GLOSSARY

Acceleration The acceleration of a moving body whose position at time t is $u(t)$ is given by

$$\frac{d^2u}{dt^2}$$

Air resistance A body moving through air (or some other medium) is slowed down by a resistive force (also called a drag or damping force) that acts opposite to the body's velocity. See also "Viscous damping" and "Newtonian damping."

Amplitude The amplitude of a periodic oscillating function $u(t)$ is half the difference between its maximum and minimum values.

Angular momentum The angular momentum vector of a body rotating about an axis is its moment of inertia about the axis times its angular velocity vector.

This is the analog in rotational mechanics of momentum (mass times velocity) in linear mechanics.

Angular velocity An angular velocity vector, $\omega(t)$, is the key to the relation between rotating body axes and a fixed coordinate system of the observer. The component ω_j of the vector $\omega(t)$ along the jth body axis describes the spin rate of the body about that axis.

Autocatalator This is a chemical reaction of several steps, at least one of which is autocatalytic.

Autocatalytic reaction In an autocatalytic reaction, a chemical species stimulates more of its own production than is destroyed in the process.

Autonomous ODE An autonomous ODE has no explicit mention of the independent variable (usually t) in the rate equations. For example, $x' = x^2$ is autonomous, but $x' = x^2 + t$ is not.

Balance law The balance law states that the net rate of change of the amount of a substance in a compartment equals the net rate of flow in minus the net rate of flow out.

Beats When two sinusoids of nearly equal frequencies are added the result appears to be a high frequency sinusoid modulated by a low frequency sinusoid called a beat. A simple example is given by the function $(\sin t)(\sin 10t)$, where the first sine produces an "amplitude modulation" of the second.

Bessel functions of the 1st kind The Bessel function of the first kind of order zero,

$$J_0(s) = 1 - \frac{1}{4}s^2 + \cdots + (-1)^n \frac{s^{2n}}{n!^2 2^{2n}} + \cdots$$

is a solution of Bessel's equation of order zero, and is bounded and convergent for all s.

Bessel functions of the 2nd kind The Bessel function of the second kind of order zero, $Y_0(s)$, is another solution of Bessel's equation of order zero. It is much more complicated than $J_0(s)$, and

$$Y_0(s) \to \infty \quad \text{as} \quad s \to 0+$$

See Chapter 11 for a complete formula for $Y_0(s)$ that involves a logarithmic term, $J_0(s)$, and a complicated (but convergent) infinite series.

Bessel's equation Bessel's equation of order $p \geq 0$ is

$$s^2 w''(s) + sw'(s) + (s^2 - p^2)w = 0$$

where p is a nonnegative constant. Module 11 considers only $p = 0$. See Chapter 11 for $p > 0$.

Bessel's equation, general solution of Bessel's equation of order zero is second order and linear. The general solution is the set of all linear combinations of $J_0(s)$ and $Y_0(s)$.

Bifurcation diagram A bifurcation diagram describes how the behavior of a dynamical system changes as a parameter varies. It can appear in studies of iteration or of differential equations.

In the case of a single real parameter, a bifurcation diagram plots a parameter versus something indicative of the behavior, such as the variable being iterated (as in Module 13, Nonlinear Behavior) or a single variable marking location and stability of equilibrium points for a differential equation.

In iteration of a function of a complex variable, two dimensions are needed just to show the parameter, but different colors can be used to show different behaviors (as in Module 13, Complex Dynamics).

Cantor Set, Cantor Dust A Cantor set was first detailed by Henry Smith in 1875, but was named in honor of Georg Cantor, the founder of set theory, after he used this bizarre construction in 1883. Now Cantor sets are found in many guises in discrete dynamical systems.

A Cantor set is a totally disconnected set, in a finite space, with uncountably many points. A typical construction is to delete a band across the middle of a set, then to delete the middle of both pieces that are left, and then to repeat this process indefinitely.

Julia sets (see glossary) for parameter values outside the Mandelbrot set (see glossary) are Cantor dusts, constructed by a similar algorithm. See Companion Book for References.

Carrying capacity The carrying capacity K of an environment is the maximum number of individuals that the environment can support at steady state. If there are fewer individuals than the carrying capacity in the environment, the population will grow; if there are more individuals, the population will decline.

A widely used model for population dynamics involving a carrying capacity is the logisitc ODE

$$\frac{dN}{dt} = rN(1 - N/K)$$

where r is the intrinsic growth rate constant.

Cascade A cascade is a compartment model where the "flow" through the compartments is all one direction.

Center A center is an equilibrium point of an autonomous planar linear system for which the eigenvalues are conjugate imaginaries $\pm i\beta$, $\beta \neq 0$. All nonconstant orbits of an autonomous planar linear system with a center are simple closed curves enclosing the equilibrium.

Centering an equilibrium If p^* is an equilibrium point of the system $x' = f(x)$ (so $f(p^*) = 0$), then the change of coordinates $x = y + p^*$ moves p^* to the origin in the y-coordinate system.

Chain rule The chain rule for differentiating a function $L(\theta(t), y(t))$ with respect to t is

$$\begin{aligned}\frac{dL}{dt} &= \frac{\partial L}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt} \\ &= L_\theta \theta' + L_y y' \\ &= L_\theta y + L_y y'\end{aligned}$$

Chaos Mathematical chaos is a technical term that describes certain nonperiodic behavior of a discrete dynamical system (Module 13) or solutions to a differential equation (Module 12). A system is said to be chaotic in a region if all of the following are true.

- It exhibits sensitive dependence on initial conditions.
- Periodic unstable orbits occur almost everywhere.
- Iterates of intervals get "mixed up."

Chaotic behavior never repeats, revisits every neighborhood infinitely often, but is not random. Each step is completely determined by the previous step.

An equivalent list of requirements appears in Module 12, Screen 1.4. Further discussion appears in Chapter 13.

