{aligned}$$

Hence

$$T_yM = \{a\varepsilon_1 + b\varepsilon_2 \,|\, a, b \in \mathbb{R}\}.$$

It follows that the **subset** $T_yM \subset \mathbb{R}^3$ is closed under vector addition and scalar multiplication: T_yM is a **subspace** of $\mathbb{R}^3$. Further, $\{\varepsilon_1, \varepsilon_2\}$ is a spanning set for this subspace. Since M is regular, this spanning set is linearly independent, hence a **basis** for T_yM. This explains why we call $\varepsilon_1, \varepsilon_2$ the coordinate basis vectors of the surface.

To summarize, we have proved the following:

Theorem 78 *T_yM is a vector space of dimension 2 spanned by $\{\varepsilon_1, \varepsilon_2\}$.* □

Definition 79 The **normal space** at $y \in M$ is

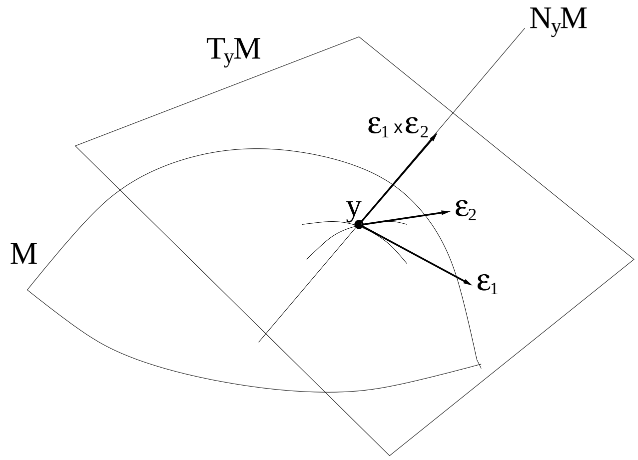
$$N_yM = \{v \in \mathbb{R}^3 : v \cdot u = 0 \text{ for all } u \in T_yM\}.$$

Any $v \in N_yM$ is said to be **normal to** M **at** y. □

Remark 80 By its definition, N_yM is also a vector space, that is, it is closed under vector addition and scalar multiplication: let $v, w \in N_yM$ and $a, b \in \mathbb{R}$. Then for all $u \in T_yM$,

$$u \cdot (av + bw) = a(u \cdot v) + b(u \cdot w) = 0 + 0$$

so $av + bw \in N_yM$. Clearly N_yM is one-dimensional, so any non-zero normal vector, for example $\varepsilon_1 \times \varepsilon_2$, is a basis for N_yM.



We can use $\varepsilon_1 \times \varepsilon_2$ to give an alternative characterization of T_yM:

Lemma 81 $T_yM = \{v \in \mathbb{R}^3 \,|\, v \cdot (\varepsilon_1 \times \varepsilon_2) = 0\}$.

Proof: Consider the linear map $L : \mathbb{R}^3 \to \mathbb{R}$, $L(v) = v \cdot (\varepsilon_1 \times \varepsilon_2)$. The lemma asserts that T_yM is the kernel of L (the set of vectors which get mapped to 0). Since $\varepsilon_1 \times \varepsilon_2 \in N_yM$, every $v \in T_yM$ is orthogonal to $\varepsilon_1 \times \varepsilon_2$, so $T_yM \subset \ker L$. Since $L(\varepsilon_1 \times \varepsilon_2) \neq 0$, $\ker L$ has dimension at most 2. Hence $\ker L \subset T_yM$. $\square$

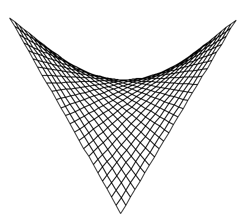
This gives us a sneaky way of checking whether a given vector is a tangent vector.

Example 82 (saddle surface) Let $M : \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$M(x_1, x_2) = (x_1, x_2, x_1x_2).$$

I claim this is a regularly parametrized surface:

(a) is M one-to-one?



(b) is M regular?

$\varepsilon_1 \quad =$ $\qquad\qquad\qquad$ $\varepsilon_2 \quad =$

$\varepsilon_1 \times \varepsilon_2 \quad =$

The point $y = (3, -2, -6) \in M$ has local coordinates $\bar{x} = (3, -2)$, so the coordinate basis for T_yM is

$\varepsilon_1 =$ $\varepsilon_2 =$

and N_yM is spanned by

$$\varepsilon_1 \times \varepsilon_2 =$$

Let's determine whether the following vectors are in T_yM, N_yM or neither:

$$u = (1, 2, 4), \qquad v = (2, 1, 1), \qquad w = (-4, 6, -2).$$

$\boxed{u:}$ $u \cdot (\varepsilon_1 \times \varepsilon_2) =$

So $v \in T_yM$. Hence $(1, 2, 4) = a\varepsilon_1 + b\varepsilon_2$ for some $a, b \in \mathbb{R}$. In fact

$(1, 2, 4) =$

so $a =$ and $b =$.

$\boxed{v:}$ $v \cdot (\varepsilon_1 \times \varepsilon_2) =$

So $v \notin T_yM$. What about N_yM?

$v \cdot \varepsilon_1 =$ $v \cdot \varepsilon_2 =$

So $v \notin N_yM$ either.

$\boxed{w:}$ $w \cdot (\varepsilon_1 \times \varepsilon_2) =$

So $w \notin T_yM$. What about N_yM?

$w \cdot \varepsilon_1 =$ $w \cdot \varepsilon_2 =$

So $w \in N_yM$. Hence $w = c\,\varepsilon_1 \times \varepsilon_2$ for some $c \in \mathbb{R}$. In fact

$(-4, 6, -2) =$

so $c =$. □

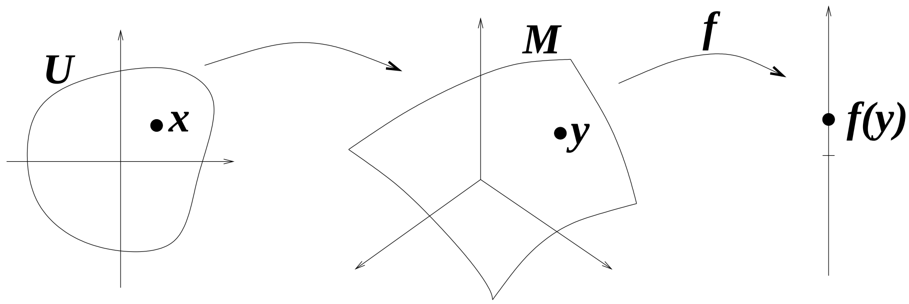
Summary

- Given a smooth mapping $M \to U \to \mathbb{R}^3$, where $U \subseteq \mathbb{R}^2$ is an open set, its **coordinate basis vectors** are
$$\varepsilon_1 = \frac{\partial M}{\partial x_1}, \qquad \varepsilon_2 = \frac{\partial M}{\partial x_2}.$$
M is **regular** if for all $x \in U$, $\varepsilon_1(x) \times \varepsilon_2(x) \neq 0$.
- A **regularly parametrized surface** is a regular, one-to-one map $M : U \to \mathbb{R}^3$.
- The **tangent space** T_yM at $y = M(\overline{x})$ is the set of all velocity vectors of curves in the surface M passing through the point y. It is a two-dimensional vector space spanned by $\{\varepsilon_1(\overline{x}), \varepsilon_2(\overline{x})\}$.
- The **normal space** N_yM at $y = M(\overline{x})$ is the set of vectors in $\mathbb{R}^3$ orthogonal to T_yM. It is a one-dimensional vector space spanned by $\varepsilon_1(\overline{x}) \times \varepsilon_2(\overline{x})$.

6 Calculus on surfaces

6.1 Directional derivatives

Given a regularly parametrized surface $M : U \to \mathbb{R}^3$, we may consider **real-valued functions** on M, $f : M \to \mathbb{R}$, that is, rules which assign to each point $y \in M \subset \mathbb{R}^3$ some real number $f(y)$.



We will need to make sense of calculus for such functions (rates of change etc.). As a first step, we define when $f : M \to \mathbb{R}$ is **smooth**. Recall that a mapping $F : \mathbb{R}^n \to \mathbb{R}$ is smooth if all its partial derivatives (of all orders) exist everywhere. It doesn't make sense to demand directly that $f : M \to \mathbb{R}$ is smooth: what does it mean to differentiate a function with repect to a point on a paraboloid, for example? To get round this, we use the parametrization provided by $M : U \to \mathbb{R}^3$. That is, loosely speaking, we think of f as a function not of $y \in M$ but of the local coordinates $x \in M$ corresponding to y.

Definition 83 Let $M : U \to \mathbb{R}^3$ be a regularly parametrized surface. The **coordinate expression** of f is $\widehat{f} : U \to \mathbb{R}$ defined by $\widehat{f} = f \circ M$, that is,

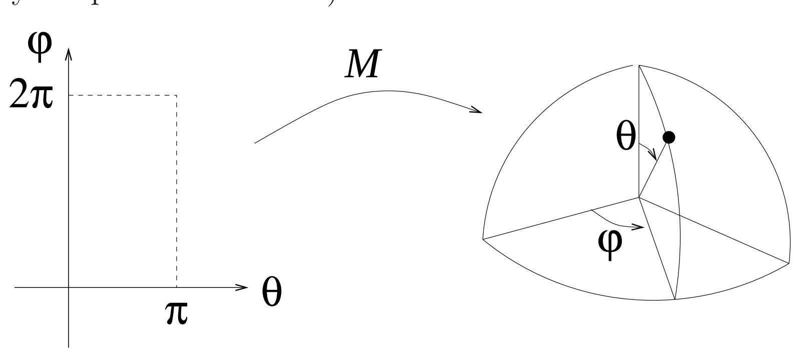
$$\widehat{f}(x_1, x_2) = f(M(x_1, x_2)).$$

We say that f is **smooth** if $\widehat{f}$ is smooth in the usual sense (all its partial derivatives, of all orders, exist at all $x \in U$).

Example 84 Consider the RPS defined in Example 71, whose image is (almost all of) the unit sphere:

$$M : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3, \quad M(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

We have denoted the local coordinates (θ, ϕ) instead of (x_1, x_2), as is traditional for this RPS (they are polar coordinates).



The following maps $M \to \mathbb{R}$

$$f : (y_1, y_2, y_3) \mapsto y_1+y_2+y_3, \quad g : (y_1, y_2, y_3) \mapsto y_1^2+y_2^2-y_3^2, \quad h : (y_1, y_2, y_3) \mapsto 1-2y_3^2.$$

have coordinate expressions

$$\begin{aligned}
\widehat{f}(\theta, \phi) = f(M(\theta, \phi)) &= \\
\widehat{g}(\theta, \phi) = g(M(\theta, \phi)) &= \\
\widehat{h}(\theta, \phi) = h(M(\theta, \phi)) &=
\end{aligned}$$

These are all smooth functions of θ and ϕ, so f, g, h are smooth functions $M \to \mathbb{R}$. Note that g and h are really the *same* function. Why?

Given a smooth function $f : M \to \mathbb{R}$ on a regularly parametrized surface M, and a tangent vector $v \in T_yM$, we can ask "what is the rate of change of f at y in the direction of v?" The answer is given (almost) by the following definition:

Definition 85 Let $f : M \to \mathbb{R}$ be a smooth function and $v \in T_yM$. The **directional derivative** of f along v is

$$v[f] = (f \circ \alpha)'(0)$$

where $\alpha : I \to M$ is a curve through y ($\alpha(0) = y$) whose velocity vector at y is $\alpha'(0) = v$. [Such a curve exists by the definition of T_yM, Definition 76.] Note that $u[f]$ is a single number, associated with the point y, not a function on M. □

Example 86 Let M be the unit sphere (parametrized as in Example 71 say), let $y = (0, 1, 0)$ and $v = (0, 0, 1)$. Then $v \in T_yM$ (**check it!**). Consider the function $f : M \to \mathbb{R}$ given by

$$f(y_1, y_2, y_3) = y_3$$

which assigns to each point on the sphere its "height" above the (y_1, y_2) plane. What is $v[f]$?

