Modeling Difficulties

- Poorly Known Processes
  - V-L and L- L thermodynamic equlb. In multi-component distillation
  - Multi-component reaction systems
  - Interaction between heat , mass and momentum

- Error in model parameters
  - U (over all heat transfer coefficient)
  - T_d
  - K_o and E in $k=k_o\, e^{E/RT}$

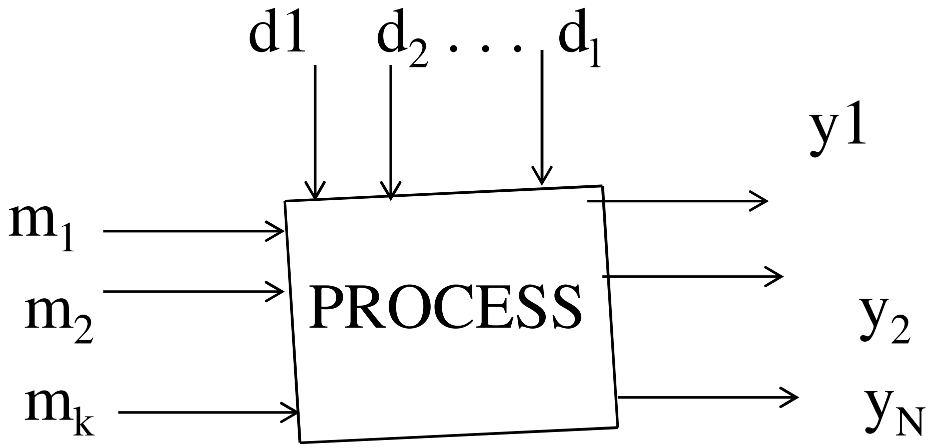
- Complexity of model (

  Distillation column with N trays need [2N+4] equations

Accurate model $\longrightarrow$ Complex

Input-Output Model

- Convenient for control purposes



$$y_i = f(m_1, m_2, \ldots, m_k, d_1, d_2, \ldots, d_l)$$

$$i = 1,2,3,\ldots\ldots\ldots\ldots,n$$

Solution of ODEs

- Modeling results in nonlinear sets of ordinary differential equations
- Solution requires numerical integration

- To get solution, we must first:
  - specify all constants (densities, heat capacities, etc, …)
  - specify all initial conditions
  - specify types of perturbations of the input variables

For the heated stirred tank,

$$\frac{dT}{dt} = \frac{F}{V}(T_{in} - T) + \frac{Q}{\rho V C_P}$$

- specify ρ, $C_{P,}$ and V
- specify T(0)
- specify Q(t) and F(t)

Input Specifications

- Study of control system dynamics

  Observe the time response of a process output to input changes

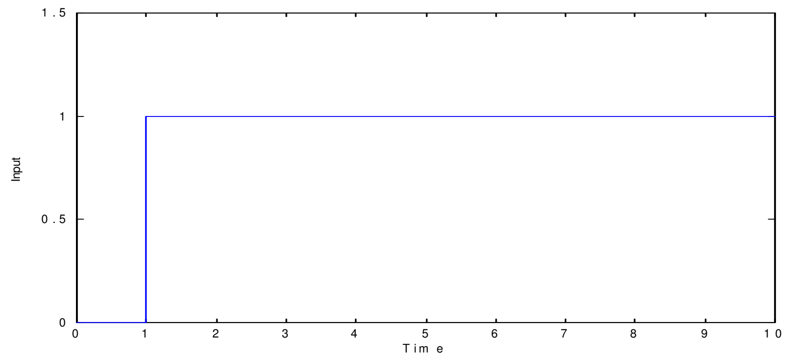
- Focus on specific inputs

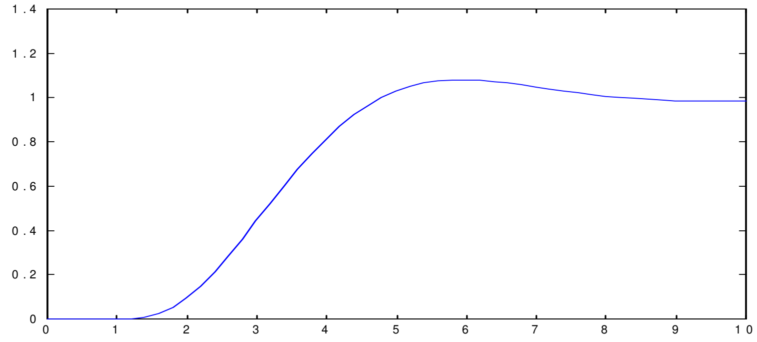
  1. Step input signals
  2. Ramp input signals
  3. Pulse and impulse signals
  4. Sinusoidal signals
  5. Random (noisy) signals