Characteristic equation The characteristic equation of a square matrix A is $\det(\lambda I - A) = |\lambda I - A| = 0$. For a 2×2 matrix, this reduces to $\lambda^2 - \operatorname{tr} A\lambda + \det A = 0$ whose solutions, called eigenvalues of A are

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{\operatorname{tr}^2 A - 4\det A}}{2}$$

Chemical law of mass action The rate of a reaction step is proportional to the product of the concentrations of the reactants.

Example: If one unit of species X produces one unit of product Y in a reaction step, the rate of the step is kx, where k is a positive constant. Thus, we have

$$x' = -kx, \qquad y' = kx$$

Example (Autocatalysis): If one unit of species X reacts with two units of Y and produces three units of Y in an autocatalytic step, the reaction rate is

$$axyy = axy^2$$

where a is a positive constant. Thus, we have

$$x' = -axy^2, \qquad y' = 3axy^2 2 - 2axy^2 = axy^2$$

because one unit of X is destroyed, while three units of Y are created, and two are consumed.

Combustion model The changing concentration $y(t)$ of a reactant in a combustion process is modeled by the IVP

$$y' = y^2(1 - y), \qquad y(0) = a, \quad 0 \leq t \leq 2/a$$

where a is a small positive number that represents a disturbance from the pre-ignition state $y = 0$. R. E. O'Malley studied the problem in his book, *Singular Perturbation Methods for Ordinary Differential Equations*, (1991: Springer).

Compartment model A compartment model is a set of boxes (the compartments) and arrows that shows the flow of a substance into and out of the different boxes.

Component graphs A component graph of a solution of a differential system is a graph of one of the dependent variables as a function of t.

Example: For the ODE system

$$x' = F(x, y)$$
$$y' = G(x, y)$$

the component graphs are the plots of a solution $x = x(t)$ and $y = y(t)$ in the respective tx- and ty-planes.

Concentration The concentration of a substance is the amount of the substance dissolved per unit volume of solution.

Connected set A connected set is a set with no islands. In the early 1980's Adrien Douady (Université Paris XI, Orsay and Ecole Normale Supérieure) and John Hubbard (Cornell University) proved that the Mandelbrot set (see Glossary) was connected. They did this by showing that its exterior could be put in a one-to-one correspondence with the exterior of a disk. They found in the process that all the angles one might note while walking around the boundary of the disk have special analogs on the Mandelbrot set. Halfway around the disk from the rightmost point corresponds to being at the tip of the Mandelbrot set, while one third or two thirds the way around the disk corresponds to the "neck" where the biggest ball attaches to the cardioid.

Conserved quantity A function $E(q, y)$ is conserved along a trajectory $q = q(t)$, $y = y(t)$, of a system $q' = f(q, y)$, $y' = g(q, y)$, if $dE(q(t), y(t))/dt = 0$.

As time changes, the value of E stays constant on each trajectory, although the value will vary from one trajectory to another. The graph of each trajectory in the qy-phase plane lies on one of the level sets $E = \text{constant}$.

This idea of a conserved quantity can be extended to any autonomous system of ODEs. An autonomous system is conservative if there is a function E that stays constant along each trajectory, but is nonconstant on every region (i.e., varies from trajectory to trajectory).

Cycle In a discrete dynamical system, including a Poincaré section, a cycle is a sequence of iterates that repeats. The number of iterates in a cycle is its period.

For an autonomous differential system, a cycle is a nonconstant solution $x(t)$ such that $x(t + T) = x(t)$, for all t, where T is a positive constant. The smallest value of T for a cycle is its period.

For a cycle in a system of 2 ODEs, see Limit Cycle.

Damped pendulum A real pendulum of length L is affected by friction or air resistance that is a function of L, θ, and θ', and acts opposite to the direction of motion.

Throughout the Linear and Nonlinear Pendulums submodule of Module 10, we assume that, if there is any damping, it is viscous (see Viscous damping); i.e., the damping force is given by $-bL\theta'$. The minus sign tells us that damping acts opposite to the velocity.

Module 4 makes a more detailed study of the effects of damping on a linear oscillator, as does Module 11 for the spring in the Robot and Egg.

Damping Damping can arise from several sources, including air resistance and friction. The most common model of damping is viscous damping—the damping force is assumed to be proportional to the velocity and acts opposite to the direction of motion. See also Newtonian damping.

Dense orbit An orbit $x(t)$ of a system of ODEs $x' = f(t, x)$ is dense in a region R of x-space if the orbit gets arbitrarily close to every point of R as time goes on.

That is, if x_1 is any point in R , and ε is any positive number, then, at some time t_1, the distance between $x(t_1)$ and x_1 is less than ε.

Determinant The determinant of the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $\det A = ad - bc$.

Deterministic A system of ODEs is said to be deterministic if the state of the system at time t is uniquely determined by the state of the system at the initial time.

For example, the single first order ODE $x' = f(t, x)$ is deterministic if f and $\partial f/\partial x$ are continuous functions of t and x, as for each set of initial data (t_0, x_0) there is exactly one solution $x(t)$.

Thus, if you were to choose the same initial data a second time and watch the solution curve trace out in time again, you would see exactly the same curve.

Dimensionless variables Suppose that a variable x is measured in units of kilograms and that x varies from 10 to 500 kilograms. If we set $y = (x \text{ kilograms})/(100 \text{ kilograms})$, y is dimensionless, and $0.1 \le y \le 5$. The smaller range of values is useful for computing. The fact that y has no units is useful because it no longer matters if the units are kilograms, grams, or some other units.

When variables are scaled to dimensionless quantities, they are typically divided by a constant somewhere around the middle of the expected range of values. For example, by dividing a chemical concentration by a "typical" concentration, we obtain a dimensionless concentration variable. Similarly, dimensionless time is obtained by dividing ordinary time by a "standard" time.

Direction field A direction field is a collection of line segments which shows the slope of the trajectories for an autonomous ODE system

$$\frac{dx}{dt} = F(x, y)$$
$$\frac{dy}{dt} = G(x, y)$$

at a representative grid of points. An arrowhead on a segment shows the direction of motion.