From the definition, we must choose a curve through $(0, 1, 0)$ in M whose initial velocity is $(0, 0, 1)$. One obvious choice is

$$\alpha(t) = (0, \cos t, \sin t).$$

$$\begin{aligned}
\text{Check:} \quad |\alpha(t)| &= \\
\alpha(0) &= \\
\alpha'(0) &=
\end{aligned}$$

$$\begin{aligned}\text{Now:}\quad (f\circ\alpha)(t) &= \\ \\ (f\circ\alpha)'(t) &= \\ \\ \text{So:}\quad v[f] &= \end{aligned}$$

Note that $v[f]$ isn't quite just the "rate of change of f in the **direction** of v," because it depends on the **length** of v. For example, let $u = 2v = (0, 0, 2) \in T_pM$. Then this points in the same direction as v, but is the initial velocity vector of the curve

$$\beta(t) = (0, \cos 2t, \sin 2t)$$

through y.

$$\begin{aligned}\text{Now:}\quad (f\circ\beta)(t) &= \\ \\ (f\circ\beta)'(t) &= \\ \\ \text{So:}\quad u[f] &= \end{aligned}$$

which is different from $v[f]$. In fact $u[f] = 2v[f]$. Coincidence? $\square$

One nice thing about Definition 85 is that it doesn't use the specific parametrization of the surface involved (we never actually had to worry about the defining map $M(\theta, \phi)$ for the unit sphere in the above example). There is something a bit worrying about it, however. To compute $v[f]$ we have to <u>choose</u> a curve α with initial velocity v. There are infinitely many such curves. In the example above, we could equally well have chosen

$$\alpha(t) = (0, \sqrt{1-t^2}, t), \quad \text{or} \quad \alpha(t) = (t^2, \sqrt{1-t^4-\sin^2 t}, \sin t)$$

for example. How do we know that the answer doesn't depend on our choice?

Lemma 87 *$v[f]$ is independent of the choice of curve α representing the tangent vector v.*

Proof: Let $\alpha, \beta : I \to M$ be two curves through $y = M(\bar{x})$ with $\alpha'(0) = \beta'(0) = v$. Then

$$\alpha(t) = M(\widehat{\alpha}(t)), \qquad \beta(t) = M(\widehat{\beta}(t))$$

where $\widehat{\alpha}, \widehat{\beta} : I \to U$ are their coordinate expressions. Hence

$$\begin{aligned} 0 = \alpha'(0) - \beta'(0) &= \left.\frac{d}{dt}\right|_{t=0} [M(\widehat{\alpha}_1(t), \widehat{\alpha}_2(t)) - M(\widehat{\beta}_1(t), \widehat{\beta}_2(t))] \\ &= \left.\frac{\partial M}{\partial x_1}\right|_{\bar{x}} [\widehat{\alpha}_1'(0) - \widehat{\beta}_1'(0)] + \left.\frac{\partial M}{\partial x_2}\right|_{\bar{x}} [\widehat{\alpha}_2'(0) - \widehat{\beta}_2'(0)] \\ &= [\widehat{\alpha}_1'(0) - \widehat{\beta}_1'(0)]\varepsilon_1(\bar{x}) + [\widehat{\alpha}_2'(0) - \widehat{\beta}_2'(0)]\varepsilon_2(\bar{x}) \\ \Rightarrow \quad \widehat{\alpha}'(0) &= \widehat{\beta}'(0) \qquad (\{\varepsilon_1, \varepsilon_2\} \text{ linearly independent}) \qquad (\clubsuit). \end{aligned}$$

Hence

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0}[f(\alpha(t))-f(\beta(t))] &= \left.\frac{d}{dt}\right|_{t=0}[f(M(\widehat{\alpha}(t)))-f(M(\widehat{\beta}(t)))] \\
&= \left.\frac{d}{dt}\right|_{t=0}[\widehat{f}(\widehat{\alpha}(t))-\widehat{f}(\widehat{\beta}(t))] \\
&= \left.\frac{\partial \widehat{f}}{\partial x_1}\right|_{\bar{x}}[\widehat{\alpha}_1'(0)-\widehat{\beta}_1'(0)]+\left.\frac{\partial \widehat{f}}{\partial x_2}\right|_{\bar{x}}[\widehat{\alpha}_2'(0)-\widehat{\beta}_2'(0)]=0
\end{aligned}$$

by (♣). □

This is reassuring. However, Definition 85 is rather cumbersome to use in practice because it's tiresome to have to keep inventing curves to represent tangent vectors. Recall that we may also think of T_yM as

$$T_yM = \{a\,\varepsilon_1 + b\,\varepsilon_2 \,|\, a,b,\in\mathbb{R}\}.$$

The next Lemma will allow us to reduce the calculation of directional derivatives to straightforward partial differentiation with respect to the local coordinates (x_1, x_2).

Lemma 88 *$v[f]$ is linear with respect to both v and f. That is, for all $u, v \in T_yM$, $a, b \in \mathbb{R}$ and $f, g : M \to \mathbb{R}$,*

(A) $(au + bv)[f] = a(u[f]) + b(v[f])$

(B) $v[af + bg] = a(v[f]) + b(v[g])$

Proof:
(A) Let $\alpha, \beta : I \to M$ be curves through $y = M(\bar{x})$ with $\alpha'(0) = u$, $\beta'(0) = v$, and let $\widehat{\alpha}, \widehat{\beta} : I \to U$ be their coordinate expressions. Then

$$\widehat{\gamma}(t) = \widehat{\alpha}(at) + \widehat{\beta}(bt) - \bar{x}$$

is the coordinate expression of a curve $\gamma(t) = M(\widehat{\gamma}(t))$ through y with $\gamma'(0) = au+bv$. [**Check it!**] Hence,

$$\begin{aligned}
(au+bv)[f] &= \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \left.\frac{d}{dt}\right|_{t=0} f(M(\widehat{\gamma}(t))) = \left.\frac{d}{dt}\right|_{t=0} \widehat{f}(\widehat{\gamma}(t)) \\
&= \left.\frac{\partial \widehat{f}}{\partial x_1}\right|_{\bar{x}} \widehat{\gamma}_1'(0) + \left.\frac{\partial \widehat{f}}{\partial x_2}\right|_{\bar{x}} \widehat{\gamma}_2'(0) \\
&= \left.\frac{\partial \widehat{f}}{\partial x_1}\right|_{\bar{x}} [a\widehat{\alpha}_1'(0)+b\widehat{\beta}_1'(0)] + \left.\frac{\partial \widehat{f}}{\partial x_2}\right|_{\bar{x}} [a\widehat{\alpha}_2'(0)+b\widehat{\beta}_2'(0)] \\
&= a\left[\left.\frac{\partial \widehat{f}}{\partial x_1}\right|_{\bar{x}} \widehat{\alpha}_1'(0) + \left.\frac{\partial \widehat{f}}{\partial x_2}\right|_{\bar{x}} \widehat{\alpha}_2'(0)\right] + b\left[\left.\frac{\partial \widehat{f}}{\partial x_1}\right|_{\bar{x}} \widehat{\beta}_1'(0) + \left.\frac{\partial \widehat{f}}{\partial x_2}\right|_{\bar{x}} \widehat{\beta}_2'(0)\right] \\
&= a(\widehat{f}\circ\widehat{\alpha})'(0) + b(\widehat{f}\circ\widehat{\beta})'(0) \\
&= a(u[f]) + b(v[f])
\end{aligned}$$

(B) Follows immediately from the definition:

$$\begin{aligned} v[af + bg] &= ((af + bg) \circ \beta)'(0) = (af \circ \beta + bg \circ \beta)'(0) \\ &= a(f \circ \beta)'(0) + b(g \circ \beta)'(0) = a(v[f]) + b(v[g]). \end{aligned} \qquad \square$$

The upshot of Lemma 88 is that if we know the directional derivatives with respect to the coordinate basis vectors, $\varepsilon_1[f], \varepsilon_2[f]$, we know $v[f]$ for *all* tangent vectors $v \in T_yM$. For any $v \in T_yM$ may be written as

$$v = a\varepsilon_1 + b\varepsilon_2$$

so applying Lemma 88 part (A) gives

$$v[f] = a\varepsilon_1[f] + b\varepsilon_2[f]. \qquad (\spadesuit)$$

But $\varepsilon_1(\bar{x})$ and $\varepsilon_2(\bar{x})$ may be represented by the x_1- and x_2-parameter curves, whose coordinate expressions are

$$(\bar{x}_1 + t, \bar{x}_2), \qquad (\bar{x}_1, \bar{x}_2 + t)$$

respectively. Hence

$$\varepsilon_1[f] = \frac{\partial \widehat{f}}{\partial x_1}, \qquad \varepsilon_2[f] = \frac{\partial \widehat{f}}{\partial x_2}.$$

Example 86 (revisited) Let's re-do Example 86 using the trick of reducing directional derivatives to partial differentiation. Recall M is the unit sphere parametrized by

$$M(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

The function is $f(y_1, y_2, y_3) = y_3$ and we wish to compute $v[f]$ where $v = (0, 0, 1) \in T_yM$ and $y = (0, 1, 0)$.

- The local coordinates of y are $\bar{x} = (\theta, \phi) = (\quad , \quad)$
  [Check: $M(\quad , \quad) =$
- The local coordinate expression for f is

  $$\widehat{f}(\theta, \phi) =$$

- The coordinate basis vectors at y are

  $$\begin{aligned} \varepsilon_1(\bar{x}) &= \\ \varepsilon_2(\bar{x}) &= \end{aligned}$$

- Hence, $v = a\varepsilon_1 + b\varepsilon_2$ where

  $$a = \qquad\qquad , b =$$

- Finally,

$$\begin{aligned} v[f] &= a\varepsilon_1[f] + b\varepsilon_2[f] = a\left.\frac{\partial \widehat{f}}{\partial \theta}\right|_{\bar{x}} + b\left.\frac{\partial \widehat{f}}{\partial \phi}\right|_{\bar{x}} \\ &= \end{aligned}$$

We got the same answer! Which calculation was easier, Example 86 or Example 86 (revisited)?

Example 89 Consider the mapping

$$M : \mathbb{R}^2 \to \mathbb{R}^3, \qquad M(x_1, x_2) = (x_1 + x_2, x_1x_2, x_1 - 2x_2).$$

(i) Verify that M is a RPS

(ii) Show that $v = (3, 1, 0)$ is tangent to M at $y = (1, 0, 1) = M(1, 0)$

(iii) Compute $v[f]$ where $f : M \to \mathbb{R}$ is $f(y_1, y_2, y_3) = (y_1 + y_3)y_2$

Solution:

(i) $M(x) = M(\bar{x})$ implies

$$\begin{aligned} x_1 + x_2 &= \bar{x}_1 + \bar{x}_2 \\ x_1 - 2x_2 &= \bar{x}_1 - 2\bar{x}_2 \end{aligned}$$

and hence $3x_2 = 3\bar{x}_2$ (subtracting the lower equation from the upper) whence $x_2 = \bar{x}_2$ and so $x_1 = \bar{x}_1$ (substituting into the upper equation). Hence M is one-to-one.

Furthermore

$$\begin{aligned} \varepsilon_1 &= \\ \varepsilon_2 &= \\ \Rightarrow \varepsilon_1 \times \varepsilon_2 &= \end{aligned}$$

which never vanishes, so M is a RPS.

(ii) At the point $y = (1, 0, 1) = M(1, 0)$ the coordinate basis and normal vectors are

$\varepsilon_1 =$ $\qquad\qquad$ $\varepsilon_2 =$ $\qquad\qquad$ $\varepsilon_1 \times \varepsilon_2 =$

so

$$v \cdot (\varepsilon_1 \times \varepsilon_2) =$$

and hence $v \in T_yM$.

(iii) We must write $v = a\varepsilon_1 + b\varepsilon_2$ for some constants a, b. That is, we seek a, b such that

$$a(\quad,\quad,\quad) + b(\quad,\quad,\quad) = (3,1,0)$$

Re-arrange this as a set (of three) linear simultaneous equations for a, b, then solve by row-reduction:

Hence $v =$

The function f has coordinate expression

$$\widehat{f}(x_1, x_2) =$$

Hence

$$\begin{aligned} v[f] &= a \left.\frac{\partial \widehat{f}}{\partial x_1}\right|_{(1,0)} + b \left.\frac{\partial \widehat{f}}{\partial x_2}\right|_{(1,0)} \\ &= \\ &= \end{aligned}$$

□

The fact that we can reduce the calculation of $v[f]$ to partial differentiation is theoretically convenient, too. For example, we know that partial derivatives obey the product (or Leibniz) rule

$$\frac{\partial}{\partial x_i}(fg) = f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}.$$

From this, we immediately obtain

Corollary 90 *For all $f, g : M \to \mathbb{R}$ and $v \in T_yM$,*

$$v[fg] = f(y)v[g] + g(y)v[f].$$

□

6.2 Vector fields

As usual, let U be an open subset of $\mathbb{R}^2$ and $M : U \to \mathbb{R}^3$ be a RPS.

Definition 91 A **vector field** on a RPS M is a smooth map $V : M \to \mathbb{R}^3$ (where smooth means that each of its component functions $V_1, V_2, V_3 : M \to \mathbb{R}$ is smooth). If $V(y) \in T_yM$ for all $y \in M$ then V is a **tangent vector field**. If $V(y) \in N_yM$ for all $y \in M$ then V is a **normal vector field**.