Common Input Signals

1. Step Input Signal: a sustained instantaneous change

e.g. Unit step input introduced at time 1

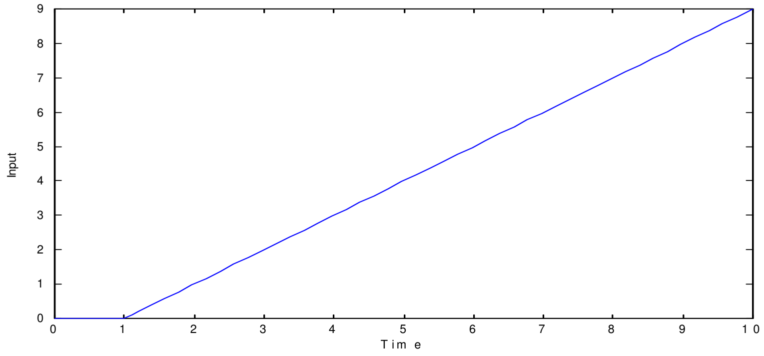


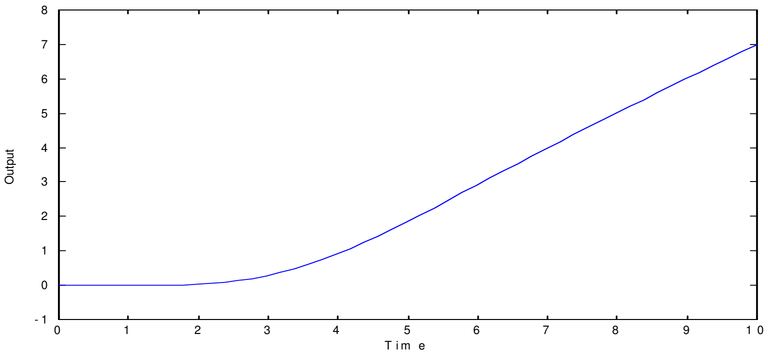


Common Input Signals

2. Ramp Input: A sustained constant rate of change

e.g.

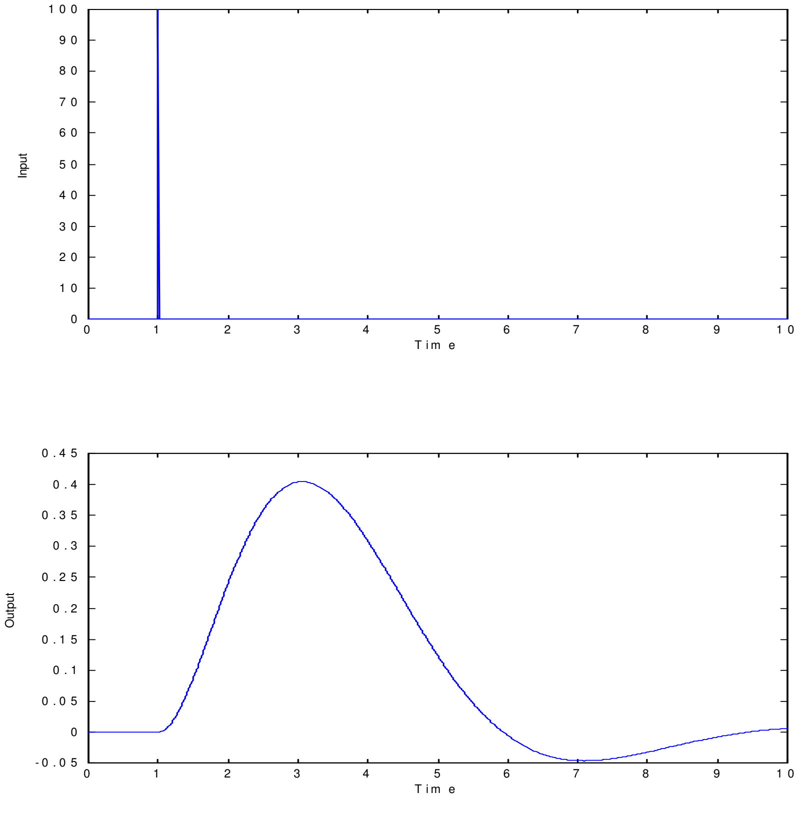




Common Input Signals

3. Pulse: An instantaneous temporary change

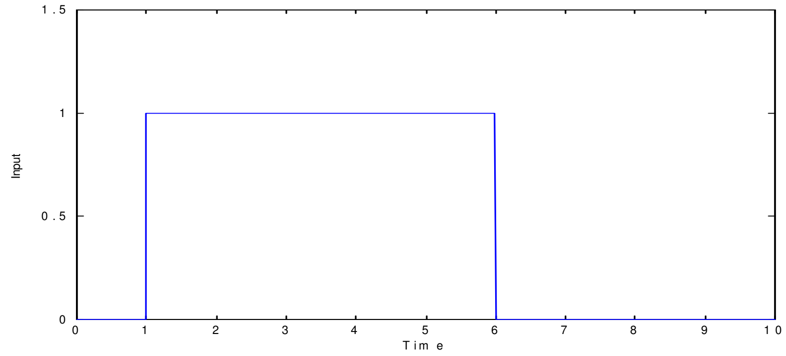
e.g. Fast pulse (unit impulse)