Disconnected Julia set A disconnected Julia set is actually a Cantor dust. It is composed entirely of totally disconnected points, which means that it is almost never possible to land on a point in the Julia set by clicking on a pixel. You will probably find that every click you can make starts an iteration that goes to infinity, only because you cannot actually land on an exact enough value to show a stable iteration.

Discrete dynamical system A discrete dynamical system takes the form $u_{n+1} = f(u_n)$, where the variable u_n gives the state of the system at "time" n, and u_{n+1} is the state of the system at time $n + 1$. See Module 13.

Eigenvalues The eigenvalues of a matrix A are the numbers λ for which

$$Av = \lambda v$$

for some nonzero vector v (the vector v is called an eigenvector). The eigenvalues λ of a 2×2 matrix A are the solutions to the characteristic equation of A:

$$\lambda^2 - \operatorname{tr} A\lambda + \det A = 0$$

$$\lambda = \frac{\operatorname{tr} A \pm \sqrt{(\operatorname{tr} A)^2 - 4\det A}}{2}$$

where $\operatorname{tr} A$ is the trace of A, and $\det A$ is the determinant of A. If a linear or linearized system of ODEs is $z' = A(z - p^*)$, and if the real parts of the eigenvalues of A are positive, then trajectories flow away from the equilibrium point, p^*. If the real parts are negative, then trajectories flow toward p^*.

Eigenvector An eigenvector of a matrix A is a nonzero vector, v, that satisfies $Av = \lambda v$ for some eigenvalue λ. The ODE Architect Tool calculates eigenvalues and eigenvectors of Jacobian matrices at any equilibrium point of an autonomous system (linear or nonlinear). Eigenvectors play a strong role in the local geometry of phase portraits at an equilibrium point.

Energy In physics and engineering, energy is defined by

$$E = \text{kinetic energy} + \text{potential energy}$$

where kinetic energy is interpreted to be the energy of motion, and the potential is the energy due to some external force, such as gravity, or (in electricity) a battery, or a magnet. If energy is conserved, i.e., stays at a constant level, then the system is said to be conservative.

If we are dealing with the autonomous differential system

$$x' = y, \qquad y' = -v(x) \tag{5}$$

we can define an "energy function" by

$$E = \frac{1}{2}y^2 + V(x)$$

where $dV/dx = v(x)$. Note that E is constant along each trajectory, because $dE/dt = y\,dy/dt + (dV/dx)(dx/dt) = y(-v(x)) + v(x)(y) = 0$, where the ODEs in system (5) have been used. The term $(1/2)y^2$ is the "kinetic energy". $V(x)$ is the "potential energy" in this context. See Chapter 10 for more on these ideas.

Epidemic An epidemic occurs in an epidemilogical model if the number of infectives, $I(t)$, increases above its initial value, I_0. Thus, an epidemic occurs if $I'(0) > 0$.

Equilibrium point An equilibrium point p^* in phase (or state) space of an autonomous ODE, is a point at which all derivatives of the state variables are zero—a stationary point—a steady-state value of the state variables. For example, for the autonomous system,

$$x' = F(x, y), \qquad y' = G(x, y)$$

if $F(x^*, y^*) = 0,\ G(x^*, y^*) = 0$, then $p^* = (x^*, y^*)$ is an equilibrium point, and $x = x^*,\ y = y^*$ (for all t) is a constant solution.

For a discrete dynamical system, an equilibrium point p^* is one for which $f(p^*) = p^*$, so that $p^*_{n+1} = p^*_n$, for all n; p^* is also called a fixed point of the system.

Estimated error For the solution $u(t)$ of the IVP $y' = f(t, y),\ y(t_0) = y_0$, the local error at the nth step of the Euler approximation is given by

$$\begin{aligned} e_n &= \text{Taylor series of } u(t) - \text{Euler approximation} \\ &= \frac{1}{2}h^2u''(t_n) + h^3u'''(t_n) + \cdots \end{aligned}$$

If the true solution, $u(t)$, is not known, we can approximate e_n for small h by

$$e_n \approx \text{Taylor approx.} - \text{Euler approx.} = \frac{1}{2}h^2 u''(t_n)$$

Euler's method Look at the IVP $y' = f(t, y),\ y(t_0) = y_0$. Euler's method approximates the solution $y(t)$ at discrete t values. For step size h , put $t_{n+1} = t_n + h$ for $n = 0, 1, 2, \ldots$. Euler's method approximates

$$y(t_1), y(t_2), \ldots$$

by the values

$$y_1, y_2, \ldots$$

where

$$y_{n+1} = y_n + hf(t_n, y_n), \quad \text{for } n = 0, 1, 2, \ldots$$

Existence and uniqueness A basic uniqueness and existence theorem says that, for the IVP,

$$x' = F(x, y, t), \qquad y' = G(x, y, t),$$
$$x(t_0) = x_0, \qquad y(t_0) = y_0$$

a unique solution $x(t),\ y(t)$ exists if $F,\ G,\ \partial F/\partial x$, $\partial F/\partial y$, $\partial G/\partial x$, and $\partial G/\partial y$ are all continuous in some region containing (x_0, y_0).

Fixed point A fixed point, p^*, of a discrete dynamical system is a point for which $x_{n+1} = f(x_n) = x_n$. That is, iteration of such a point simply gives the same point.

A fixed point can also be called an equilibrium or a steady state. A fixed point may be a sink, a source, or a saddle, depending on the character of the eigenvalues of the associated linearization matrix of the iterating function.

Forced damped pendulum A forced, viscously damped pendulum has the modeling equation

$$mx'' + bx' + ksinx = F(t)$$

The beginning of Module 10 explains the terms and parameters of this equation using θ instead of x. Module 12 examines a case where chaos can result, with $b = 0.1,\ \ m = 1,\ \ k = 1,\ \ A = 1,$ and $F(t) = \cos t$. All three submodules of Module 12 are involved in explaining the behaviors, and the introduction to the Tangled Basins submodule shows a movie of what happens when b is varied from 0 to 0.5.