Remark 92 Just as for functions, we define the **coordinate expression** of a vector field $V : M \to \mathbb{R}^3$ to be

$$\widehat{V} : U \to \mathbb{R}^3, \qquad \widehat{V}(x_1, x_2) = V(M(x_1, x_2)).$$

It follows from Theorem 78 that the coordinate expression of any **tangent** vector field V takes the form

$$\widehat{V}(x_1, x_2) = f(x_1, x_2)\varepsilon_1(x_1, x_2) + g(x_1, x_2)\varepsilon_2(x_1, x_2)$$

where f, g are smooth real-valued functions on U. Similarly, it follows from Remark 80 that any **normal** vector field $V : M \to \mathbb{R}^3$ takes the form

$$\widehat{V}(x_1, x_2) = f(x_1, x_2)\varepsilon_1(x_1, x_2) \times \varepsilon_2(x_1, x_2)$$

where f is a smooth real-valued function on U. It is standard in Differential Geometry to elide the difference between a vector field and its coordinate expression, so we will often refer to $\varepsilon_1(x), \varepsilon_2(x)$ as tangent vector fields and $\varepsilon_1(x) \times \varepsilon_2(x)$ as a normal vector field.

Given a tangent vector $v \in T_yM$ and a function $f : M \to \mathbb{R}$, we can find the directional derivative $v[f]$, a **single real number**. If instead of a single tangent vector we are given a **tangent vector field** $V : M \to \mathbb{R}^3$, and a function $f : M \to \mathbb{R}$, we can define a new **function** $V[f] : M \to \mathbb{R}$,

$$V[f](y) = V(y)[f]$$

which assigns to each point $y \in M$ the directional derivative of f with respect to $V(y) \in T_yM$. Note this only makes sense for **tangent** vector fields.

Example 93 On the unit sphere, $M : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$,

$$M(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

$$V(y_1, y_2, y_3) = (2y_2, -2y_1, 0)$$

is a tangent vector field. To check this, construct its coordinate expression:

$$\widehat{V}(\theta, \phi) = V(M(\theta, \phi)) \quad =$$

$$=$$

where we have used the fact that

$\varepsilon_\theta =$ $\qquad\qquad\qquad\qquad$ $\varepsilon_\phi =$

Let $f : M \to \mathbb{R}$ such that $f(y_1, y_2, y_3) = y_1 + y_2^2$. What is $V[f]$?
Coordinate expression of f:

$$\widehat{f}(\theta, \phi) =$$

Hence

$$\begin{aligned} V[f] &= \\ \\ &= \\ \\ &= \end{aligned}$$

Note that, strictly speaking, this is not $V[f]$ but rather its coordinate expression. To obtain $V[f]$ we must change back from the local coordinates (θ, ϕ) to (y_1, y_2, y_3):

$V[f](y_1, y_2, y_3) =$ $\square$

We may also define directional derivatives **of** vector fields:

Definition 94 Let $V : M \to \mathbb{R}^n$ be a vector field on M and $u \in T_yM$. Then the **directional derivative** of V with respect to u is

$$\nabla_u V = (V \circ \alpha)'(0)$$

where $\alpha : I \to M$ is a curve through $y \in M$ with $\alpha'(0) = u$. $\square$

Notes:

- This assigns to a single **tangent** vector u and a vector **field** V a single vector $\nabla_u V \in \mathbb{R}^3$.
- V can be any vector field (not necessarily tangent). u must be a tangent vector.
- Even if V is a tangent vector field, there is no reason to expect $\nabla_u V$ to be a tangent vector.
- $V(y) = (V_1(y), V_2(y), V_3(y))$ where the component functions $V_i : M \to \mathbb{R}$, $i = 1, 2, 3$ are smooth functions on M. Hence

  $$\nabla_u V = ((V_1 \circ \alpha)'(0), (V_2 \circ \alpha)'(0), (V_3 \circ \alpha)'(0)) = (u[V_1], u[V_2], u[V_3]). \qquad (\diamondsuit)$$

  It follows immediately from Lemma 87 that $\nabla_u V$ is well defined (independent of the choice of the generating curve α).

Many other convenient properties of $\nabla_u V$ follow immediately from equation $(\diamondsuit)$ and our previous work on $u[f]$.

Lemma 95 *Let V, W be vector fields on M, $u, v \in T_y M$, f be a smooth function on M and $a, b \in \mathbb{R}$. Then*

(a) $\nabla_{au+bv} W = a\nabla_u W + b\nabla_v W$.

(b) $\nabla_u (aV + bW) = a\nabla_u V + b\nabla_u W$.

(c) $\nabla_u (fW) = u[f]W(y) + f(y)\nabla_u W$.

(d) $u[V \cdot W] = (\nabla_u V) \cdot W(y) + V(y) \cdot (\nabla_u W)$.

Proof: (a),(b) follow from Lemma 88 and ($\diamondsuit$). (c) follows from Corollary 90 and ($\diamondsuit$). To see part (d), note that

$$\begin{aligned}
u[V \cdot W] &= u[V_1W_1 + V_2W_2 + V_3W_3] \\
&= u[V_1W_1] + u[V_2W_2] + u[V_3W_3] \qquad \text{(Lemma 88)} \\
&= u[V_1]W_1(y) + V_1(y)u[W_1] + u[V_2]W_2(y) + V_2(y)u[W_2] + u[V_3]W_3(y) + V_3(y)u[W_3] \\
&\qquad \text{(Corollary 90)} \\
&= (u[V_1], u[V_2], u[V_3]) \cdot W(y) + V(y) \cdot (u[W_1], u[W_2], u[W_3]) \\
&= (\nabla_u V) \cdot W(y) + V(y) \cdot (\nabla_u W). \qquad \square
\end{aligned}$$

As with directional derivatives of functions, we can use Lemma 95 to reduce calculation of $\nabla_u V$ to partial differentiation with respect to (x_1, x_2). We just need to express u in terms of the coordinate basis and find the coordinate expression for V.

Example 96 Let V be the vector field on the unit sphere defined in Example 93. Let $u = (-1, 0, 1) \in T_{(0,1,0)}M$. Calculate $\nabla_u V$.
Task 0: Find the local coordinates of the basepoint $y = (0, 1, 0) \in M$.
In this case $M(\pi/2, \pi/2) = (0, 1, 0)$, so $(\theta, \phi) = (\pi/2, \pi/2)$.
Task 1: Write u in terms of the coordinate basis vectors.
At $y = (0, 1, 0)$ the coordinate basis is

$\varepsilon_\theta =$ $\qquad\qquad\qquad$ ε_ϕ

Hence

$u =$

Task 2: Find the coordinate expression for V.
We've already done this; it's

$\widehat{V}(\theta, \phi) =$

Task 3: Calculate the partial derivatives.

$\nabla_u V \;=$

$=$

$=$ $\square$

Given a tangent vector **field** $U : M \to \mathbb{R}^3$ and another vector field $V : M \to \mathbb{R}^3$ (tangent, normal or neither), we can define a 3rd vector **field** $W : M \to \mathbb{R}^3$ as follows: at each $y \in M$,

$$W(y) = \nabla_{U(y)} V.$$

We shall denote this vector field $W = \nabla_U V$. In fact, it will be almost exclusively in this context that we will make use of the directional derivative ∇.

Example 97 For M as in Example 93, let N be the unit normal vector field

$$N = \frac{\varepsilon_\theta \times \varepsilon_\phi}{|\varepsilon_\theta \times \varepsilon_\phi|}.$$

Calculate the vector fields

$$\nabla_{\varepsilon_\theta}\varepsilon_\theta,\ \nabla_{\varepsilon_\theta}\varepsilon_\phi,\ \nabla_{\varepsilon_\phi}\varepsilon_\theta,\ \nabla_{\varepsilon_\phi}\varepsilon_\phi,\ \nabla_{\varepsilon_\theta}N,\ \nabla_{\varepsilon_\phi}N.$$

In terms of coordinate expressions,

$$\varepsilon_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \quad \varepsilon_\phi = (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0),$$

$$\varepsilon_\theta \times \varepsilon_\phi \ =$$

$$N \ =$$

Hence:

$$\nabla_{\varepsilon_\theta}\varepsilon_\theta = \frac{\partial \varepsilon_\theta}{\partial \theta} =$$

$$\nabla_{\varepsilon_\theta}\varepsilon_\phi = \frac{\partial \varepsilon_\phi}{\partial \theta} =$$

$$\nabla_{\varepsilon_\phi}\varepsilon_\theta = \frac{\partial \varepsilon_\theta}{\partial \phi} =$$

$$\nabla_{\varepsilon_\phi}\varepsilon_\phi = \frac{\partial \varepsilon_\phi}{\partial \phi} =$$

$$\nabla_{\varepsilon_\theta}N = \frac{\partial N}{\partial \theta} =$$

$$\nabla_{\varepsilon_\phi}N = \frac{\partial N}{\partial \phi} =$$

$\square$

These results have a couple of interesting features. First, we see that $\nabla_{\varepsilon_\theta}\varepsilon_\phi = \nabla_{\varepsilon_\phi}\varepsilon_\theta$. Second, we see that both $\nabla_{\varepsilon_\theta}N$ and $\nabla_{\varepsilon_\phi}N$ are *tangent* vector fields. These observations are not special to the unit sphere, and will, in fact, have important consequences. The first is very easy to prove:

Lemma 98 *For all i, j, $\nabla_{\varepsilon_i}\varepsilon_j = \nabla_{\varepsilon_j}\varepsilon_i$.*

Proof: The equation holds trivially if $i = j$. Hence we need only check the case $i = 1$, $j = 2$. Then

$$\nabla_{\varepsilon_1}\varepsilon_2 = \frac{\partial \varepsilon_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial M}{\partial x_2}\right) = \frac{\partial^2 M}{\partial x_1 \partial x_2}.$$

But for any smooth function $f : U \to \mathbb{R}$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}.$$

Hence

$$\nabla_{\varepsilon_1}\varepsilon_2 = \frac{\partial^2 M}{\partial x_2 \partial x_1} = \frac{\partial \varepsilon_1}{\partial x_2} = \nabla_{\varepsilon_2}\varepsilon_1$$

which was to be proved. □

Summary

- A function $f : M \to \mathbb{R}$ is **smooth** if its coordinate expression $\widehat{f} = f \circ M : U \to \mathbb{R}$ is smooth.
- Given a smooth function $f : M \to \mathbb{R}$ and a tangent vector $v \in T_y M$, the **directional derivative** of f with respect to (or along) v is

  $$v[f] = (f \circ \alpha)'(0)$$

  where $\alpha(t)$ is any generating curve for v.
- The directional derivative is linear, that is

  $$(au + bv)[f] = a(u[f]) + b(v[f]), \qquad u[af + bg] = a(u[f]) + b(u[g]).$$

- Directional derivatives along coordinate basis vectors reduce to partial derivatives

  $$\varepsilon_1[f] = \frac{\partial \widehat{f}}{\partial x_1}, \qquad \varepsilon_2[f] = \frac{\partial \widehat{f}}{\partial x_2}$$

- Vector **fields** are smooth maps $V : M \to \mathbb{R}^3$.
- We can extend the definition of directional derivative to vector fields. The directional derivative of a vector field V along a tangent vector field U is denoted $\nabla_U V$.

7 Curvature of an oriented surface

7.1 The shape operator on an oriented surface

Definition 99 An **orientation** on a RPS $M : U \to \mathbb{R}^3$ is a choice of unit normal vector field N, i.e. a smooth assigment of a unit vector in $N_y M$ to each $y \in M$. Only two orientations are possible (if M is connected). We can choose N to be

$$N = \frac{\varepsilon_1 \times \varepsilon_2}{|\varepsilon_1 \times \varepsilon_2|}$$

or we can choose it to be

$$\tilde{N} = -\frac{\varepsilon_1 \times \varepsilon_2}{|\varepsilon_1 \times \varepsilon_2|} = \frac{\varepsilon_2 \times \varepsilon_1}{|\varepsilon_2 \times \varepsilon_1|}.$$

We call the first of these the **canonical orientation** on M. Unless otherwise stated, we will always use this orientation rather than $\tilde{N}$. Note that this convention depends on the ordering of the coordinates x_1, x_2. □

The key point about oriented surfaces is that the unit normal at y, $N(y)$, determines the tangent space at y:

$$T_y M = \{v \in \mathbb{R}^3 : v \cdot N(y) = 0\},$$

by Lemma 81. So we can glean information about how the tangent space varies with y by computing directional derivatives $\nabla_u N$ where $u \in T_y M$. Just as with the curvature of a regularly parametrized curve, it's essential that N is a *unit* normal vector field – if it weren't then $\nabla_u N$ would contain information about the rate of change of the *length* of N in the direction u, as well as the rate of change of the direction of N.