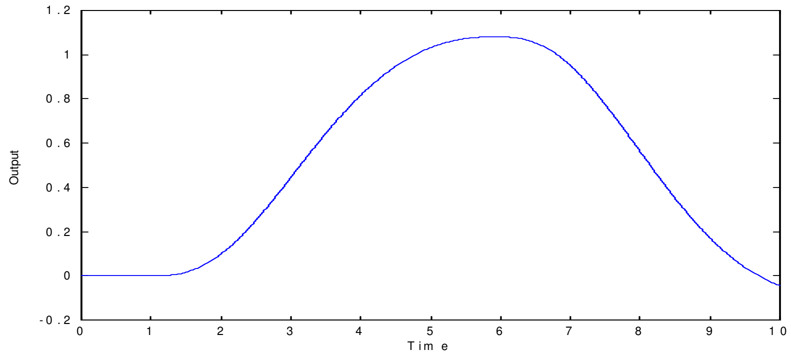


Common Input Signals

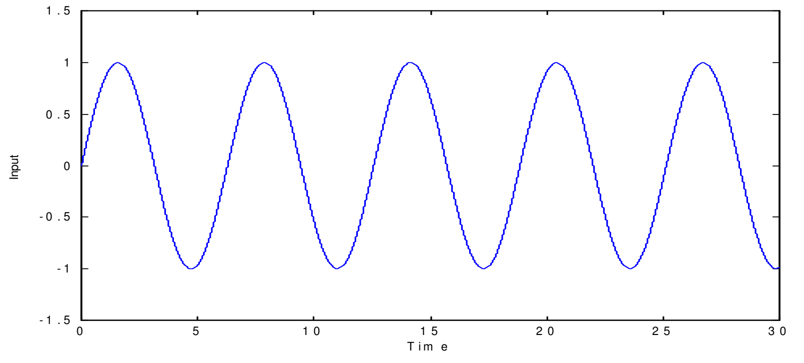
3. Pulses:

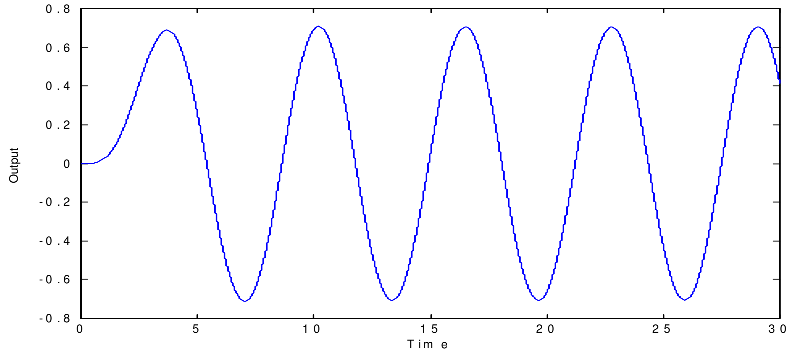
e.g. Rectangular Pulse





4. Sinusoidal input





Laplace Transform

$$F(s) = L(f(t)) = \int_0^\infty f(t)e^{-st}\,dt$$

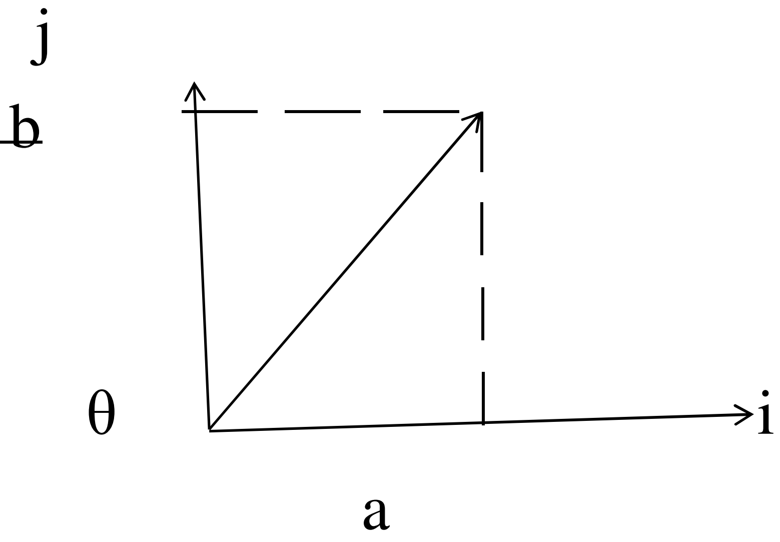
f(t) has to be at least stepwise continues

It is transform. from time domain (t) to Laplace domain (s)

S= a+bj

$$|s| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}(b/a)$$



Common Transforms

1. Constant

$$f(t) = a$$

$$\Im[a] = \int_0^\infty a e^{-st} dt = (-ae^{-st}/s)_0^\infty = a/s$$

2. Step

$$f(t) = \begin{cases} 0 & t < 0 \\ a & t \geq 0 \end{cases}$$

$$\Im[f(t)] = \int_0^\infty a e^{-st} dt = (-ae^{-st}/s)_0^\infty = a/s$$

3. Ramp function

$$f(t) = \begin{cases} 0 & t < 0 \\ at & t \geq 0 \end{cases}$$

$$\Im[f(t)] = \int_0^\infty ate^{-st} dt = -\frac{e^{-st}at}{s}\Bigg]_0^\infty + \int_0^\infty \frac{ae^{-st}}{s} dt = \frac{a}{s^2}$$

4. Rectangular Pulse

$$f(t) = \begin{cases} 0 & t < 0 \\ a & 0 \le t < t_w \\ 0 & t \ge t_w \end{cases}$$

$$\Im[f(t)] = \int_0^{t_w} a e^{-st} dt = \frac{a}{s}(1 - e^{-t_w s})$$

5. Unit impulse

$$\Im[\delta(t)] = \lim_{t_w \to 0} \frac{1}{t_w s}(1 - e^{-t_w s})$$

$$\Im[\delta(t)] = \lim_{t_w \to 0} \frac{s e^{-t_w s}}{s} = 1$$

1. Exponential

$$f(t) = e^{-bt}$$

$$\Im[e^{-bt}] = \int_0^\infty e^{-bt} e^{-st} dt = \int_0^\infty e^{-(s+b)t} dt$$

$$\Im[e^{-bt}] = -\left.\frac{e^{-(s+b)t}}{s+b}\right]_0^\infty = \frac{1}{s+b}$$