Forced pendulum Some of the most complex and curious behavior occurs when the pendulum is driven by an external force. In Module 10, The Pendulum and its Friends, you can experiment with three kinds of forces in the Linear and Nonlinear Pendulums submodule, and an internal pumping force in the Child on a Swing submodule. But, for truly strange behavior, take a look at Module 12, Chaos and Control.

Fractal dimension Benoit Mandelbrot in the early 1980's coined the word "fractal" to apply to objects with dimensions between integers. The boundary of the Mandelbrot set (see glossary) is so complicated that its dimension is surely greater than one (the dimension of any "ordinary" curve). Just how much greater remained an open question until 1992 when the Japanese mathematician Mitsuhiro Shishikura proved it is actually dimension two!

Frequency The frequency of a function of period T is $1/T$. Another widely used term is "circular frequency", which is defined to be $2\pi/T$. For example, the periodic function $sin(3t)$ has period $T = 2\pi/3$, frequency $3/(2\pi)$, and circular frequency 3.

General solution Consider the linear system $x' = Ax + u$ [where x has 2 components, A is a 2×2 matrix of constants, and u is a constant vector or a function only of t]. Let A have distinct eigenvalues $\lambda_1,\ \lambda_2$ with corresponding eigenvectors $v_1,\ v_2$. All solutions of the system are given by the so-called general solution:

$$x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 + \tilde{x}$$

where $\tilde{x}$ is any one particular solution of the system and C_1 and C_2 are arbitrary constants.. If u is a constant vector, then $\tilde{x} = p^*$, the equilibrium of the system. If x has more than two dimensions, terms of the same form are added until all dimensions are covered. Note that, if $u = 0,\ \ p^* = 0$ is an equilibrium.

Geodesic Any smooth curve can be reparameterized to a unit speed curve $x(t)$, where $|x'(t)| = 1$. Unit-speed curves $x(t)$ on a surface are geodesics if the acceleration vector $x''(t)$ is perpendicular to the surface at each point x(t).

It can be shown that a geodesic is locally length-minimizing, so, between any two points sufficiently close, the geodesic curve is the shortest path.

GI tract The gastro-intestinal (GI) tract consists of the stomach and the intestines.

Gravitational force The gravitational force is the force on a body due to gravity. If the body is near the earth's surface, the force has magnitude mg, where m is the body's mass, and the force acts downward. The value of acceleration due to gravity, g, is 32 ft/sec^2 (English units), 9.8 meters/sec^2 (metric units).

Great circle A great circle on a sphere is an example of a geodesic. You can test this with a ball and string. Hold one end of the string fixed on a ball. Choose another point some distance away, and find the geodesic or shortest path by pulling the string tight between the two points. You will find that it always is along a circle centered at the center of the ball, which is the definition of a great circle.

Hooke's law Robert Hooke, an English physicist in the seventeenth century, stated the law that a spring exerts a force, on an attached mass, which is proportional to the displacement of the mass from the equilibrium position and points back toward that position.

Initial condition An initial condition specifies the value of a state variable at some particular time, usually at $t = 0$.

Initial value problem An initial value problem (IVP) consists of a differential equation or a system of ODEs and an initial condition specifying the value of the state variables at some particular time, usually at $t = 0$.

Integral surfaces The surface S defined by $F(x, y, z) = C$, where C is a constant, is an integral surface of the autonomous system

$$x' = f(x, y, z), \qquad y' = g(x, y, z), \qquad z' = h(x, y, z)$$

if

$$\begin{aligned}\frac{d}{dt}F(x, y, z) &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial F}{\partial x}f + \frac{\partial F}{\partial y}g + \frac{\partial F}{\partial z}h = 0\end{aligned}$$

for all x, y, z. We get a family of integral surfaces by varying the constant C. An orbit of the system that touches an integral surface stays on it. The function F is called an integral of the system.

For example, the family of spheres

$$F = x^2 + y^2 + z^2 = \text{constant}$$

is a family of integral surfaces for the system

$$x' = y, \qquad y' = z - x, \qquad z' = -y$$

because

$$2xx' + 2yy' + 2zz' = 2xy + 2y(z - x) + 2z(-y) = 0$$

Each orbit lies on a sphere, and each sphere is covered with orbits.

Intermediate An intermediate is a chemical produced in the course of a reaction which then disappears as the reaction comes to an end.

Intrinsic growth rate At low population sizes, the net rate of growth is essentially proportional to population size, so that $N' = rN$. The constant r is called the intrinsic growth rate constant. It gives information about how fast the population is changing before resources become limited and reduce the growth rate.

Iteration Iteration generates a sequence of numbers by using a given number x_0 and the rule $x_{n+1} = f(x_n)$, where $f(x)$ is a given function. Sometimes, x_n is written as $x(n)$.

IVP See initial value problem.

Jacobian matrix The system $x' = F(x, y)$, $y' = G(x, y)$, has the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}$$

The eigenvalues and eigenvectors of this matrix at an equilibrium point p^* help determine the local geometry of the phase portrait.

Jacobian matrices J can be defined for autonomous systems of ODEs with any number of state variables.

The ODE Architect Tool will find eigenvalues and eigenvectors of J at any equilibrium point.

Julia Set In complex dynamics, a Julia set for a given function $f(z)$ separates those points that iterate to infinity from those that do not. See the third submodule of Module 13 Dynamical Systems.

Julia sets were discovered about 1910 by two French mathematicians, Pierre Fatou and Gaston Julia. But, without computer graphics, they were unable to see the details of ragged structure that today display Cantor sets, self-similarity and fractal properties.

Kinetic energy of rotation The kinetic energy of rotation of a gyrating body is

$$E = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

where I_j and ω_j are, respectively, the moment of inertia and the angular velocity about the body axis, j, for $j = 1, 2, 3$.

Lift The lift force on a body moving through air is a force that acts in a direction orthogonal to the motion. Its magnitude may be modeled by a term which is proportional to the speed or to the square of the speed.

Limit cycle A cycle is a closed curve orbit of the system

$$x' = F(x, y)$$
$$y' = G(x, y)$$

A cycle is the orbit of a periodic solution.