Lemma 100 *Let $M : U \to \mathbb{R}^3$ be a surface oriented by N, and $u \in T_y M$. Then $\nabla_u N \in T_y M$ also.*

Proof: It suffices to show that $N(y) \cdot (\nabla_u N) = 0$. Now N is a unit vector field so $N \cdot N = 1$. Taking the directional derivative of this (constant) function with respect to u gives

$$0 = u[N \cdot N] = (\nabla_u N) \cdot N(y) + N(y) \cdot (\nabla_u N) = 2N(y) \cdot (\nabla_u N),$$

where we have used Lemma 95 part (d). The result immediately follows. □

Reminder 101 A map $L : V \to V$, where V is a vector space, is said to be **linear** if for all $a, b \in \mathbb{R}$ and $u, v \in V$,

$$L(au + bv) = aL(u) + bL(v).$$

In this case we usually write Lu instead of $L(u)$. □

Definition 102 Let $M : U \to \mathbb{R}^3$ be a surface oriented by N. The **shape operator** at $y \in M$ is the map

$$S_y : T_y M \to T_y M, \qquad S_y : u \mapsto -\nabla_u N.$$

S_y really does map $T_y M$ to itself, by Lemma 100, and is a linear map by Lemma 95. It is often called the **Weingarten map** in honour of its discoverer. □

Example 103 (unit sphere) Recall that for the unit sphere in $\mathbb{R}^3$ with its "polar coordinate" parametrization, one has the directional derivatives

$$\nabla_{\varepsilon_\theta} N = \varepsilon_\theta, \qquad \nabla_{\varepsilon_\phi} N = \varepsilon_\phi,$$

(see Example 97). Any tangent vector $u = a\varepsilon_\theta + b\varepsilon_\phi$. Hence

$$\begin{aligned} S_y u &= S_y(a\varepsilon_\theta + b\varepsilon_\phi) = aS_y\varepsilon_\theta + bS_y\varepsilon_\phi \\ &= -a\nabla_{\varepsilon_\theta} N - b\nabla_{\varepsilon_\phi} N = -a\varepsilon_\theta - b\varepsilon_\phi = -u. \end{aligned}$$

So the shape operator on the unit sphere is $S_y = -\mathrm{Id}$, that is, *minus* the identity map on $T_y M$. □

Example 104 (a plane) Let $M : \mathbb{R}^2 \to \mathbb{R}^3$ such that $M(x_1, x_2) = (x_1, x_2, 0)$. This is a surface. We may orient it by $N(x) = e_3 = (0, 0, 1)$. This is a *constant* vector field, so for all $u \in T_y M$, $\nabla_u N = 0$. Hence the shape operator on a plane is $S_y = 0$ (the trivial linear map which maps every $u \in T_y M$ to the zero vector). □

Example 105 (a cylinder) Let $M : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ such that $M(x_1, x_2) = (x_2, c\cos x_1, c\sin x_1)$. This is a cylinder of radius $c > 0$, symmetric about the y_1-axis.

$$\varepsilon_1 = \qquad\qquad \varepsilon_2 =$$

$$\varepsilon_1 \times \varepsilon_2 =$$

$$N =$$

$$\begin{aligned} S_y\varepsilon_1 &= -\nabla_{\varepsilon_1} N = -\frac{\partial N}{\partial x_1} = \\ S_y\varepsilon_2 &= -\nabla_{\varepsilon_2} N = -\frac{\partial N}{\partial x_2} = \end{aligned}$$

From this, we can construct a 2×2 matrix representing the linear map S_y relative to the basis $\varepsilon_1, \varepsilon_2$ for $T_y M$: we identify any vector $v = a\varepsilon_1 + b\varepsilon_2$ with the column 2-vector $\begin{bmatrix} a \\ b \end{bmatrix}$. Then

$S_y(a\varepsilon_1 + b\varepsilon_2) = aS_y\varepsilon_1 + bS_y\varepsilon_2 =$

so the matrix representing this linear map with respect to the basis $\{\varepsilon_1, \varepsilon_2\}$ is

$$\widehat{S}_y = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

More briefly, $\widehat{S}_y$ is the matrix whose columns are $S_y\varepsilon_1$ and $S_y\varepsilon_2$, thought of as 2-vectors relative to $\{\varepsilon_1, \varepsilon_2\}$. $\square$

Our measure(s) of the curvature of a surface will be defined in terms of the *eigenvalues* of the linear map $S_y : T_yM \to T_yM$. It turns out to be crucial that S_y has a property called *self-adjointness*.

Definition 106 A linear map $L : T_yM \to T_yM$ is **self adjoint** if for all $u, v \in T_yM$, $u \cdot Lv = v \cdot Lu$. $\square$

Theorem 107 *Let M be an oriented surface and $S_y : T_yM \to T_yM$ be its shape operator at the point y. Then S_y is self adjoint.*

Proof: Since $\{\varepsilon_1, \varepsilon_2\}$ spans T_yM, it suffices to show that for all i, j,

$$\varepsilon_i \cdot S_y\varepsilon_j = \varepsilon_j \cdot S_y\varepsilon_i.$$

To see this, note that

$$\begin{aligned}
\varepsilon_i \cdot S_y\varepsilon_j - \varepsilon_j \cdot S_y\varepsilon_i &= -\varepsilon_i \cdot \nabla_{\varepsilon_j} N + \varepsilon_j \cdot \nabla_{\varepsilon_i} N \\
&= -\varepsilon_j[\varepsilon_i \cdot N] + (\nabla_{\varepsilon_j}\varepsilon_i) \cdot N + \varepsilon_i[\varepsilon_j \cdot N] - (\nabla_{\varepsilon_i}\varepsilon_j) \quad \text{(Lemma 95(d))} \\
&= -\varepsilon_j[0] + \varepsilon_i[0] + (\nabla_{\varepsilon_j}\varepsilon_i - \nabla_{\varepsilon_i}\varepsilon_j) = 0 \quad \text{(Lemma 98)} \quad \square
\end{aligned}$$

Example 108 (a saddle surface) Let $M : \mathbb{R}^2 \to \mathbb{R}^3$ such that $M(x_1, x_2) = (x_1, x_2, x_1x_2)$. This is a RPS, as we have shown previously. Let's construct its shape operator $S_y : T_yM \to T_yM$ at the point $y = M(1, 0) = (1, 0, 0)$.

$$\varepsilon_1 = \qquad\qquad\qquad \varepsilon_2 =$$

$$\varepsilon_1 \times \varepsilon_2 =$$

$$N =$$

$$\begin{aligned}
S_y\varepsilon_1 &= -\nabla_{\varepsilon_1} N = -\left.\frac{\partial N}{\partial x_1}\right|_{(1,0)} = -\left.\frac{d}{dt}\right|_{t=1} N(t, 0) = -\left.\frac{d}{dt}\right|_{t=1} \frac{(0, -t, 1)}{\sqrt{1+t^2}} \\
&= \frac{1}{2^{\frac{3}{2}}}(0, 1, 1) = \frac{1}{2^{\frac{3}{2}}}\varepsilon_2. \\
S_y\varepsilon_2 &= -\nabla_{\varepsilon_2} N = -\left.\frac{\partial N}{\partial x_2}\right|_{(1,0)} = -\left.\frac{d}{dt}\right|_{t=0} N(1, t) = -\left.\frac{d}{dt}\right|_{t=0} \frac{(-t, 1, 1)}{\sqrt{2+t^2}} \\
&= -\frac{1}{\sqrt{2}}(-1, 0, 0) = \frac{1}{\sqrt{2}}\varepsilon_1 \\
\widehat{S}_y &= \begin{bmatrix} \uparrow & \uparrow \\ S_y\varepsilon_1 & S_y\varepsilon_2 \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad \end{bmatrix}.
\end{aligned}$$

Let's check that S_y is self adjoint: at $y = M(1,0)$,

$|\varepsilon_1|^2 =$, $|\varepsilon_2|^2 =$.

Hence

$$\varepsilon_1 \cdot S_y \varepsilon_2 =$$

$$\varepsilon_2 \cdot S_y \varepsilon_1 =$$

$\square$

7.2 The principal curvatures of an oriented surface

Let V be a vector space (e.g. $V = T_yM$) and $L : V \to V$ be a linear map (e.g. $S_y : T_yM \to T_yM$). An **eigenvalue** of L is a number λ such that

$$Lu = \lambda u \qquad (\diamondsuit)$$

for some $u \in V$, $u \neq 0$. The vector u is called an **eigenvector** corresponding to the eigenvalue λ. Clearly, given one such u any multiple au ($a \neq 0$) is also an eigenvector, so we are free to choose our eigenvectors to have *unit length*. Let $\mathrm{Id} : V \to V$ be the identity map on V ($\mathrm{Id}\, u = u$). Then we may re-arrange ($\diamondsuit$) to read

$$(L - \lambda\,\mathrm{Id})u = 0, \qquad u \neq 0,$$

so λ is an eigenvalue if and only if the linear map $L - \lambda\,\mathrm{Id}$ *fails to be one-to-one* (both $u \neq 0$ and 0 get mapped to 0 by $L - \lambda\,\mathrm{Id}$), that is, if and only if $L - \lambda\,\mathrm{Id}$ is *not invertible*. Choose a basis $e_1, \ldots, e_m$ for V and let $\widehat{L}$ be the $m \times m$ matrix representing L. Now Id is represented by the $m \times m$ identity matrix $\mathbb{I}_m$, so the linear map $L - \lambda\mathrm{Id}$ is represented by the square matrix $\widehat{L} - \lambda\mathbb{I}_m$. Hence, it fails to be invertible if and only if

$$\det(\widehat{L} - \lambda\mathbb{I}_m) = 0. \qquad (\clubsuit)$$

Equation ($\clubsuit$) is a degree m polynomial equation in λ, called the **characteristic equation** of the linear map L. Although it *looks* like it depends on the matrix $\widehat{L}$ chosen to represent L (i.e. the choice of *basis* for V), in fact it doesn't. The equation has exactly m solutions, counted with multiplicity. However, even though the coefficients of the polynomial are all real, these solutions may, in general, be **complex**. In this case, the solution λ is still called an eigenvalue of L, but its interpretation is rather subtle.

Example 109 On $\mathbb{R}^2$ consider the linear map $L : (a,b) \mapsto (-b,a)$. Relative to the standard basis $e_1 = (1,0)$, $e_2 = (0,1)$ this has matrix representation

$$\widehat{L} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so the characteristic equation is

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

So L has only complex eigenvalues, namely $\pm i$. $\square$

Luckily, this can never happen for the shape operator $S_y : T_yM \to T_yM$ because it is self-adjoint.

Theorem 110 *Let $L : T_yM \to T_yM$ be a self-adjoint linear map. Then*

(A) *its eigenvalues are all real, and*

(B) *its eigenvectors form an orthonormal basis for T_yM.*

Proof: (A) Let e_1, e_2 be an orthonormal basis for T_yM and

$$\widehat{L} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

be the matrix representing L relative to this basis. Then, by definition,

$$e_1 \cdot (Le_2) = e_1 \cdot (L_{12}e_1 + L_{22}e_2) = L_{12},$$

and

$$e_2 \cdot (Le_1) = e_2 \cdot (L_{11}e_1 + L_{21}e_2) = L_{21}.$$

But L is self adjoint, so $L_{12} = L_{21}$ (that is, the matrix $\widehat{L}$ is *symmetric*). Hence, the characteristic equation of L is

$$0 = \det(\widehat{L} - \lambda \mathbb{I}_2) = \begin{vmatrix} L_{11} - \lambda & L_{12} \\ L_{12} & L_{22} - \lambda \end{vmatrix} = \lambda^2 - (L_{11} + L_{22})\lambda + L_{11}L_{22} - L_{12}^2. \qquad (\spadesuit)$$

The discriminant of this quadratic polynomial is

$$\text{“}b^2 - 4ac\text{”} = (L_{11} + L_{22})^2 - 4L_{11}L_{22} + 4L_{12}^2 = (L_{11} - L_{22})^2 + 4L_{12}^2 \geq 0.$$

Hence ($\spadesuit$) has two real solutions.
(B) Let these real eigenvalues be λ_1, λ_2 and denote their corresponding eigenvectors u_1, u_2. If $\lambda_1 = \lambda_2$, equation ($\spadesuit$) has a repeated root, hence its discriminant vanishes, so $L_{11} = L_{22} = \lambda$ and $L_{12} = 0$. But then $\widehat{L} = \lambda \mathbb{I}_2 \quad \Rightarrow \quad L = \lambda\,\mathrm{Id}$, so *every* vector $u \in T_yM$ is an eigenvector corresponding to eigenvalue λ. So we may choose $u_1 = e_1$ and $u_2 = e_2$, which are orthonormal.