2. Cosine

$$f(t) = \cos(\omega t) = \frac{e^{-j\omega t} + e^{j\omega t}}{2}$$

$$\Im[\cos(\omega t)] = \frac{1}{2}\left\{\int_0^\infty e^{-(s-j\omega)t} dt + \int_0^\infty e^{-(s+j\omega)t} dt\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right\} = \frac{s}{s^2+\omega^2}$$

3. Sine (ωt)

$$f(t) = \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\Im[\sin(\omega t)] = \frac{1}{2j}\left\{\int_0^\infty e^{-(s-j\omega)t}dt - \int_0^\infty e^{-(s+j\omega)t}dt\right\}$$

$$= \frac{1}{2j}\left\{\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right\} = \frac{\omega}{s^2+\omega^2}$$

Common Transforms

1. Derivative of a function f(t)

$$\frac{df(t)}{dt}$$

$$du = df$$

$$v = e^{-st}$$

$$\Im[\frac{df}{dt}] = uv]_0^\infty - \int_0^\infty u\,dv = f(t)e^{-st}\Big]_0^\infty - \int_0^\infty (-sf(t)e^{-st})dt$$

$$\Im[\frac{df}{dt}] = s\int_0^\infty f(t)e^{-st}dt - f(0) = sF(s) - f(0)$$

2. Integral of a function f(t)

$$\Im\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)d\tau\right)dt = \frac{F(s)}{s}$$

Final Value Theorem

$$\lim_{t \to \infty} [y(t)] = \lim_{s \to 0} [sY(s)]$$

Initial Value Theorem

$$y(0) = \lim_{s \to \infty} [sY(s)]$$

Algorithm for Solution of ODEs

- Take Laplace Transform of both sides of ODE and boundary conditions.
- Solve for $Y(s)=p(s)/q(s)$
- Factor the characteristic polynomial $q(s)$
- Perform partial fraction expansion
- Inverse Laplace using Tables of Laplace Transforms

Partial fraction Expansions

1. q(s) has real and distinct roots

$$q(s) = \prod_{i=1}^{n} (s + b_i)$$

expand as

$$r(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s + b_i}$$

2. q(s) has real but repeated roots

$$q(s) = (s + b)^n$$

expanded

$$r(s) = \frac{\alpha_1}{s + b} + \frac{\alpha_2}{(s + b)^2} + \quad + \frac{\alpha_n}{(s + b)^n}$$

- q(s) has imajnary roots

roots comes in the form of complex conjugates:

$$r_1, r_2 = a \pm\ b\, i$$

a = real part

b = imaginary part

$$r(s) = \frac{\alpha 1}{s + a\, i} + \frac{\alpha 1}{s - bi}$$

General solution :

$$Y\,(t) = C_1 e^{-p1t} + C_2 e^{-p2\,t} + \ldots\ldots C_n e^{-pnt} + C_{11} e^{-p11t} + C_{12} t\, e^{-p12t} + \ldots\ldots$$

$$C_{1m} t_{m-1} e^{-pm-1\ t} + C_{21} e^{-at} + C^{*}_{21}\ e^{bjt}$$

Example 8.1

$a_2\, d^2y/dt^2 + a_1\, dy/dt + a_o y(t) = f(t)$

f(t) is a unit step (f(t)= 1)

$y(o) = y`(o) = 0$

cas 1: $a_1=4$, $a_2= 1$, $a_o= 3$

$r_1= 0$, $r_2 = r_3 =-1$

cas 2: $a_1=2$, $a_2= 1$, $a_o= 1$

$r_1 = 0$, $r_2 =-1$, $r_3= -3$

cas 3: $a_1=2$, $a_2= 2$, $a_o= 1$

$r_1 = 0$, $r_2 =-1$, $r_3= -3$

Transfer Functions

A linear, n^{th}-order system is:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \quad + a_1 \frac{dy}{dt} + a_0 y = b f(t)$$

With initial conditions: $y(0)= y`(0)=y``(0)=\ldots\ldots y^{(n-1)`}(0) =0$

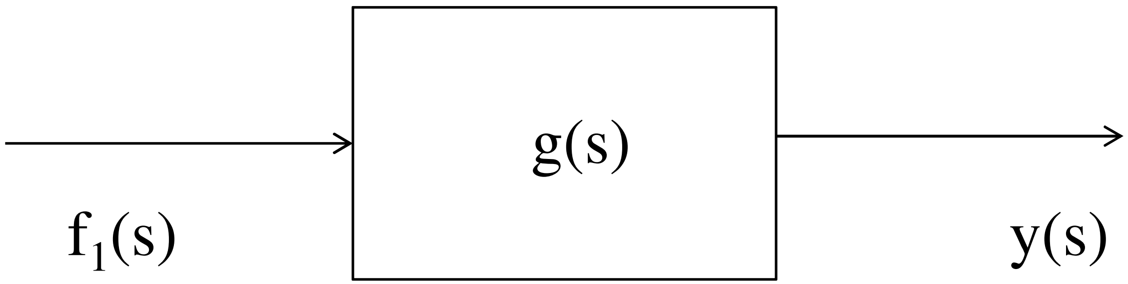
$$a_n s^n y(s)+a_{n-1} s^{n-1} y(s)+s^{n-2} y(s)\ldots\ldots+a_1 s y(s)+ a_0\, y(s) = b\, f(s)$$