An attracting limit cycle is a cycle that attracts all nearby orbits as time increases, and a repelling limit cycle if it repels all nearby orbits as time increases.

Linearization For a nonlinear ODE, a linearization (or linear approximation) can be made about an equilibrium, $p^* = (x^*, y^*)$, as follows:

For $x' = F(x, y),\ y' = G(x, y)$, the linearized system is $z' = J(z - p^*)$, where J is the Jacobian matrix evaluated at p^*, i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{bmatrix}_{p^*} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}$$

The eigenvalues and eigenvectors of the Jacobian matrix, J, at an equilibrium point, p^*, determine the geometry of the phase portrait close to the equilibrium point p^*. These ideas can be extended to any autonomous system of ODEs. A parallel definition applies to a discrete dynamical system.

Linear pendulum Pendulum motion can be modeled by a nonlinear ODE, but there is an approximating linear ODE that works well for small angles θ, where $sin\theta \approx \theta$. In that case, the mathematics is the same as that discussed for the mass on a spring in Module 4.

Linear system A linear system of first-order ODEs has only terms that are linear in the state variables. The coefficients can be constants or functions (even nonlinear) of t.

Example: Here is a linear system with state variables x and y, and constant coefficients $a, b, \dots, h$:

$$x' = ax + by + c$$
$$y' = fx + gy + h$$

This can be written in matrix/vector form as:

$$z' = Az + k$$

$$z = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ f & g \end{bmatrix}, \qquad k = \begin{bmatrix} c & h \end{bmatrix}$$

The example can be extended to n state variables and an $n \times n$ matrix A. If $z = p^*$ is an equilibrium point of a linear system, then $k = -Ap^*$ and the system may be written as

$$z' = A(z - p^*)$$

What is special about a constant coefficient linear system is that linear algebra can be applied to find the general solution. See General solution (for linear ODEs).

Lissajous figures Jules Antoine Lissajous was a 19th-century French physicist who devised ingenious ways to visualize wave motion that involves more than one frequency. For example, try plotting the parametric curve $x_1 = \sin 2t,\ \ x_2 = \sin 3t$ in the x_1x_2-plane with $0 \le t \le 320$.

The graph of a solution $x_1 = x_1(t),\ x_2 = x_2(t)$ of

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{for } B \text{ a } 2 \times 2 \text{ constant matrix}$$

in the x_1x_2-plane is a Lissajous figure if the vector $(x_1(0), x_2(0))$ is not an eigenvector of B.

See also "Normal modes and frequencies."

Local IVP One-step methods for approximating solutions to the IVP

$$y' = f(t, y), \qquad y(t_0) = y_0$$

generate the $(n + 1)$st approximation, y_{n+1}, from the nth, y_n, by solving the local IVP

$$u' = f(t, u), \qquad u(t_n) = y_n$$

This is exactly the same ODE, but the initial condition is different at each step.

Logistic model The logistic equation is the fundamental model for population growth in an environment with limited resources. Many advanced models in ecology are based on the logistic equation.

For continuous models, the logistic ODE is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{k}\right)$$

where r is the intrinsic growth rate constant, and K is the carrying capacity.

For discrete models, the logistic map is

$$f_\lambda(x) = \lambda x\left(1 - \frac{x}{K}\right)$$

where λ is the intrinsic growth rate constant, and K is again the carrying capacity.

Mandelbrot Set In complex dynamics, for $f_c(z) = z^2 + c$, the Mandelbrot set is a bifurcation diagram in the complex c-plane, computed by coloring all c-values for which z does not iterate to infinity. It acts as a catalog of all the Julia sets for individual values of c.

The boundary of the Mandelbrot set is even more complicated than the boundary of a given Julia set. More detail appears at every level of zoom, but no two regions are exactly self-similar.

Two mathematicians at UCLA, R. Brooks and J. P. Matelsky, published the first picture in 1978. It is now called the Mandelbrot set, because Benoit Mandelbrot of the Thomas J. Watson IBM Research Center made it famous in the early 80s.

You can experiment with the Mandelbrot set in Module 13, on screens 3.1 and 3.4.

Matrix An $n \times n$ square matrix A of constants, where n is a positive integer, is an array of numbers arranged into n rows and n columns. The entry where the ith row meets the jth column is denoted by a_{ij}.

In ODEs we most often see matrices A as the array of coefficients of a linear system. For example, here is a planar linear system with a 2×2 coefficient matrix A:

$$\begin{aligned} x' &= 2x - 3y \\ y' &= 7x + 4y \end{aligned} \qquad A = \begin{bmatrix} 2 & -3 \\ 7 & 4 \end{bmatrix}$$

Mixing A function $f : R \to R$ is "mixing" if given any two intervals I and J there exists an $n > 0$ such that the nth iterate of I intersects J.

Modeling A mathematical model is a collection of variables and equations representing some aspect of a physical system. In our case, the equations are differential equations. Steps involved in the modeling process are:

1. State the problem.
2. Identify the quantities to which variables are to be assigned; choose units.
3. State laws which govern the relationships and behaviors of the variables.
4. Translate the laws and other data into mathematical notation.
5. Solve the resulting equations.
6. Apply the mathematical solution to the physical system.
7. Test to see whether the solution is reasonable.
8. Revise the model and/or restate the problem, if necessary.

Moment of inertia The moment of inertia, I, of a body B about an axis is given by

$$I = \iiint_B r^2 \rho(x, y, z)\, dV(x, y, z)$$

where r is the distance from a general point in the body to the axis and ρ is the density function for B. Each moment of inertia plays the same role as mass does in nonrotational motion, but, now, the shape of the body and the position of the axis play a role.

Newtonian damping A body moving through air (or some other medium) is slowed down by a resistive force that acts opposite to the body's velocity, v. In Newtonian damping (or Newtonian drag), the magnitude of the force is proportional to the square of the magnitude of the velocity, i.e., to the square of the speed:

$$\text{force} = -k|v|v \quad \text{for some positive constant } k$$

Newton's law of cooling The temperature, T, of a warm body immersed in a cooler outside medium of temperature T_{out} changes at a rate proportional to the temperature dfference,

$$\frac{dT}{dt} = k(T_{out} - T)$$

where T_{out} is assumed to be unaffected by T (unless stated otherwise). The same ODE works if T_{out} is larger than T (Newton's law of warming).

