Basic Counting Principle

Permutations

$$P_r^n = (n)_r = \frac{n!}{(n-r)!},\ r \le n$$

$$n! = n(n-1)(n-2)\times\ldots\times 1$$

$$\binom{n}{n_1\ n_2\ \ldots n_k} = \frac{n!}{n_1!\times n_2!\times\ldots\times n_k!}$$

Combinations

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!},\ r \le n$$

Probability

Classical probability

$$P(E) = \frac{\text{numberof elements in } E}{\text{numberof elements in } S}$$

Empirical probability

$$P(E) = \frac{m}{n}$$

Addition Rule

If A and B are mutually exclusive events in S:

$$P(A \cup B) = P(A) + P(B)$$

If A and B are not mutually exclusive events in S:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Multiplication Rule

If A and B are independent events in S:

$$P(A \cap B) = P(A) \times P(B)$$

If A and B are dependent events in S:

$$P(A \cap B) = P(A) \times P(B|A)$$

Conditional Probability

$$P(B|A) = \frac{P(A \cap B)}{P(A)},\ P(A) > 0$$

The law of total probability

Let $A_1, A_2, \ldots, A_n$ be a partition of S. Let D be an event defined on S , then :

$$P(D) = \sum_{i=1}^{n} P(A_i)P(D|A_i)$$

Baye's Theorem

Let $A_1, A_2, \ldots, A_n$ be a partition of S. Let D be an event defined on S , then :

$$P(A_k|D) = \frac{P(A_k)P(D|A_k)}{\sum_{i=1}^{n} P(A_i)P(D|A_i)},\ k = 1,2,3,\ldots,n$$

Discrete random variable

Expected value

$$E(X) = \mu = \sum x\ f(x).$$

Variance

$$\sigma^2 = V(X) = [\sum x^2 f(x)] - [E(X)]^2$$

Standard deviation

$$\sigma = \sqrt{V(X)}.$$

r^{th} non-central moments

$$\mu_r' = E(X^r) = \sum x^r\ f(x)$$

Cumulative distribution function

$$F_x(x) = P(X \le x)$$

$$f(x_i) = F(x_i) - F(x_{i-1})$$

$$P(a < X \le b) = F(b) - F(a)$$

$$P(a \le X \le b) = F(b) - F(a) + f(a)$$

$$P(a \le X < b) = F(b) - F(a) + f(a) - f(b)$$

$$P(a < X < b) = F(b) - F(a) - f(b).$$

Probability generating function

$$G(t) = E(t^X) = \sum t^X f(x)$$

$$G'(t)\big|_{t=1} = G'(1) = E(x)$$

$$.G''(t)\big|_{t=1} = G''(1) = E[X(X-1)]$$

$$Var(X) = G_x''(1) + G_x'(1) - (G_x'(1))^2.$$

Moment generating function

$$M(t) = E(e^{tx}) = \sum e^{tx}\ f(x)$$

$$M'(t) = E(X e^{tx})\big|_{t=0} = M'(0) = E(X)$$

$$M''(t) = E(X^2 e^{tx})\big|_{t=0} = M''(0) = E(X^2)$$

Discrete Probability Distributions

Bernoulli Distribution(p)

$$f_X(x) = p^x q^{1-x}, x = 0,1$$

$$E(X) = p, V(X) = pq$$

$$G(t) = q + pt$$

$$M(t) = q + pe^t$$

Binomial Distribution (n, p)

$$f_X(x) = C_x^n p^x\ q^{n-x},\ x = 0,1,\ldots,n,$$

$$E(X) = np, V(X) = npq$$

$$G(t) = (q + pt)^n$$

$$M(t) = (q + pe^t)^n$$

Poisson Distribution (λ)

$$f_X(x) = \frac{\lambda^x\ e^{-\lambda}}{x!},\ x = 0,1,\ldots$$

$$E(X) = \lambda,\ V(X) = \lambda$$

$$M(t) = e^{\lambda(e^t - 1)}$$

Geometric Distribution (p)

$$f_X(x) = p\, q^{x-1},\ x = 1, 2, \ldots$$

$$E(X) = 1/p, V(X) = q/p^2$$

$$G(t) = \frac{p\,t}{1-q\,t}$$

$$M(t) = \frac{p\,e^t}{1-q\,e^t}$$

Negative Binomial Distribution (k, p)

$$f_X(x) = \binom{x-1}{k-1} p^k\ q^{x-k},\ k = 1,2,\ldots$$

$$x = k, k+1, \ldots, E(X) = k/p,\ V(X) = kq/p^2$$

$$G(t) = \left(\frac{p\,t}{1-q\,t}\right)^k$$

$$M(t) = \left(\frac{p\,e^t}{1-q\,e^t}\right)^k$$

Hyper geometric Distribution (n,k,N)

$$f_X(x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, \qquad x = 0,1,\ldots,n$$

$$E(X) = n\frac{k}{N},\ V(X) = n\frac{k}{N}\left(1-\frac{k}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Continuous random variable

Expected value

$$E(X) = \mu = \int_X x\ f(x)\,dx$$

Variance

$$\sigma^2 = Var(X) = \left[\int_X x\ f(x)\,dx\right] - [E(X)]^2$$

Standard deviation

$$\sigma = \sqrt{Var(X)}.$$

r^th non-central moments

$$\mu_r' = E(X^r) = \int_X x^r\ f(x)\,dx$$

Cumulative Function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

$$f(x) = \frac{d}{dx}F(x)$$

Continuous Distributions

Uniform Distribution

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & ,a < x < b \\ 0 & ,\text{otherwise} \end{cases}$$

$$E(X) = \frac{b+a}{2},\ V(X) = \frac{(b-a)^2}{12}$$

$$M(t) = \frac{e^{bt}-e^{at}}{t(b-a)}$$

Gamma Distribution

$$f_X(x) = \frac{\theta^m}{\Gamma m} x^{m-1}\, e^{-\theta x}\ ,\ x \ge 0,\ \theta > 0$$

$$E(X) = m/\theta,\ V(X) = m/\theta^2$$

$$M(t) = \left(1-\frac{t}{\theta}\right)^{-m}, t < \theta$$

$$\Gamma m = \int_0^\infty x^{m-1} e^{-x}\,dx = (m-1)!$$

Exponential Distribution

$$f_X(x) = \lambda\, e^{-\lambda x}\ ,\ x \ge 0,\ \lambda > 0$$

$$E(X) = 1/\lambda,\ V(X) = 1/\lambda^2,\ \ M(t) = \left(1-\frac{t}{\lambda}\right)^{-1}, t < \lambda$$

Beta Distribution

$$f_X(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1}\ ,\ 0 < x < 1,\ a,b > 0$$

$$E(X) = \frac{a}{a+b},\ V(X) = \frac{ab}{(a+b+1)(a+b)^2}$$

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

Normal Distribution

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

$$-\infty < \mu < \infty, \sigma > 0$$

$$E(X) = \mu,\ V(X) = \sigma^2,\ M(t) = e^{\left(\frac{\sigma^2 t^2}{2}\right)+\mu t}$$

Standard Normal Distribution

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, -\infty < z < \infty$$

$$Z = \frac{X-\mu}{\sigma},$$

$$E(Z) = 0,\ V(Z) = 1,$$

$$M_Z(t) = e^{\frac{t^2}{2}}$$

Empirical Normal Rule

68% of data lies between(μ-σ,μ+σ)

95% of data lies between(μ-2σ,μ+2σ)

99.7% of data lies between(μ-3σ,μ+3σ)