$$y(s)(a_n s^n+a_{n-1} s^{n-1}+s^{n-2}\ldots\ldots+a_1 s+ a_{0)} = b\, f(s)$$

$$\frac{y(s)}{f(s)} = \frac{b}{(a_n s^n+a_{n-1} s^{n-1}+\ldots+a_1 s+ a_0)} = g(s) = \frac{\text{Output}}{\text{Input}}$$

g(s) is the transfer function

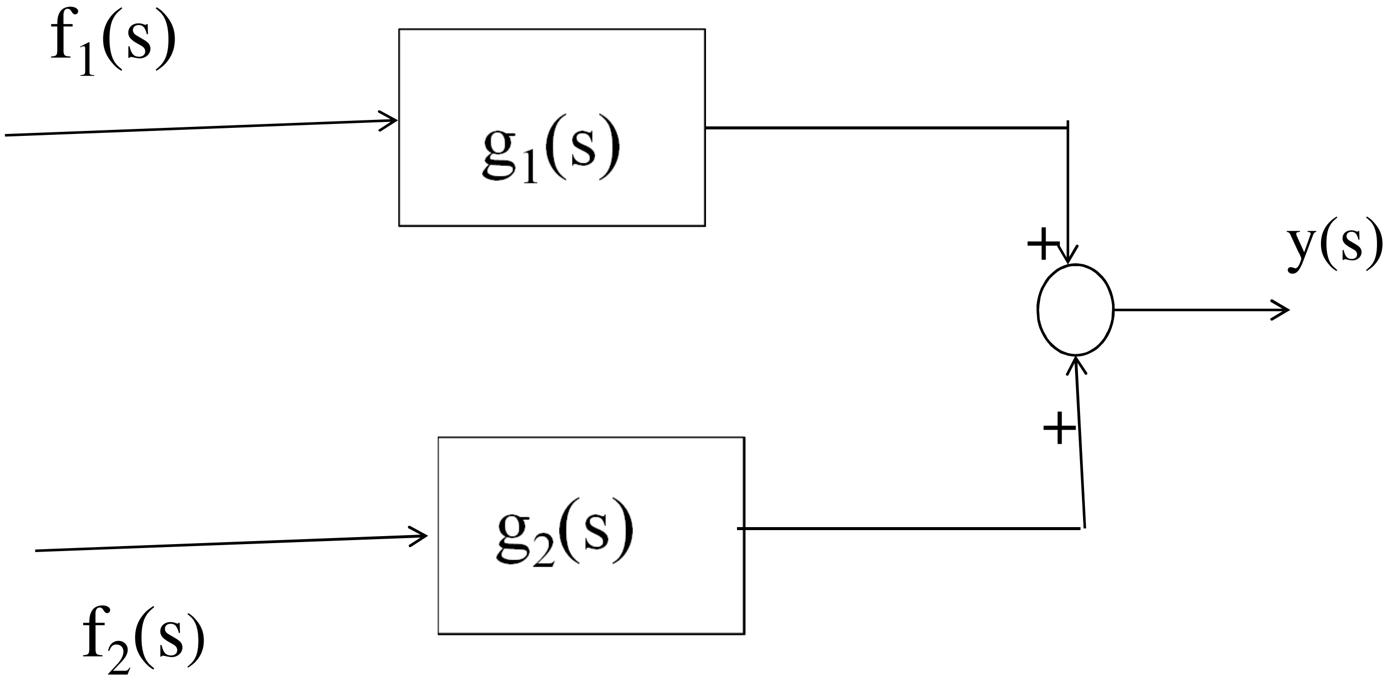
$y(s) = g(s) * f(s)$



For two inputs f_1 and f_2:

$$y(s) = \frac{b_1 \; f_1(s)}{(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0)} + \frac{b_2 \; f_2(s)}{(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0)}$$

$$y(s) = g_1(s) * f_1(s) + g_2(s) * f_2(s)$$



Note that the denominator of $g_1(s)$ and $g_2(s)$ are the same.

It is called characteristic equation.

Poles and zeros of Trasfer Function

$$g(s) = \frac{Q(s)}{P(s)}$$

$P(s) \equiv$ Characteristic Polynomial

$Q(s)$ has a lower order than $P(s)$

Roots of $Q(s)$ are called zeros of T. F. ($g(s) = 0$ at zeros)

Roots of $P(s)$ are called poles of T. F. ($g(s) = \infty$ at poles)