If $\lambda_1 \neq \lambda_2$ then

$$0 = u_1 \cdot Lu_2 - u_2 \cdot Lu_1 = (\lambda_2 - \lambda_1)u_1 \cdot u_2$$

so $u_1 \cdot u_2 = 0$. $\square$

This allows us to make the following definition:

Definition 111 Let $M : U \to \mathbb{R}^3$ be an oriented surface, $S_y : T_yM \to T_yM$ be its shape operator. Then the **principal curvatures** of M at y are κ_1, κ_2, the eigenvalues of S_y. The **principal curvature directions** of M at y are the corresponding (normalized) eigenvectors u_1, u_2. By Theorem 110, κ_1, κ_2 are real and the eigenvectors u_1, u_2 form an orthonormal basis for T_yM. □

Example 112 (saddle surface) Recall (Example 108) that the saddle surface

$$M(x_1, x_2) = (x_1, x_2, x_1x_2)$$

at $y = M(1, 0) = (1, 0, 0)$ has shape operator

$$\widehat{S}_y = \begin{bmatrix} 0 & 2^{-\frac{1}{2}} \\ 2^{-\frac{3}{2}} & 0 \end{bmatrix}$$

relative to the coordinate basis $\varepsilon_1 = (1, 0, 0), \varepsilon_2 = (0, 1, 1)$ for $T_{(1,0,0)}M$. Note this matrix is not symmetric, because the coordinate basis is not orthonormal. Nonetheless, its eigenvalues must still be real, and its eigenvectors orthonormal. Let's check. The characteristic equation is

$$0 = \begin{vmatrix} -\lambda & 2^{-\frac{1}{2}} \\ 2^{-\frac{3}{2}} & -\lambda \end{vmatrix} =$$

so the principal curvatures are:

$$\kappa_1 = \qquad\qquad , \kappa_2 = \qquad\qquad .$$

To find u_1, solve the linear system $\widehat{S}_y u_1 = \kappa_1 u_1$:

We won't worry about the length of u_1 yet. Similarly for u_2:

Are these vectors orthogonal? They certainly don't *look* orthogonal. But you must remember that these 2×1 column matrices represent u_1 and u_2 with respect to the basis $\varepsilon_1, \varepsilon_2$. The vectors u_1, u_2 themselves lie in $T_yM \subset \mathbb{R}^3$, that is, they are 3-dimensional vectors. In fact,

$$u_1 =$$

$$u_2 =$$

whence we see that $u_1 \cdot u_2 = 0$ as required. Note that when we normalize them to have unit length, we must again think of them as vectors in $\mathbb{R}^3$, not 2×1 column matrices:

$\square$

Example 113 (unit sphere) Recall (Example 103) that the unit sphere

$$M(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

has a very simple shape operator, namely $S_y = -\mathrm{Id}$. It follows that every vector $u \in T_yM$ is an eigenvector with eigenvalue -1. Since $\varepsilon_\theta, \varepsilon_\phi$ are orthogonal, and $|\varepsilon_\theta| = 1$, we may choose as our principal curvature directions

$$u_1 = \varepsilon_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \qquad u_2 = \frac{\varepsilon_\phi}{|\varepsilon_\phi|} = (-\sin\phi, \cos\phi, 0).$$

Note that $u_1 \cdot u_2 = 0$ as it should. $\square$

Example 114 (a cylinder) Recall (Example 105) that the cylinder of radius c, $M(x_1, x_2) = (x_2, c\cos x_1, c\sin x_1)$, has shape operator

$$\widehat{S}_y = \begin{bmatrix} -\frac{1}{c} & 0 \\ 0 & 0 \end{bmatrix}$$

relative to the coordinate basis $\varepsilon_1 = (0, -c\sin x_1, c\cos x_1), \varepsilon_2 = (1, 0, 0)$. So its characteristic equation is

$$\begin{vmatrix} -\frac{1}{c} - \lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \lambda(\lambda + \frac{1}{c}) = 0.$$

It follows that the principal curvatures are $\kappa_1 = -1/c$ and $\kappa_2 = 0$. What are the corresponding principal curvature directions?

$\widehat{S}_y u_1 = -u_1/c \quad \Rightarrow$ $\qquad\qquad$ $\widehat{S}_y u_2 = 0 \quad \Rightarrow$

Hence, after normalizing, $u_1 = \varepsilon_1/c$ and $u_2 = \varepsilon_2$. Orthogonal? Yes.

Note that if we make the cylinder smaller (reduce c) then $|\kappa_1|$ gets larger. This makes intuitive sense: a tightly rolled cylinder should be more highly curved than a loosely rolled one. $\square$

7.3 Normal curvature

Definition 115 Let $M : U \to \mathbb{R}^3$ be a regularly parametrized surface. The **unit tangent space** at $y \in M$ is

$$U_yM = \{v \in T_yM : |v| = 1\},$$

the set of unit vectors in T_yM. Geometrically, we can think of U_yM as the unit circle in T_yM. □

Note that U_yM is *not* a vector space.

Definition 116 Let M be an oriented surface. The **normal curvature function** of M at $y \in M$ is

$$k_y : U_yM \to \mathbb{R}, \qquad k_y(u) = u \cdot S_y u$$

where S_y denotes the shape operator as usual. □

Why call this "normal curvature"?

Lemma 117 *Let M be a surface oriented by N and $u \in U_yM$. Let $\alpha : I \to M$ be any unit speed curve in M through y generating u. Then the component of the curvature vector of α in the direction of $N(y)$ is $k_y(u)$.*

Proof: We have a curve in M with $\alpha(0) = y$, $\alpha'(0) = u$. Since α is a unit speed curve, its curvature vector at y is $k(0) = \alpha''(0)$. Now α stays in M, so its velocity is always tangent to M, and hence

$$\alpha'(t) \cdot N(\alpha(t)) = 0 \qquad (\clubsuit)$$

for all $t \in I$. Differentiate ($\clubsuit$) with respect to t and set $t = 0$:

$$\begin{aligned} \alpha''(0) \cdot N(\alpha(0)) + \alpha'(0) \cdot (N \circ \alpha)'(0) &= 0 \\ \Rightarrow \quad k(0) \cdot N(y) + u \cdot (N \circ \alpha)'(0) &= 0. \end{aligned}$$

But $(N \circ \alpha)'(0) = \nabla_u N$ by the definition of directional derivatives (Definition 94). Hence

$$k(0) \cdot N(y) = -u \cdot \nabla_u N = u \cdot S_y u = k_y(u)$$

as was to be proved. □

Example 118 (hyperboloid of one sheet) The hyperboloid

$$H = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 - y_3^2 = 1\}$$

may be parametrized by

$$\begin{aligned} &M : (-\pi, \pi) \times \mathbb{R} \to \mathbb{R}^3 \\ &M(\theta, z) = (\sqrt{1 + z^2} \cos\theta, \sqrt{1 + z^2} \sin\theta, z). \end{aligned}$$

The coordinate basis is

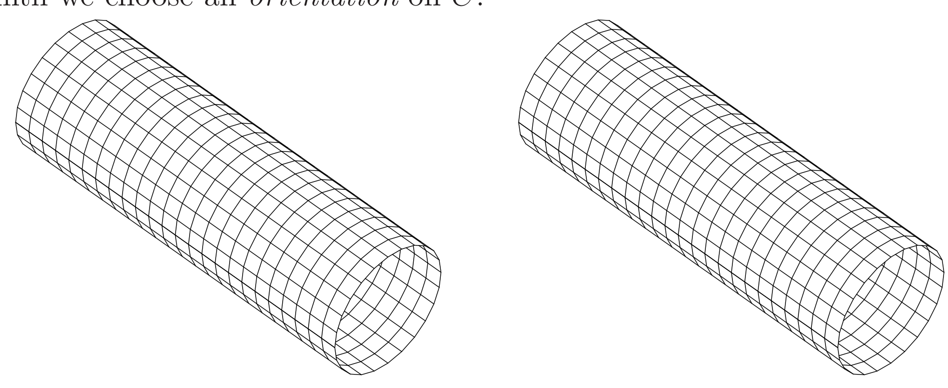
$$\left.\begin{array}{l}\varepsilon_\theta = (-\sqrt{1+z^2}\sin\theta, \sqrt{1+z^2}\cos\theta, 0) \\ \varepsilon_z = (\frac{z}{\sqrt{1+z^2}}\cos\theta, \frac{z}{\sqrt{1+z^2}}\sin\theta, 1)\end{array}\right\} \quad \Rightarrow \quad N(\theta, z) = \frac{(\sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, -z)}{\sqrt{1+2z^2}}.$$

At $y = (1,0,0) = M(0,0)$, $\varepsilon_\theta = (0,1,0)$ and $\varepsilon_z = (0,0,1)$ happen to be in U_yM (they're unit vectors). What are $k_y(\varepsilon_\theta)$ and $k_y(\varepsilon_z)$?

$$\begin{aligned} S_y\varepsilon_\theta &= -\nabla_{\varepsilon_\theta} N = -\left.\frac{\partial N}{\partial \theta}\right|_{(0,0)} = -(0,1,0) = -\varepsilon_\theta \\ \Rightarrow \quad k_y(\varepsilon_\theta) &= \varepsilon_\theta \cdot (-\varepsilon_\theta) = -1 \\ S_y\varepsilon_z &= -\nabla_{\varepsilon_z} N = -\left.\frac{\partial N}{\partial z}\right|_{(0,0)} = -(0,0,-1) = \varepsilon_z \\ \Rightarrow \quad k_y(\varepsilon_z) &= \varepsilon_z \cdot \varepsilon_z = 1. \end{aligned}$$

The *sign* of the normal curvature $k_y(u)$ tells us whether the surface is curving towards its unit normal ($k_y(u) > 0$) or away from its unit normal ($k_y(u) < 0$) as we move in the direction u. Note that this depends on the choice of orientation N. $\square$

Example 119 (cylinder) Consider the cylinder $C = \{y \in \mathbb{R}^3 : y_1^2 + y_3^2 = 1\}$. $y = (1,0,0) \in C$ and $u_1 = (0,1,0)$, $u_2 = (0,0,1)$ are both unit tangent vectors at y. Can we figure out the sign of $k_y(u_1)$ and $k_y(u_2)$ without any calculations at all? **No!** Not until we choose an *orientation* on C:



N = outward pointing unit normal $\quad$ N = inward pointing unit normal

. $\square$

Normal curvature allows us to give a new interpretation of the principal curvatures of a surface.

Theorem 120 *Let $M : U \to \mathbb{R}^3$ be an oriented surface and κ_1, κ_2 be its principal curvatures at $y \in M$, ordered so that $\kappa_1 \leq \kappa_2$. Then*

$$\begin{aligned}\kappa_1 &= \min\{k_y(u) : u \in U_yM\} \\ \kappa_2 &= \max\{k_y(u) : u \in U_yM\}.\end{aligned}$$

Proof: Let u_1, u_2 be the orthonormal pair of eigenvectors corresponding to κ_1, κ_2 (see Theorem 110). These span T_yM so any unit vector $u \in U_yM$ may be written

$$u = \cos\theta\, u_1 + \sin\theta\, u_2$$

for some angle θ. The normal curvature of u is

$$\begin{aligned}k_y(u) &= (\cos\theta\, u_1 + \sin\theta\, u_2) \cdot S_y(\cos\theta\, u_1 + \sin\theta\, u_2) \\ &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \qquad\qquad (\spadesuit) \\ &= \kappa_2 - (\kappa_2 - \kappa_1)\alpha,\end{aligned}$$

where $\alpha = \cos^2\theta \in [0, 1]$. Clearly, this function attains its maximum at $\alpha = 0$ and its minimum at $\alpha = 1$ (recall $\kappa_2 - \kappa_1 \geq 0$), and these values are κ_2 and κ_1 respectively. □

Example 121 (hyperboloid of one sheet) For M and y as in Example 118, construct unit vectors with the following properties:

$$\text{(a)}\quad k_y(u) = 0, \quad \text{(b)}\quad k_y(u) = 2, \quad \text{(c)}\quad S_yu = 0.$$

(a) Recall that $S_y\varepsilon_\theta = -\varepsilon_\theta$ and $S_y\varepsilon_z = \varepsilon_z$, so the principal curvatures are $\kappa_1 = -1$, $\kappa_2 = 1$, and the principal curvature directions are $u_1 = \varepsilon_\theta$, $u_2 = \varepsilon_z$ (N.B.: these already have unit length). We seek a unit vector $u = \cos\theta\, u_1 + \sin\theta\, u_2$ such that $k_y(u) = 0$. But then by ($\spadesuit$),

$$\begin{aligned}0 = k_y(u) &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta = -\cos^2\theta + \sin^2\theta \\ \Rightarrow \quad \cos^2\theta &= \sin^2\theta\end{aligned}$$

which has 4 solutions, $\cos\theta = \pm\frac{1}{\sqrt{2}}$, $\sin\theta = \pm\cos\theta$. So any one of the 4 unit vectors

$$u = \frac{1}{\sqrt{2}}(\varepsilon_\theta + \varepsilon_z), \quad u = \frac{1}{\sqrt{2}}(\varepsilon_\theta - \varepsilon_z), \quad u = \frac{1}{\sqrt{2}}(-\varepsilon_\theta + \varepsilon_z), \quad u = \frac{1}{\sqrt{2}}(-\varepsilon_\theta - \varepsilon_z)$$

solves the problem. Substituting in the explicit basis vectors ε_θ, ε_z:

$$u = \frac{1}{\sqrt{2}}(0, 1, 1), \quad u = \frac{1}{\sqrt{2}}(0, 1, -1), \quad u = \frac{1}{\sqrt{2}}(0, -1, 1), \quad u = \frac{1}{\sqrt{2}}(0, -1, -1).$$

(b) No such u can exist by Theorem 120. For all $u \in U_yM$, $-1 \leq k_y(u) \leq 1$.