Newton's second law Newton's second law states that, for a body of constant mass,

$$\text{mass} \cdot \text{acceleration} = \text{sum of forces acting on body}$$

This is a differential equation, because acceleration is the rate of change of velocity, and velocity is the rate of change of position.

Nodal equilibrium The behavior of the trajectories of an autonomous system of ODEs is nodal at an equilibrium point if all nearby trajectories approach the equilibrium point with definite tangents as $t \to +\infty$ (nodal sink), or as $t \to -\infty$ (nodal source).

If the system is linear with the matrix of coefficients A, then the equilibrium is a nodal sink if all eigenvalues of A are negative, a nodal source if all eigenvalues are positive. This also holds at an equilibrium point of any nonlinear autonomous system, where A is the Jacobian matrix at the equilibrium point.

Nonautonomous ODE A system of ODEs with t occurring explicitly in the expressions for the rates is nonautonomous.

Nonlinear center point An equilibrium point of a nonlinear system, $x' = F(x, y)$, $y' = G(x, y)$, is a center if all nearby orbits are simple closed curves enclosing the equilibrium point.

Nonlinear ODE A nonlinear ODE or system has at least some dependent variables appearing in nonlinear terms (e.g., xy, $\sin x$, $\sqrt{x}$). Thus, linear algebra cannot be applied to the system overall. But, near an equilibrium (of which there are usually more than one for a nonlinear system of ODEs), a linearization is (usually) a good approximation, and allows analysis with the important roles of the eigenvalues and eigenvectors.

Nonlinear pendulum Newton's laws of motion give us

$$\text{force} = \text{mass} \times \text{acceleration}$$

In the circular motion of a pendulum of fixed length, L, at angle θ, acceleration is given by $L\theta''$. The only forces acting on the undamped pendulum are those due tension in the rod and gravity. The component of force in the direction in which the pendulum bob is moving:

$$F = mL\theta'' = -mgsin\theta$$

where m is the mass of the pendulum bob, and g is the acceleration due to gravity. The mass of the rigid support rod is assumed to be negligible.

Normal modes and frequencies The normal modes of a second order system $z'' = Bz$ (where B is a 2×2 matrix with negative eigenvalues μ_1, μ_2) are eigenvectors v_1, v_2 of B. The general solution is all linear combinations of the periodic oscillations z_1, z_2, z_3, z_4 along the normal modes.

$$z_1 = v_1 \cos\omega_1 t, \quad z_2 = v_1 \sin\omega_1 t,$$
$$z_3 = v_2 \cos\omega_2 t, \quad z_4 = v_2 \sin\omega_2 t$$

where $\omega_1 = \sqrt{-\mu_1}$, $\omega_2 = \sqrt{-\mu_2}$, are the normal frequencies.

See also "Second order systems."

Normalized ODE In a normalized differential equation, the the highest order derivative appears alone in a separate term and has a coefficient equal to one.

ODE See ordinary differential equation.

On-off function See square wave.

Orbit See trajectory.

Order of the method A method of numerical approximation to a solution of an IVP is order p, if there exists a constant C such that

$$\max(|\text{global error}|) < Ch^p$$

as $h \to 0$.

Ordinary differential equation An ordinary differential equation (ODE) is an equation involving an unknown function and one or more of its derivatives. The order of the ODE is the order of the highest derivative in the ODE. Examples:

$$\frac{dy}{dt} = 2t, \qquad \text{(first order, unknown } y(t))$$
$$\frac{dy}{dt} = 2y + t, \qquad \text{(first order, unknown } y(t))$$
$$x'' - 4x' + 7x = 4\sin 2t, \qquad \text{(second order, unknown } x(t))$$

Oscillation times Oscillation times of a solution curve $x(t)$ of an ODE that oscillates around $x = 0$ are the times between successive crossings of $x = 0$ in the same direction. If the solution is periodic, the oscillation times all equal the period.

Oscillations A scalar function $x(t)$ oscillates if $x(t)$ alternately increases and decreases as time increases. The oscillation is periodic of period T if $x(t+T) = x(t)$ for all t and if T is the smallest positive number for which this is true.

Parametrization Each coordinate of a point in space may sometime be given in terms of other variable(s) or parameter(s). A single parameter suffices to describe a curve in space. Two parameters are required to describe a two-dimensional surface.

Period The period of a periodic function $u(t)$ is the smallest time interval after which the graph of u versus t repeats itself. It can be found by estimating the time interval between any two corresponding points, e.g., successive absolute maxima.

The period of a cycle in a discrete dynamical system is the minimal number of iterations after which the entire cycle repeats.

Periodic phase plane The periodic xx'-phase plane for the pendulum ODE $x'' = 0.1x' + \sin x = \cos t$ is plotted periodically in x. An orbit leaving the screen on the right comes back on the left. In other words, the horizontal axis represents $x \bmod 2\pi$. This view ignores how many times the pendulum bob has gone over the top. See Module 12, screen 1.4.

Phase angle The phase angle, δ, of the oscillatory function $u(t) = A\cos(\omega_0 t + \delta)$ shifts the the graph of $u(t)$ from the position of a standard cosine graph $u = \cos\omega_0 t$ by the amount δ/ω_0 . The phase angle may have either sign and must lie in the interval $-\pi/\omega_0 < \delta < \pi/\omega_0$.

Phase plane The phase plane, or state plane, is the xy plane for the dependent variables x and y of the system

$$x' = F(x, y)$$
$$y' = G(x, y)$$

The trajectory, or orbit, of a solution

$$x = x(t), \qquad y = y(t)$$

of the system is drawn in this plane with t as a parameter. A graph of trajectories is called a phase portrait for the system.

The higher dimensional analog is called phase space, or state space.

Pitch The pitch (frequency) of an oscillating function $u(t)$ is the number of oscillations per unit of time t.