They play important roles in dynamics

$$\frac{y(s)}{f(s)} = g(s) \longrightarrow y(s) = g(s)^* f(s)$$

If $f(t)$ is given take L. T. of $f(t) \longrightarrow y(t) = L^{-1}[g(s)^* f(s)]$

$$g(s) = \frac{Q(s)}{P(s)} = \frac{Q(s)}{(s-p_1)\,(s-p_2)\,(s-p_3)^m\,(s-p_4)\,(s-p^*_4)}$$

wher p_1, p_2, p_3, p_4, p^*_4 are roots of P(s)

$$g(s) = \frac{Q(s)}{P(s)} = \frac{C_1}{(s-p_1)} + \frac{C_2}{(s-p_2)} + \frac{C_{31}}{(s-p_3)} + \frac{C_{32}}{(s-p_3)^2} + \ldots + \frac{C_{3m}}{(s-p_3)^m} + \frac{C_4}{(s-p_4)} + \frac{C_{*4}}{(s-p^*_4)}$$

$$f(t) = C \longrightarrow f(s) = C/s$$

$$y(t) = L^{-1}\,[g(s) * f(s)] = L^{-1}\,[g(s) * C/s]$$

$$y(t) = C_0 + C_1 e^{p1t} + C_2 e^{p2t} + \left[C_{31} + \frac{C_{32}}{1!} t + \frac{C_{33}}{2!} t^2 + \ldots + \frac{C_{3m}}{(m-1)!} t^{m-1}\right] e^{p3t} + C_4 e^{p4t} + C^*_4 e^{p^*4t}$$

$p_4 = a + bj$

$p^*_4 = a - bj$

$$e^{p4t} = e^{(a+bj)t} = e^{at} . e^{bjt} = e^{at} [\cos(bt) + j \sin (bt)]$$

$$e^{p^*4t} = e^{(a-bj)t} = e^{at} . e^{-bjt} = e^{at} [\cos(bt) - j \sin (bt)]$$

y(t) is stable if all poles and the real parts of the complex conjugates are negatives (p_i and $a_i < 0$) $\longrightarrow$ $y(t) = C_0$ as $t \rightarrow \infty$

y(t) is unstable if any of p_i and a_i are > 0 $\longrightarrow$ $y(t) \rightarrow \infty$ as $t \rightarrow \infty$

y(t) is marginally stable of all p_i and $a_i = 0$

First-order systems

Time-domain model

First order process

$$a_1 \frac{\mathrm{d}\,y}{\mathrm{d}\,t} + a_0 y = b\,u(t)$$

($a_0 \neq 0$, dividing by a_0)

$$\tau_P \frac{\mathrm{d}\,y}{\mathrm{d}\,t} + y = K_P\,u(t)$$

Laplace-domain model

$$y(s) = \left(\frac{K_P}{\tau_P s + 1} \right) u(s)$$

- K_P (b/a_0 is the process ***steady state gain*** (it can be >0 or <0)

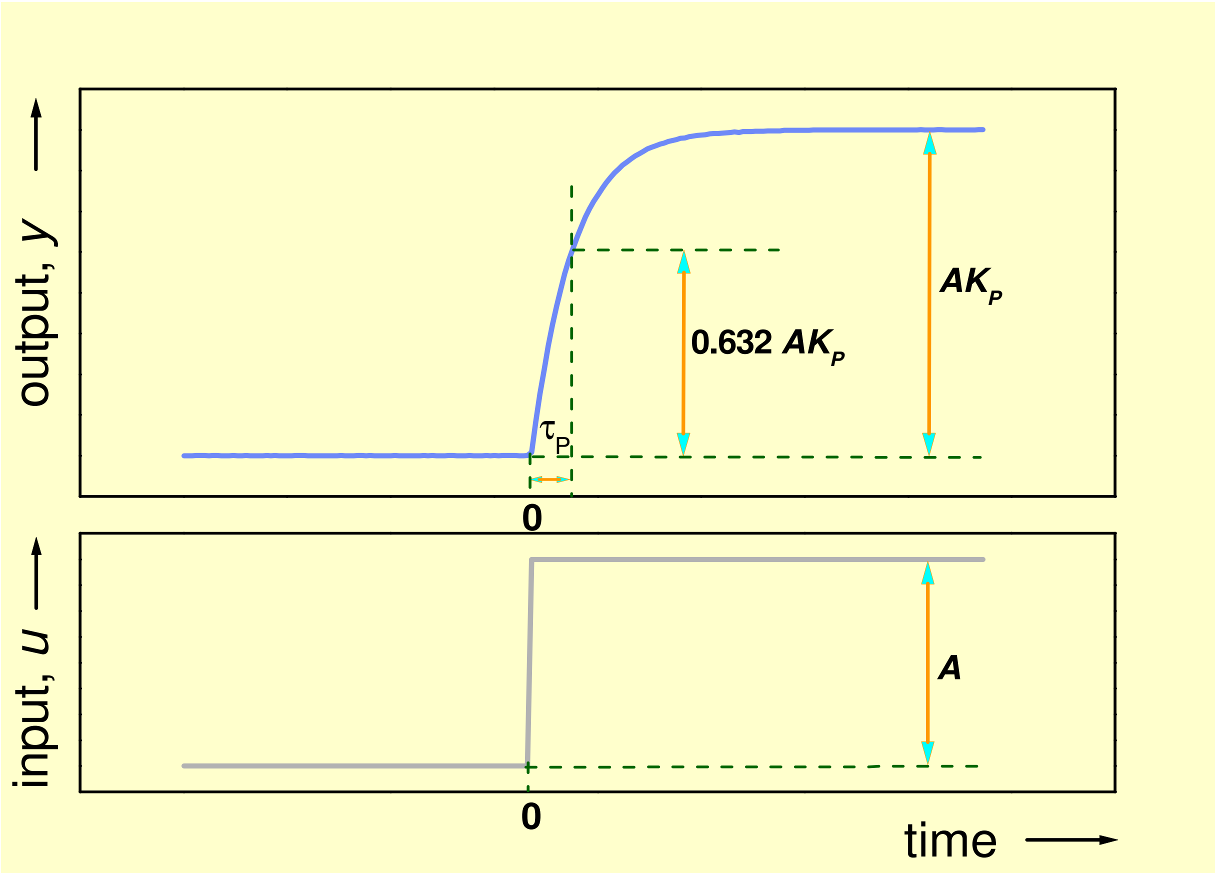
- τ_P (a_1/a_0) is the process ***time constant*** (it is always >0)