(c) Again, no such u can exist. If it did, then since $u \neq 0$, it follows that 0 is an eigenvalue of S_y, and hence a principle curvature. But the principal curvatures are just ± 1, not 0. □

7.4 Mean and Gauss curvatures

Definition 122 The **mean curvature** of M at $y \in M$ is

$$H(y) = \frac{1}{2}(\kappa_1 + \kappa_2)$$

where κ_1, κ_2 are the principal curvatures of M at y. The **Gauss curvature** of M at y is

$$\sigma(y) = \kappa_1\kappa_2 \qquad \square$$

One need not solve the eigenvalue problem for S_y to compute $H(y)$ and $\sigma(y)$:

Proposition 123 *For all $y \in M$,*

$$H(y) = \frac{1}{2}\mathrm{tr}\,\widehat{S}_y, \quad \sigma(y) = \det \widehat{S}_y,$$

where $\widehat{S}_y$ is the matrix representing $S_y : T_yM \to T_yM$ relative to some choice of basis for T_yM.

[Recall that the **trace** of a square matrix L is the sum of its diagonal elements.]

Proof: Let $\widehat{S}_y = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ relative to our chosen basis. Then κ_1, κ_2 are roots of the polynomial

$$p(\lambda) = \begin{vmatrix} S_{11} - \lambda & S_{12} \\ S_{21} & S_{22} - \lambda \end{vmatrix} = \lambda^2 - (S_{11} + S_{22})\lambda + (S_{11}S_{22} - S_{12}S_{22}) = \lambda^2 - \mathrm{tr}\,\widehat{S}_y\,\lambda + \det \widehat{S}_y. \quad (\clubsuit)$$

But $p(\kappa_1) = p(\kappa_2) = 0$ and the coefficient of λ^2 is unity, so

$$p(\lambda) = (\lambda - \kappa_1)(\lambda - \kappa_2) = \lambda^2 - (\kappa_1 + \kappa_2)\lambda + \kappa_1\kappa_2. \qquad (\spadesuit)$$

Comparing coefficients of λ and unity in $(\spadesuit)$ and $(\clubsuit)$, we see that $\kappa_1 + \kappa_2 = \mathrm{tr}\,\widehat{S}_y$ while $\kappa_1\kappa_2 = \det \widehat{S}_y$. $\square$

Example 124 (a) The unit sphere has shape operator $S_y : v \mapsto -v$ at every point y (see Example 103). Hence, relative to *any* basis for T_yM,

$$\widehat{S}_y = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It follows that

$$H(y) = \frac{1}{2}(-1 \, + \, -1) = -1$$
$$\sigma(y) = (-1)(-1) = 1$$

for all $y \in M$.

(b) The saddle surface $M(x_1, x_2) = (x_1, x_2, x_1x_2)$ at $y = (0, 1, 0) = M(0, 1)$ has shape operator

$$\left.\begin{aligned} S_y\varepsilon_1 &= \frac{1}{\sqrt{2}}\varepsilon_2 \\ S_y\varepsilon_2 &= \frac{1}{2\sqrt{2}}\varepsilon_1 \end{aligned}\right\}\text{see Example 108.}$$

$$\text{Hence:}\qquad \widehat{S}_y = \begin{bmatrix} 0 & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad\Rightarrow\quad \begin{aligned} H(y) &= 0 \\ \sigma(y) &= -\frac{1}{4} \end{aligned}$$

at this particular point. □

Two more ways of interpreting mean curvature $H(y)$:

Proposition 125 *$H(y)$ is the average value of the normal curvature $k_y : U_yM \to \mathbb{R}$, thought of as a function on the unit circle in T_yM.*

Proof: Let κ_1, κ_2 be the principal curvatures, u_1, u_2 be the associated orthonormal principal curvature directions. Then for any $u \in U_yM$ there exists θ such that $u = u_1 \cos\theta + u_2 \sin\theta$. The normal curvature of u is

$$k_y(u) = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

Hence, the average value of k_y as θ varies over $[0, 2\pi]$ is

$$\begin{aligned} \langle k_y(u(\theta))\rangle &= \frac{1}{2\pi}\int_0^{2\pi}(\kappa_1\cos^2\theta + \kappa_2\sin^2\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}[\frac{\kappa_1}{2}(1+\cos 2\theta) + \frac{\kappa_2}{2}(1-\cos 2\theta)]\,d\theta = \frac{1}{2}(\kappa_1+\kappa_2). \qquad \square \end{aligned}$$

Clearly $k_y(u_1) = \kappa_1$, $k_y(u_2) = \kappa_2$, so $H(y) = \frac{1}{2}(k_y(u_1) + k_y(u_2))$. In fact we have the following:

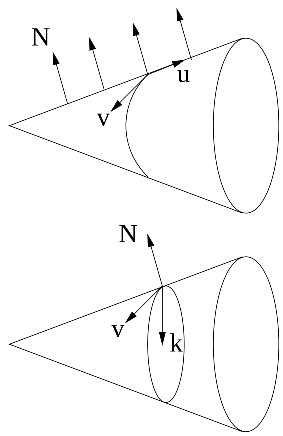
Proposition 126 *Let v_1, v_2 be* <u>*any*</u> *orthogonal pair of unit vectors in T_yM. Then $H = \frac{1}{2}(k_y(v_1) + k_y(v_2))$.*

Proof: Using the orthonormal basis of principal curvature directions u_1, u_2 again, there must exist θ such that $v_1 = u_1\cos\theta + u_2\sin\theta$. Consider the vector $\bar{v} = -u_1\sin\theta + u_2\cos\theta$. Clearly $\bar{v}\cdot v_1 = 0$ and $|\bar{v}|^2 = 1$, so either $v_2 = \bar{v}$ or $v_2 = -\bar{v}$. For any vector $u \in U_yM$, $k_y(-u) = (-u)\cdot S_y(-u) = u \cdot S_y\, u = k_y(u)$, so in either case $k_y(v_2) = k_y(\bar{v})$. Hence

$$\begin{aligned} k_y(v_1) + k_y(v_2) &= (u_1\cos\theta + u_2\sin\theta)\cdot(\cos\theta\, S_y\, u_1 + \sin\theta\, S_y\, u_2) + \\ &\qquad (-u_1\sin\theta + u_2\cos\theta)\cdot(-\sin\theta\, S_y\, u_1 + \cos\theta\, S_y\, u_2) \\ &= (u_1\cos\theta + u_2\sin\theta)\cdot(\kappa_1 u_1\cos\theta + \kappa_2 u_2\sin\theta) + \\ &\qquad (-u_1\sin\theta + u_2\cos\theta)\cdot(-\kappa_1 u_1\sin\theta + \kappa_2 u_2\cos\theta) \\ &= \kappa_1\cos^2\theta + \kappa_2\sin^2\theta + \kappa_1\sin^2\theta + \kappa_2\cos^2\theta = \kappa_1 + \kappa_2. \end{aligned}$$

Hence $H(y) = (\kappa_1 + \kappa_2)/2 = (k_y(v_1) + k_y(v_2))/2$. □

Example 127 Let M = a cone, outwardly oriented. What can we deduce about σ and H just by *looking* at the surface?



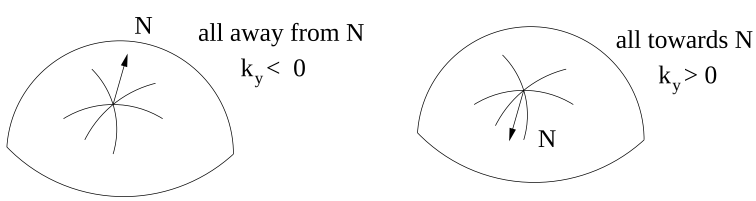
Pick orthonormal pair u, v at any point $y \in M$. Then N is constant along the straight line in M generating $u \Rightarrow \nabla_u N = 0 \Rightarrow S_y u = 0$. Hence at least one principal curvature $\kappa_2 = 0 \Rightarrow \sigma(y) = 0$. It follows that $k_y(u) = u \cdot S_y u = 0$.

But $k_y(v) = k \cdot N < 0$ since the circle generating v curves *away* from the outward unit normal. Hence, by Proposition 126, $H(y) < 0$. $\square$

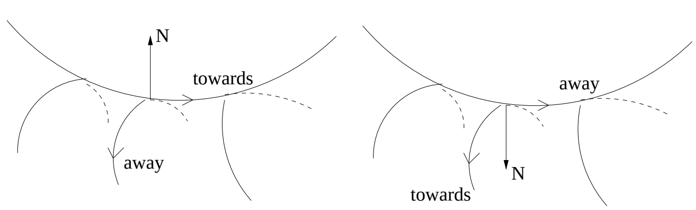
Fact 128 If we change orientation $N \mapsto \widetilde{N} = -N$ then $S_y \mapsto \widetilde{S}_y = -S_y \quad \Rightarrow \quad \kappa_i \mapsto \widetilde{\kappa}_i = -\kappa_i \quad \Rightarrow \quad H \mapsto \widetilde{H} = -H$ **but** $\sigma \mapsto \widetilde{\sigma} = \sigma$. The only measure of curvature we've introduced so far which is independent of the choice of orientation on M is the *Gauss curvature*. For all other types of curvature (including normal curvature), the *sign* of the curvature has no intrinsic meaning (i.e. no meaning independent of our more or less arbitrary choice of orientation).

The sign of σ **does** have an intrinsic geometric meaning!

$\sigma(y) > 0 \Rightarrow$ κ_1, κ_2 have same sign
$\Rightarrow$ $k_y(u)$ strictly positive (if $\kappa_1, \kappa_2 > 0$) <u>or</u>
$k_y(u)$ strictly negative (if $\kappa_1, \kappa_2 < 0$)
$\Rightarrow$ all curves through y curve **towards** $N(y)$ <u>or</u>
all curves through y curve **away from** $N(y)$
$\Rightarrow$ all curves curve in the same "sense".

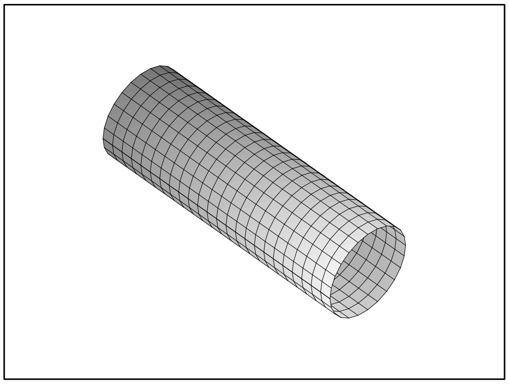


$\sigma(y) < 0 \Rightarrow$ κ_1, κ_2 differ in sign
$\Rightarrow$ $k_y(u)$ takes both positive <u>and</u> negative values
$\Rightarrow$ $\exists$ curves through y which curve in opposite senses.