Poincaré Henri Poincaré (1854–1912) was one of the last mathematicians to have a universal grasp of all branches of the subject. He was also a great popular writer on mathematics. Poincaré's books sold over a million copies.

Poincaré section A Poincaré section of a second order ODE $x'' = f(x, x', t)$, where f has period T in t, is a strobe picture of the xx'-phase plane that plots only the points of an orbit that occur at intervals separated by a period of T time units, i.e., the sequence of points

$$\begin{aligned} P_0 &= (x(0), x'(0)) \\ P_1 &= (x(T), x'(T)) \\ &\vdots \\ P_n &= (x(nT), x'(nT)) \\ &\vdots \end{aligned}$$

This view of phase space was developed by Henri Poincaré in the early twentieth century, because it is especially useful for analyzing nonautonomous differential equations. For further detail, see the entire second submodule of Module 12, Chaos and Control.

A Poincaré section is a two-dimensional discrete dynamical system. Another example of such a system is discussed in some detail in the second submodule of Module 13.

Population quadrant In a two-species population model, the population quadrant of the phase plane is the one where both dependent variables are non-negative.

Post-image In a discrete dynamical system, a post-image of a set S_0 is another set of points, S_1 , where the iterates of S_0 land in one step.

For a Poincaré section of an ODE, S_1 would be the set of points arriving at S_1 when the ODE is solved from S_0 over one time period of the Poincaré section. See submodule 3 of Module 12.

Pre-image In a discrete dynamical system, a pre-image of a set S_0 is another set of points, S_{-1}, that iterate to S_0 in one step.

For a Poincaré section of an ODE, S_{-1} would be the set of points arriving at S_0 when the ODE is solved from S_{-1} over one time period of the Poincaré section. See submodule 3 of Module 12.

Products The products of a chemical reaction are the species produced by a reaction step. The end products are the species that remain after all of the reaction steps have ended.

Proportional Two variables are proportional if their ratio is constant. Thus, the circumference, c, of a circle is proportional to the diameter, because $c/d = 1$.

The basic linear differential equation

$$\frac{dy}{dt} = ky$$

represents a quantity y whose derivative is proportional to its value.

Random Random motion is the opposite of deterministic motion. In random motion, there is no way to predict the future state of a system from knowledge of the initial state. For example, if you get heads on the first toss of a coin, you cannot predict the outcome of the fifth toss.

Rate constant Example: The constant coefficients a, b, and c in the rate equation

$$x'(t) = ax(t) - by(t) - cx^2(t)$$

are often called rate constants.

Rates of chemical reactions The rate of a reaction step is the speed at which a product species is created or (equivalently) at which a reactant species is destroyed in the step.

Reactant A chemical reactant produces other chemicals in a reaction.

Resonance This phenomenon occurs when the amplitude of a solution of a forced second order ODE becomes either unbounded (in an undamped ODE) or relatively large (in a damped ODE) after long enough times.

Rotation system As Lagrange discovered in the 18th century, the equations of motion governing a gyrating body are

$$\omega_1' = \frac{(I_2 - I_3)\omega_2\omega_3}{I_1}$$
$$\omega_2' = \frac{(I_3 - I_1)\omega_1\omega_3}{I_2}$$
$$\omega_3' = \frac{(I_1 - I_2)\omega_1\omega_2}{I_3}$$

where I_j is the principal moment of inertia, and ω_j is the component of angular velocity about the jth body axis.

Saddle An equilibrium point of a planar autonomous ODE, or a fixed point of a discrete two-dimensional dynamical system, with the property that, in one direction (the unstable one), trajectories move away from it, while, in another direction (the stable one), trajectories move toward it.

At a saddle of an ODE, one eigenvalue of the associated linearization matrix must be real and positive, and at least one eigenvalue must be real and negative.

Scaling Before computing or plotting, variables are often scaled for convenience.

See also “Dimensionless variables.”

Second order systems Second order systems of the form $z'' = Bz$ often arise in modeling mechanical structures with no damping, (and hence, no loss of energy). Here, z is an n-vector state variable, z'' denotes d^2z/dt^2, and B is an $n \times n$ matrix of real constants.

Although numerical solvers usually require that we introduce $v = z'$ and enter the system of $2n$ first order ODEs, $z' = v, \; v' = Bz$, we can learn a lot about solutions directly from the eigenvalues and eigenvectors of the matrix B.

See also “Normal modes and frequencies” and Screen 3.4 in Module].

Sensitivity An ODE model contains elements, such as initial data, environmental parameters, and functions, whose exact values are experimentally determined. The effect on the solution of the model ODEs when these factors are changed is called sensitivity.

Sensitivity to initial conditions A dynamical system has sensitive dependence on initial conditions if every pair of nearby points eventually gets mapped to points far apart.

Separatrix Separatrices are trajectories of a planar autonomous system that enter or leave an equilibrium point p with definite tangents as $t \to \pm\infty$, and divide a neighborhood of p into distinct regions of quite different long-term trajectory behavior as t increases or decreases.

For more on separatrices see “Separatrices and Saddle Points” in Chapter 7.

Sink A sink is an equilibrium point of a system of ODEs, or a fixed point of a discrete dynamical system, with the property that all trajectories move toward the equilibrium.

If all eigenvalues of the associated linearization matrix at an equilibrium of a system of ODEs have negative real part, then the equilibrium is a sink.

Slope The slope of a line segment in the xy-plane is given by the formula

$$m = \frac{\text{change in } y}{\text{change in } x}$$

The slope of a function $y = f(x)$ at a point is the value of the derivative of the function at that point.

Slope field See direction field.

Solution A solution to a differential equation is any function which gives a true statement when plugged into the equation. Such a function is called a particular solution. Thus,

$$y = t^2 - 2$$

is a particular solution to the equation

$$\frac{dy}{dt} = 2t$$

The set of all possible solutions to a differential equation is called the general solution. Thus,

$$y = t^2 + C$$

is the general solution to the equation

$$\frac{dy}{dt} = 2t$$

Source A source is an equilibrium of a system of ODEs, or a fixed point of a discrete dynamical system, with the property that all trajectories move away from the equilibrium.