- Transfer function of a first-order system:

$$G(s) = \frac{K_P}{\tau_P s + 1}$$

Response of first-order systems

- We only consider the response to a *step* forcing function of amplitude *A*



The time-domain response is:

$$y(t) = AK_P\left(1 - e^{-\frac{t}{\tau_P}}\right)$$

Determining the process gain

An open-loop test can be performed starting from the reference steady state:

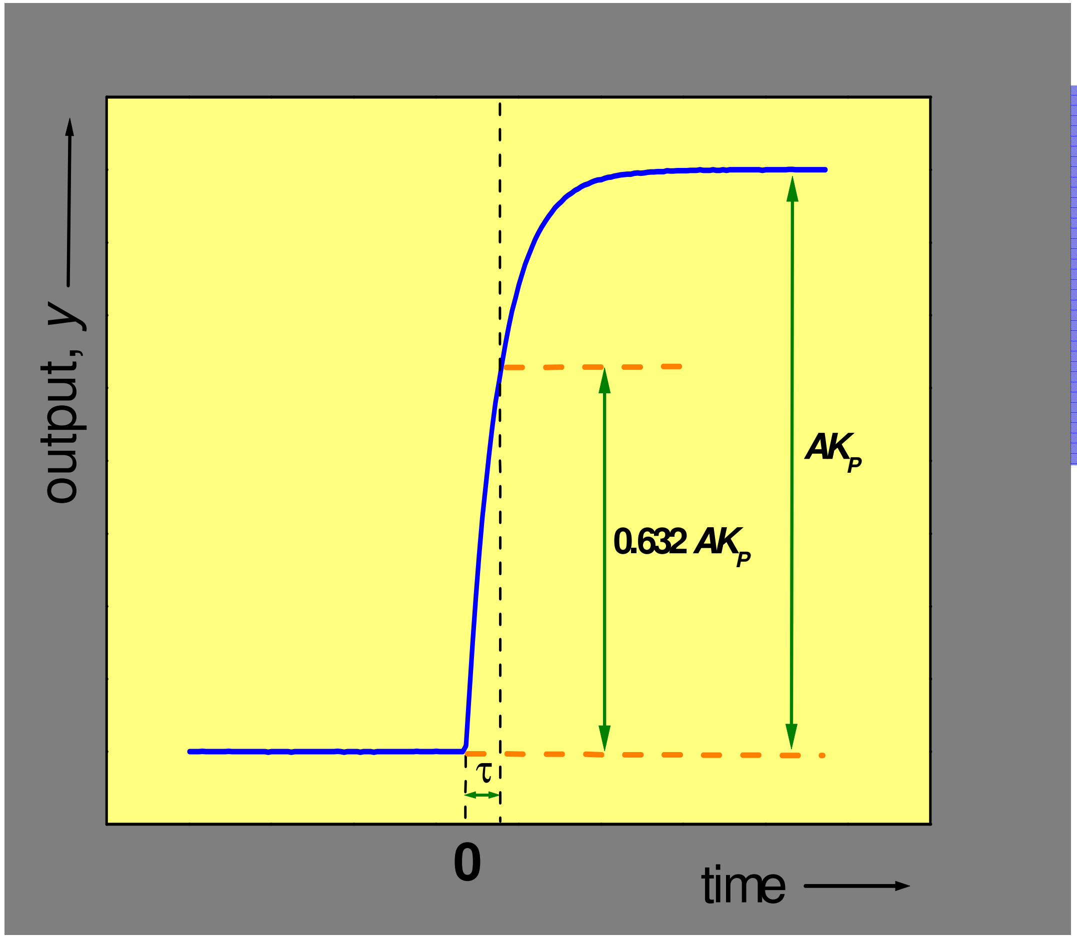
- step the input to the process
- record the time profile of the measured output until a new steady state is approached
- check if this profile resembles $y(t) = AK_P(1 - e^{-t/\tau_P})$
- if so, calculate K_P as:

$$K_P = \frac{y_{ss,new} - y_{ss,ref}}{u_{new} - u_{ref}} = \left(\frac{\Delta(\text{output})}{\Delta(\text{input})} \right)_{steady\,state}$$

The process gain can be determined from <u>steady state</u> information only

Determining the time constant

- From the same open-loop test:
  - determine τ_P graphically (note: it has the dimension of time)



You need <u>dynamic</u> information to determine the process time constant

Determining the values of K_P and τ_P from process data is known as *process identification*

Pure capacity process

$$a_1 \frac{\mathrm{d}\, y}{\mathrm{d}\, t} + a_0 y = b\, f(t)$$

Let $a_0 = 0$ then

$$a_1 \frac{\mathrm{d}\, y}{\mathrm{d}\, t} = b\, f(t)$$ divide by a_1 gives $$\frac{\mathrm{d}\, y}{\mathrm{d}\, t} = (b / a_1)\, f(t)$$

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, t} = K`\, f(t)$$ take L. T.