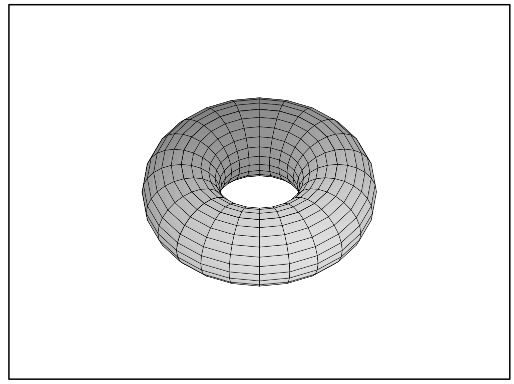


Example 129 Let's test our intuition on the following surfaces:

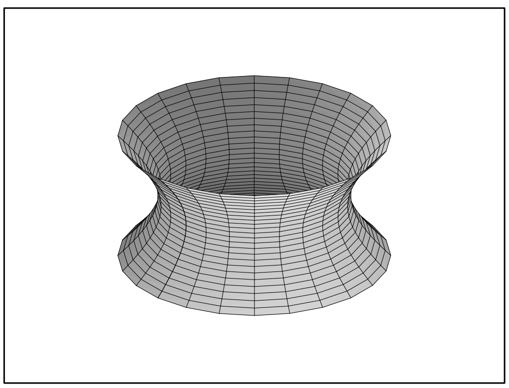
(a)



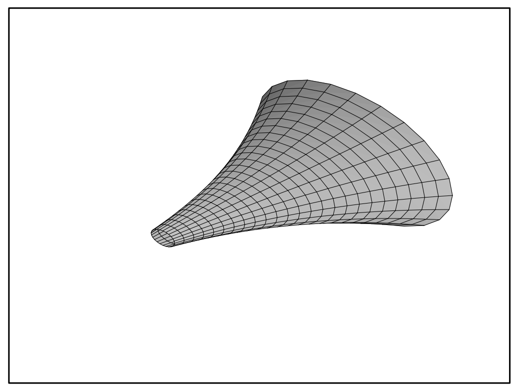
(b)



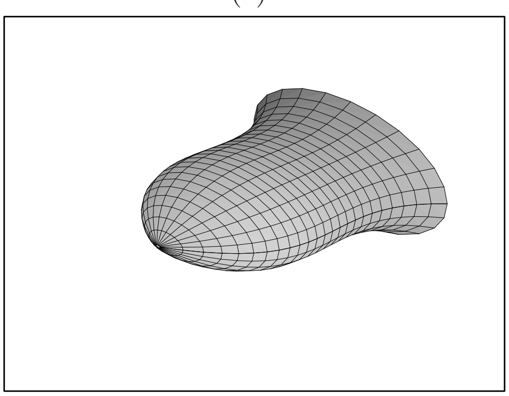
(c)



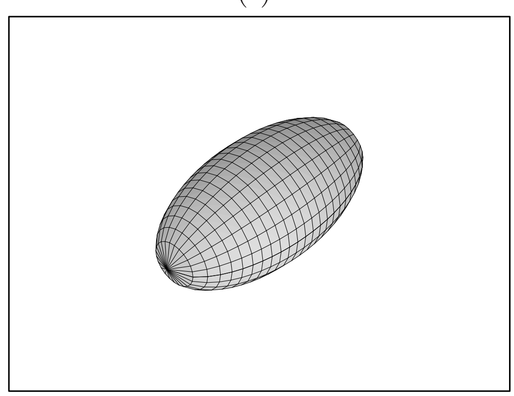
(d)



(e)



(f)



Summary

- An **oriented** surface is a RPS together with a choice of unit normal vector field N. Usually we choose
$$N = \frac{\varepsilon_1 \times \varepsilon_2}{|\varepsilon_1 \times \varepsilon_2|}.$$
- The **shape operator** of an oriented surface is
$$S_y : T_yM \to T_yM, \qquad S_y(v) = -\nabla_v N.$$
The shape operator is **linear**,
$$S_y(au + bv) = aS_y(u) + bS_y(v),$$
and **self adjoint**,
$$u \cdot S_y(v) = v \cdot S_y(u).$$
- The **principal curvatures** of M at y are the eigenvalues κ_1, κ_2 of S_y. The **principal curvature directions** are the corresponding (normalized) eigenvectors.
- The **normal curvature** of a **unit vector** $u \in T_yM$ is
$$k_y(u) = u \cdot S_y(u).$$
This coincides with $k(0) \cdot N(y)$ where $k(t)$ is the curvature vector of any generating curve for u. The principle curvatures are the maximum and minimum values of $S_y(u)$ as u takes all values in the unit tangent space at y.
- The **mean curvature** at y is
$$H = \frac{1}{2}(\kappa_1 + \kappa_2).$$
This coincides with $\frac{1}{2}(k_y(v_1) + k_y(v_2))$ where v_1, v_2 is any orthogonal pair of unit vectors in T_yM.
- The **Gauss curvature** at y is
$$\sigma = \kappa_1 \kappa_2.$$
The sign of σ has intrinsic meaning, independent of the choice of N: if $\sigma(y) > 0$ then either all curves in M through y curve towards $N(y)$, or they all curve away from $N(y)$; if $\sigma(y) < 0$ then some curves curve towards $N(y)$ and some curve away.
- Both H and σ can be computed directly from any matrix $\widehat{S}_y$ representing S_y:
$$H = \frac{1}{2}\operatorname{tr} \widehat{S}_y, \qquad \sigma = \det \widehat{S}_y.$$

Homework Deadlines MATH2051, 2011/2012

(i) Friday 14 October 2011

(ii) Friday 28 October 2011

(iii) Friday 11 November 2011

(iv) Friday 25 November 2011

(v) Friday 9 December 2011

University of Leeds, School of Mathematics
MATH 2051 Geometry of Curves and Surfaces.
Exercises 1: Regularly Parametrized Curves

Submit on Friday 14 October 2011

1. Determine which of the following are regularly parametrized curves, carefully explaining your reasoning:

   (a) $\gamma : \mathbb{R} \to \mathbb{R}^3$, $\gamma(t) = (t^2, t^3, \sin t)$ (b) $\gamma : (0, \infty) \to \mathbb{R}^4$, $\gamma(t) = (\sqrt{t}, e^t, t^2, 3)$

   (c) $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \int_0^t e^{r^2}\, dr)$ (d) $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(t) = (t + \cos t, \sin t)$.

2. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(t) = (t^3, t^2 - t)$.
   (a) Show that γ is a regularly parametrized curve.
   (b) Construct its tangent line at time $t = -1$, $\widehat{\gamma}_{-1} : \mathbb{R} \to \mathbb{R}^2$.
   (c) Find all intersection points between γ and $\widehat{\gamma}_{-1}$.

3. (a) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ such that $\gamma(t) = (t, \cos t, \sin t)$ (a vertical helix). Calculate the arc length along γ from $t = 0$ to $t = 2\pi$.
   (b) Let $\gamma : (0, 2\pi) \to \mathbb{R}^2$ such that $\gamma(t) = (t + \sin t, 1 - \cos t)$ (this is the trajectory described by a point on the rim of a wheel of unit radius rolling through one revolution without slipping). Calculate the total length of this curve.
   (c) Let $\gamma(t) = (t + 2\cos t, 2\sin t)$ and $\delta(t) = (4t + 2\cos t, 2\sin t)$. Given that the arc length along γ from $t = 0$ to $t = 2\pi$ is 13.36 to 2 decimal places, calculate the arc length along δ from $t = 0$ to $t = 2\pi$. To how many decimal places is your answer accurate?
   (d) Let $\gamma(t) = (t, \frac{1}{2}t^2)$. Compute $\sigma_0(t)$, the arc length function for γ based at $t_0 = 0$.
   [Hint: you'll need a clever substitution in the integral. Try hyperbolic trig functions.]

4. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(t) = e^t(\cos t, \sin t)$.
   (a) Construct the signed arc length function $\sigma_0 : \mathbb{R} \to J$ and its inverse function $\tau_0 : J \to \mathbb{R}$.
   (b) Hence find a unit speed reparametrization $\beta : J \to \mathbb{R}^2$ of γ. *Clearly state the domain J of β.*

5. The trajectory $\gamma : \mathbb{R} \to \mathbb{R}^3$ of an electrically charged particle (for example, an electron) moving in a uniform magnetic field satisfies the differential equation

   $$\gamma''(t) = B \times \gamma'(t)$$

   where $B = (B_1, B_2, B_3)$ is a constant 3-vector describing the magnetic field, and $\times$ denotes the vector product.
   (a) Show that the particle travels at constant speed.
   (b) Show that the component of the particle's velocity in the direction of the magnetic field B is also constant.
   (c) Without loss of generality, we can choose our Cartesian coordinate system so that $B = (0, 0, B_3)$, $\gamma(0) = (0, 0, 0)$ and $\gamma'(0) = (v_1, 0, v_3)$. From the previous parts, it follows that $|\gamma'(t)| = |v|$, and $\gamma_3'(t) = v_3$ for all time, so

   $$\gamma'(t) = (v_1 \cos\psi(t), v_1 \sin\psi(t), v_3)$$

   where $\psi : \mathbb{R} \to \mathbb{R}$ is some unknown function. Show that $\psi'(t) = B_3$. Hence compute the trajectory $\gamma(t)$ with these initial data. Describe the motion of the particle.

University of Leeds, School of Mathematics

MATH 2051 Geometry of Curves and Surfaces.
Exercises 2: Curvature of Parametrized Curves

Submit on Friday 28 October 2011

1. For each of the following RPCs $\gamma : \mathbb{R} \to \mathbb{R}^n$, compute the curvature vector $k : \mathbb{R} \to \mathbb{R}^n$:
   (a) $\gamma(t) = (t, t^3)$,
   (b) $\gamma(t) = (\sin t, 2t, 2t, \cos t)$,
   (c) $\gamma(t) = (e^{(t^2)}, \int_0^t e^{(\alpha^2)} d\alpha)$,
   (d) $\gamma(t) = (t - \cos t, \sin t, t)$.

2. (a) Given that $f : (0, \infty) \to \mathbb{R}$ is smooth, show that $\gamma : (0, \infty) \to \mathbb{R}^3$, $\gamma(t) = (t^2, \log t, \int_0^t f(\alpha) d\alpha)$ is a RPC.
   (b) Given the extra information that $f(1) = 1$ and $f'(1) = -1$, calculate $k(1)$, the curvature vector of γ at $t = 1$.

3. For each of the following planar RPCs $\gamma : \mathbb{R} \to \mathbb{R}^2$, compute the signed curvature $\kappa : \mathbb{R} \to \mathbb{R}$ and find all the inflexion points.
   (a) $\gamma(t) = (t, t^3)$,
   (b) $\gamma(t) = (t, t + \sin t)$,
   (c) $\gamma(t) = (e^{(t^2)}, \int_0^t e^{(\alpha^2)} d\alpha)$,

   [Hint: for (a), (c) you can use your answer to question 1!]

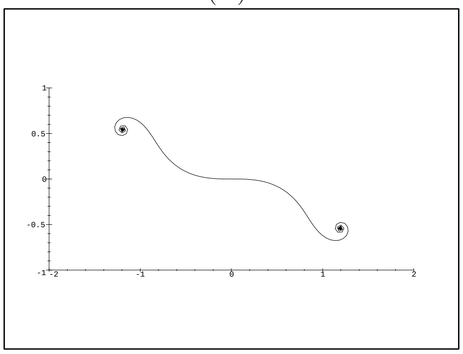
4. Given a prescribed function $\kappa : I \to \mathbb{R}$, there exists a unique unit speed curve $\gamma : I \to \mathbb{R}^2$ with $\gamma(0) = (0,0)$, $\gamma'(0) = (1,0)$ and signed curvature κ. The curves corresponding to the signed curvature functions ($I = \mathbb{R}$ in all cases)

$$\kappa_1(s) = s^4 - 5s^2 + 4, \qquad \kappa_2(s) = 4(s^3 + s), \qquad \kappa_3(s) = e^{(s^2)},$$

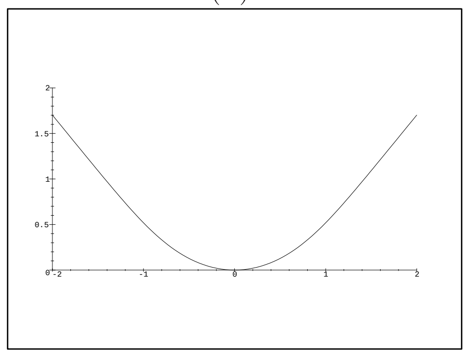
$$\kappa_4(s) = e^{-(s^2)}, \qquad \kappa_5(s) = 4(s^3 - s),$$

are depicted below in the wrong order, figures (A) to (E). Determine which curve corresponds to which signed curvature. In each case, briefly explain your reasoning. (Unexplained answers will *not* receive full credit.)