If all eigenvalues of the associated linearization matrix at an equilibrium of a system of ODEs have positive real part, then the equilibrium is a source.

Spiral equilibrium An equilibrium point of a planar autonomous system of ODEs is a spiral point if all nearby orbits spiral toward it (or away from it) as time increases.

If the system is linear with the matrix of coefficients A, then the equilibrium is a spiral sink if the eigenvalues of A are complex conjugates with negative real part, a spiral source if the real part is positive. This also holds at an equilibrium point of any nonlinear planar autonomous system, where A is the Jacobian matrix at the equilibrium.

Spring A Hooke's law restoring force (proportional to displacement, x, from equilibrium) and a viscous damping force (proportional to velocity, but oppositely directed) act on a body of mass m at the end of spring. By Newton's Second Law,

$$mx'' = -kx - bx'$$

where k and b are the constants of proportionality.

Spring force The spring force is often assumed to obey Hooke's law—the magnitude of the force in the spring is proportional to the magnitude of its displacement from equilibrium, and the force acts in the direction opposite to the displacement.

The proportionality constant, k, is called the spring constant. A large value of k corresponds to a stiff spring.

Square wave An on-off function (also called a square wave) is a periodic function which has a constant nonzero value for a fraction of each period; otherwise, it has a value of 0. For example, $y = A\mathrm{SqWave}(t, 6, 2)$ is a square wave of amplitude A and period 6, which is "on" for the first 2 units of its period of 6 units, then is off the next 4 time units.

Stable An equilibrium point p^* of an autonomous system of ODEs is stable if trajectories that start near p^* stay near p^*, as time advances. The equilibrium point $p^* = 0$ of the linear system $z' = Az$, where A is a matrix of real constants, is stable if all eigenvalues of A are negative or have negative real parts.

State space The phase plane, or state plane, is the xy-plane for the dependent variables x and y of the system

$$x' = F(x, y)$$
$$y' = G(x, y)$$

The trajectory, or orbit, of a solution

$$x = x(t), \qquad y = y(t)$$

of the system is drawn in this plane with t as a parameter. A graph of trajectories is called a phase portrait for the system.

The higher dimensional analog is called phase space, or state space.

State variables These are dependent variables whose values at a given time can be used with the modeling ODEs to determine the state of the system at any other time.

Steady state A steady state of a system of ODEs is an equilibrium position where no state variable changes with time.

Surface A surface of a three-dimensional object is just its two- dimensional "skin," and does not include the space or volume enclosed by the surface.

Taylor remainder For an $n+1$ times differentiable function $u(t)$, the difference (or Taylor remainder)

$$u(t) - [u(t_0) + hu'(t_0) + \cdots + \frac{1}{n!}h^n u^{(n)}(t_0) + \cdots]$$

can be written as

$$\frac{1}{(n+1)!}h^n u^{(n+1)}(c)$$

for some c in the interval $[t_0, t_0 + h]$, a fact which gives useful estimates.

Taylor series expansion For an infinitely differentiable function $u(t)$, the Taylor series expansion at t_0 for $u(t_0 + h)$ is

$$u(t_0) + hu'(t_0) + \frac{1}{2}h^2 u''(t_0) + \cdots + \frac{1}{n!}h^n u^{(n)}(t_0) + \cdots$$

Taylor series method Look at the IVP $y' = f(t, y)$, $y(t_0) = y_0$. For a step size h, the three-term Taylor series method approximates the solution $y(t)$ at $t_{n+1} = t_n + h$, for $n = 0, 1, 2, \ldots$, using the algorithm

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{1}{2}h^2 f_t(t_n, y_n)$$

Trace The trace of a square matrix is the sum of its diagonal entries. So

$$\mathrm{tr}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$$

Trace-determinant parabola The eigenvalues λ_1, λ_2 of a 2×2 matrix A are given by

$$\lambda_1, \lambda_2 = \frac{\mathrm{tr}\,A \pm \sqrt{\mathrm{tr}^2 A - 4 \det A}}{2}$$

The trace-determinant parabola, $4 \det A = \mathrm{tr}^2 A$, divides the $\mathrm{tr}\,A - \det A$ plane into the upper region where A's eigenvalues are complex conjugates and the lower region where they are real. The two eigenvalues are real and equal on the parabola.

Trajectory A trajectory (or orbit, or path) is the parametric curve drawn in the xy-plane, called the phase plane or state plane, by $x = x(t)$ and $y = y(t)$ as t changes, where $x(t)$, $y(t)$ is a solution of

$$x' = F(x, y, t)$$
$$y' = G(x, y, t)$$

The trajectory shows how $x(t)$ and $y(t)$ play off against each other as time changes.

For a higher dimensional system, the definition extends to parametric curves in higher dimensional phase space or state space.

Unstable An equilibrium point p^* of an autonomous system of ODEs is unstable if it is not stable. That means there is a neighborhood N of p^* with the property that, starting inside each neighborhood M of p^*, there is at least one trajectory that goes outside N as time advances.

Vector A vector is a directed quantity with length. In two dimensions, a vector can be written in terms of unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, directed along the positive x and y axes.

Viscous damping A body moving through air (or some other medium) is slowed down by a resistive force that acts opposite to the body's velocity, v. In viscous damping (or viscous drag), the force is proportional to the velocity:

$$\text{force} = -kv$$

for some positive constant k.

Wada property The Wada property, as described and illustrated on Screen 3.2 of Module 12 is the fact that:

> Any point on the boundary of any one of the areas describe on Screen 3.2 is also on the boundary of all the others.

The geometry/topology example constructed by Wada was the first to have this property; we can now show that the basins of attraction for our forced, damped pendulum ODE have the same property. See Module 12 and Chapter 12.

All we know about Wada is that a Japanese manuscript asserts that someone by that name is responsible for constructing this example, showing that for three areas in a plane, they can become so utterly tangled that every boundary point touches all three areas!