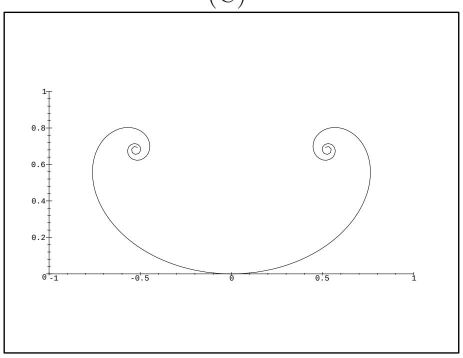
(A)



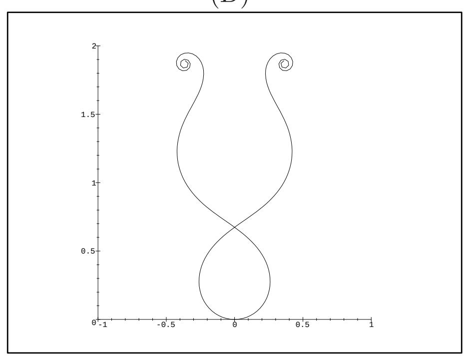
(B)



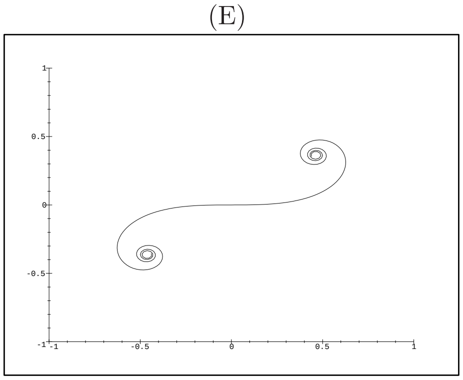
(C)



(D)



(E)



[This question is taken verbatim from a past final exam. I like this type of question very much...]

5. Find the centre of curvature of the curve $\gamma(t) = (t^2 - t, e^t)$ at $t = 0$.

6. Let $\gamma(t) = (t, \cosh t)$. Compute $\kappa(t)$, the signed curvature of γ, and $E_\gamma(t)$, the evolute of γ. Sketch the curves γ and E_γ. Is E_γ a RPC?

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MATH 2051 Geometry of Curves and Surfaces.
Exercises 3: Frenet Frame, Surfaces

Submit on Friday 11 November 2011

1. Given that $\gamma : \mathbb{R} \to \mathbb{R}^3$ has $\gamma'(0) = \frac{1}{\sqrt{2}}(1,1,0)$ and $\gamma''(0) = (1,0,1)$ construct $[u(0), n(0), b(0)]$, the Frenet frame for γ at time $t = 0$. Is it possible to tell from the given information whether γ is a USC?

2. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ such that $\gamma(s) = \frac{1}{\sqrt{5}}(2\sin s, 2\cos s, s)$. Show that γ is a USC of nonvanishing curvature. Construct its Frenet frame $[u(s), n(s), b(s)]$ and determine its torsion $\tau(s)$.

3. A curve $\gamma : I \to \mathbb{R}^3$ is *spherical* if its image $\gamma(I)$ lies entirely on the surface of some sphere $S \subset \mathbb{R}^3$; that is, if there exist constants $p \in \mathbb{R}^3$ and $r \in (0, \infty)$ such that for all $s \in I$, $|\gamma(s) - p| = r$.
   The sphere S in question is then centred at p and has radius r.

   Let $\gamma : I \to \mathbb{R}^3$ be a spherical unit speed curve. Prove that

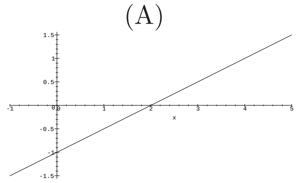
   (a) γ has nonvanishing curvature and hence a well defined Frenet frame and torsion τ.

   (b) If $\tau(s) \neq 0$ for all $s \in I$ then $$\frac{1}{\kappa(s)^2} + \left(\frac{\kappa'(s)}{\tau(s)\kappa(s)^2}\right)^2 = r^2 \qquad (\clubsuit)$$
   a constant (in fact, r is the radius of the sphere containing γ).

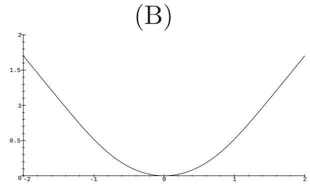
   Does the converse of this "theorem" hold: if $\gamma : I \to \mathbb{R}^3$ is a unit speed curve of nonvanishing curvature whose curvature and torsion satisfy equation ($\clubsuit$), does it follow that γ is a spherical curve? [Hint: look at question 2.]

4. For each of the curves depicted below, determine whether the curve's **evolute** is:
(i) not globally defined, (ii) globally defined but not regular, (iii) regular.

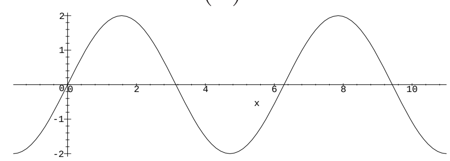
(A)



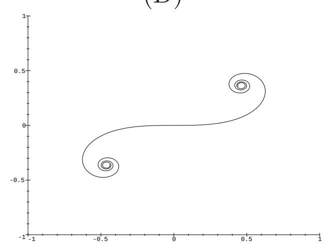
(B)



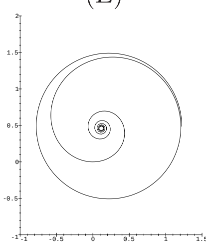
(C)



(D)



(E)



One and only one of these curves has *no* regular parallel curves (except for itself). Which one?

5. Let $\gamma : I \to \mathbb{R}^2$ such that $\gamma(t) = (t, \cosh t)$. Construct $I_\gamma : \mathbb{R} \to \mathbb{R}^2$, the involute of γ starting at $t = 0$.

6. Determine whether each of the following maps $M : U \to \mathbb{R}^3$ is a regularly parametrized surface.

   (a) $M : \mathbb{R}^2 \to \mathbb{R}^3$, $M(x_1, x_2) = (x_1, x_1 x_2, e^{x_1 + x_2})$

   (b) $M : \mathbb{R}^2 \to \mathbb{R}^3$, $M(x_1, x_2) = (x_2 \cos x_1, x_1 x_2^2, x_1 \cosh x_2)$

   (c) $M : \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$, $M(z, \psi) = (\cosh z \cos \psi, \cosh z \sin \psi, \sinh z)$.

7. Verify that the image set of M as defined in question 6c is contained in the hyperboloid of one sheet:
$$H = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 - y_3^2 = 1\}.$$
Is the image set **all** of H? If not, what subset of H is missing? Find local coordinates (z, ψ) for the point $(1, 1, 1) \in H$.

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MATH 2051 Geometry of Curves and Surfaces.
Exercises 4: Tangent and Normal Spaces

Submit on Friday 25 November 2011

1. Given a point $y \in \mathbb{R}^3$, find expressions for the distance from y to $(0,0,1)$ and the distance from y to the plane $y_3 = 0$. Let $P \subset \mathbb{R}^3$ be the set of points which are equidistant from the plane $y_3 = 0$ and the point $(0,0,1)$. Find a RPS whose image set is precisely P.

2. Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve of nonvanishing curvature, and consider the map $M : \mathbb{R} \times (0,\infty) \to \mathbb{R}^3$ defined by

$$M(s,t) = \gamma(s) + t\gamma'(s).$$

Show that M is regular. In the case that γ is the helix

$$\gamma(s) = \frac{1}{\sqrt{2}}(\cos s, \sin s, s)$$

show that M is a RPS. [Hint: you need to show that M is one-to-one, that is, $M(s,t) = M(\bar{s},\bar{t})$ implies $(s,t) = (\bar{s},\bar{t})$. Now $M(s,t) = M(\bar{s},\bar{t})$ implies $M_3(s,t) = M_3(\bar{s},\bar{t})$ and $|M(s,t)|^2 = |M(\bar{s},\bar{t})|^2$. This gives a pair of (polynomial) equations for $s, t, \bar{s}, \bar{t}$, which you should be able to show are solved only if $t = \bar{t}$ and $s = \bar{s}$.]

3. Let $M : \mathbb{R}^2 \to \mathbb{R}^3$ such that $M(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_1^2 - x_2^2)$. Show that M is a RPS. Find the local coordinates of $y = (3,1,3) \in M$. Determine whether the following vectors are tangent, normal or neither at y:

$$(2,2,-1), \quad (-1,3,8), \quad (-1,-3,1), \quad (0,0,0).$$

4. With M and y as in question 3, let $v = (1,0,1)$. You are given that $v \in T_yM$. Express v in the form $a\varepsilon_1 + b\varepsilon_2$. Hence compute the directional derivatives $v[f]$, $v[g]$ and $v[fg]$ where $f, g : M \to \mathbb{R}$ are the functions

$$f(y_1,y_2,y_3) = y_1 + y_2 + y_3, \qquad g(y_1,y_2,y_3) = y_1^2 + y_2^2 + y_3^2.$$

5. With M as in question 3, let N be the unit normal vector field

$$N(x_1,x_2) = \frac{\varepsilon_1(x_1,x_2) \times \varepsilon_2(x_1,x_2)}{|\varepsilon_1(x_1,x_2) \times \varepsilon_2(x_1,x_2)|}.$$

Calculate the vector fields $\nabla_{\varepsilon_1}\varepsilon_1$, $\nabla_{\varepsilon_1}\varepsilon_2$, $\nabla_{\varepsilon_1}N$, $\nabla_{\varepsilon_2}\varepsilon_1$, $\nabla_{\varepsilon_2}\varepsilon_2$, $\nabla_{\varepsilon_2}N$. Determine which of these are tangent vector fields, and write those which are in the form $a(x)\varepsilon_1(x) + b(x)\varepsilon_2(x)$.

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MATH 2051 Geometry of Curves and Surfaces.

Exercises 5: Curvature of Surfaces

Submit on Friday 9 December 2011

1. Show that $M : \mathbb{R}^2 \to \mathbb{R}^3$, $M(x_1, x_2) = (x_1, x_2, \frac{1}{2}x_2^2)$ is a regularly parametrized surface. Sketch the image set of M. Construct the unit normal vector field $N(x_1, x_2)$. Compute $S_y\varepsilon_1$ and $S_y\varepsilon_2$ at a general point $y = M(x)$. Deduce the principal curvatures and principal curvature directions at a general point.

2. You are given that the tangent space T_yM of a certain RPS M is spanned by a pair of vectors $\varepsilon_1, \varepsilon_2$ and that the shape operator at y satisfies

$$S_y\varepsilon_1 = \varepsilon_1 - 2\varepsilon_2, \qquad S_y\varepsilon_2 = -\varepsilon_1 + 3\varepsilon_2.$$

   Deduce the principal curvatures of M at y. If $\varepsilon_1, \varepsilon_2$ are both unit vectors, what is $\varepsilon_1 \cdot \varepsilon_2$?

3. The set $\bar{H} = \{y \in \mathbb{R}^3 \,|\, y_1^2 + y_2^2 - y_3^2 = -1\}$ is a hyperboloid of two sheets. You are given that

$$N(y_1, y_2, y_3) = \frac{(y_1, y_2, -y_3)}{\sqrt{y_1^2 + y_2^2 + y_3^2}}$$

   is a unit normal vector field on N. Verify that $u = (0, 1, 0)$ is tangent to $\bar{H}$ at the specific point $\tilde{y} = (1, 0, \sqrt{2}) \in \bar{H}$. Write down a generator curve $\alpha : I \to \bar{H}$ for the tangent vector u. Hence compute $S_{\tilde{y}}u$. Deduce **one** of the principal curvatures of $\bar{H}$ at $\tilde{y}$, and **both** of the principal curvature directions.

4. A certain oriented surface M has, at a certain point y, principal curvatures $\kappa_1 = 0$, $\kappa_2 = 4$ and principal curvature directions $u_1 = (1, 0, 0)$, $u_2 = (0, 1/\sqrt{2}, -1/\sqrt{2})$. In each of the following cases, either construct a unit vector $u \in U_yM$ with the stated property, or explain why no such u exists.

   (a) $k_y(u) = 0$
   (b) $k_y(u) = 1$
   (c) $k_y(u) = -1$
   (d) $S_yu = u$
   (e) $|S_yu| = 2\sqrt{3}$.

5. Consider the mapping $M : \mathbb{R} \times (-\pi, \pi) \to \mathbb{R}^3$ defined by

$$M(t, \theta) = (t, (1 + \frac{t^2}{2})\cos\theta, (1 + \frac{t^2}{2})\sin\theta).$$

   You are given that M is a RPS. Compute its Gauss curvature at the point $p = M(0, 0) = (0, 1, 0)$.

6. An oriented surface $M : U \to \mathbb{R}^3$ is said to be **minimal** if its mean curvature $H(y)$ is zero for all $y \in M$. Show that a minimal surface must have non-positive Gauss curvature, that is $\sigma(y) \leq 0$ for all $y \in M$.

   Look at the surfaces depicted below. Two of them are minimal surfaces, while the other 4 are not. Identify which two are minimal. [Hint: think about the Gauss curvature of the surfaces...]