$y(s) = K`/s$ Pure capacity process

Second-order systems

- Time-domain representation:

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = bu(t)$$

Dividing by a_0 where $a_0 \neq 0$

$$(a_2 / a_0)\frac{d^2 y}{dt^2} + (a_1 / a_0)\frac{dy}{dt} + y = (b / a_0)u(t)$$

$$\tau^2 \frac{d^2 y}{dt^2} + 2\zeta\tau \frac{dy}{dt} + y = K_p u(t)$$

K_p = process gain

τ = natural period

ζ = damping coefficient

Laplace-domain representation:

$$\frac{Y(s)}{U(s)} = \frac{K_p}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Three cases of processes:

Overdamped $\xi > 1$

Critically Damped $\xi = 1$

Underdamped $\xi < 1$

Note: Chemical processes are typically overdamped or critically damped

Second Order Processes

- Roots of the characteristic polynomial

$$\frac{-2\xi\tau \pm \sqrt{4\xi^2\tau^2 - 4\tau^2}}{2\tau^2}$$

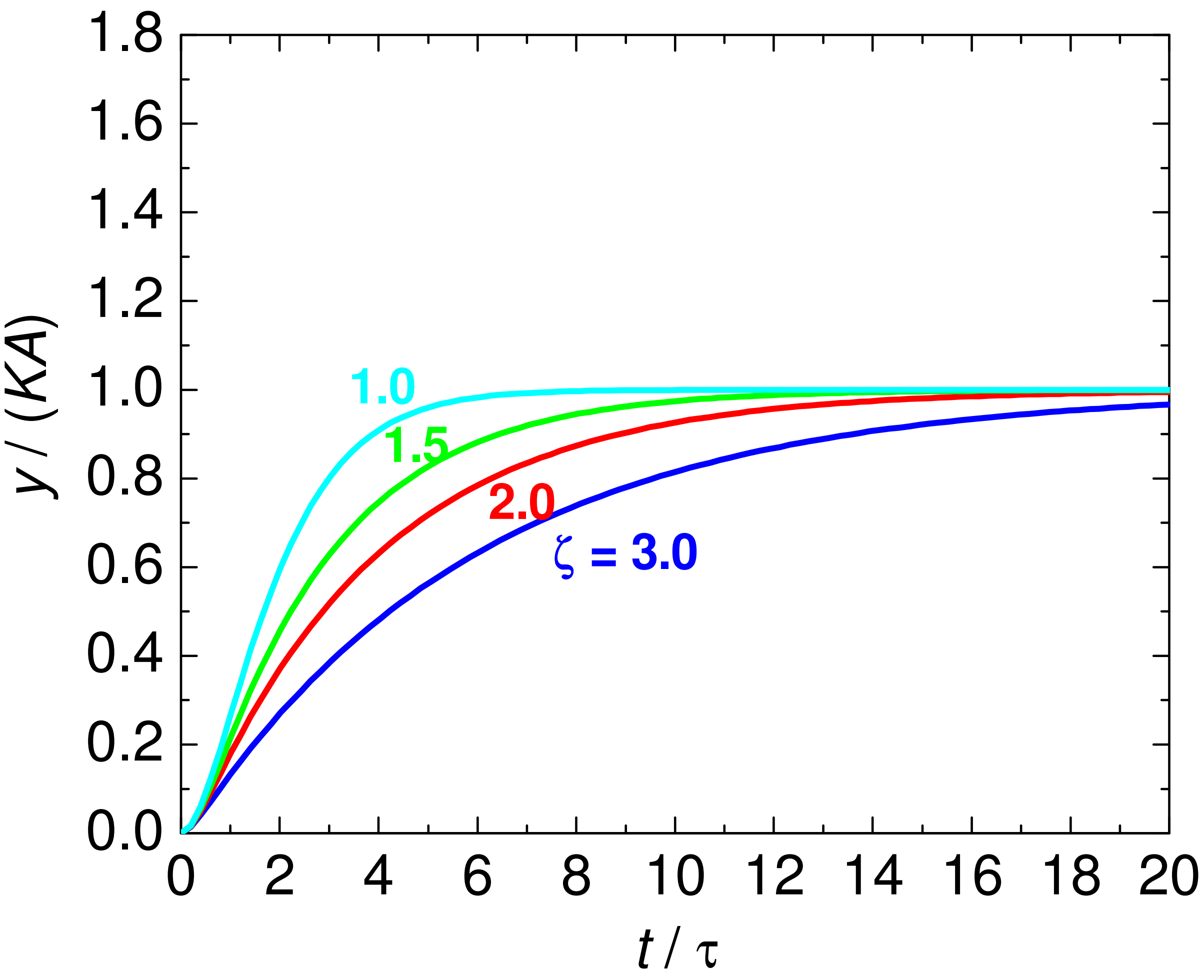
$$-\frac{\xi}{\tau} \pm \frac{1}{\tau}\sqrt{\xi^2 - 1}$$

Case 1) $\xi>1$: Two distinct real roots
System has an exponential behavior

Case 2) $\xi=1$: One multiple real root
Exponential behavior

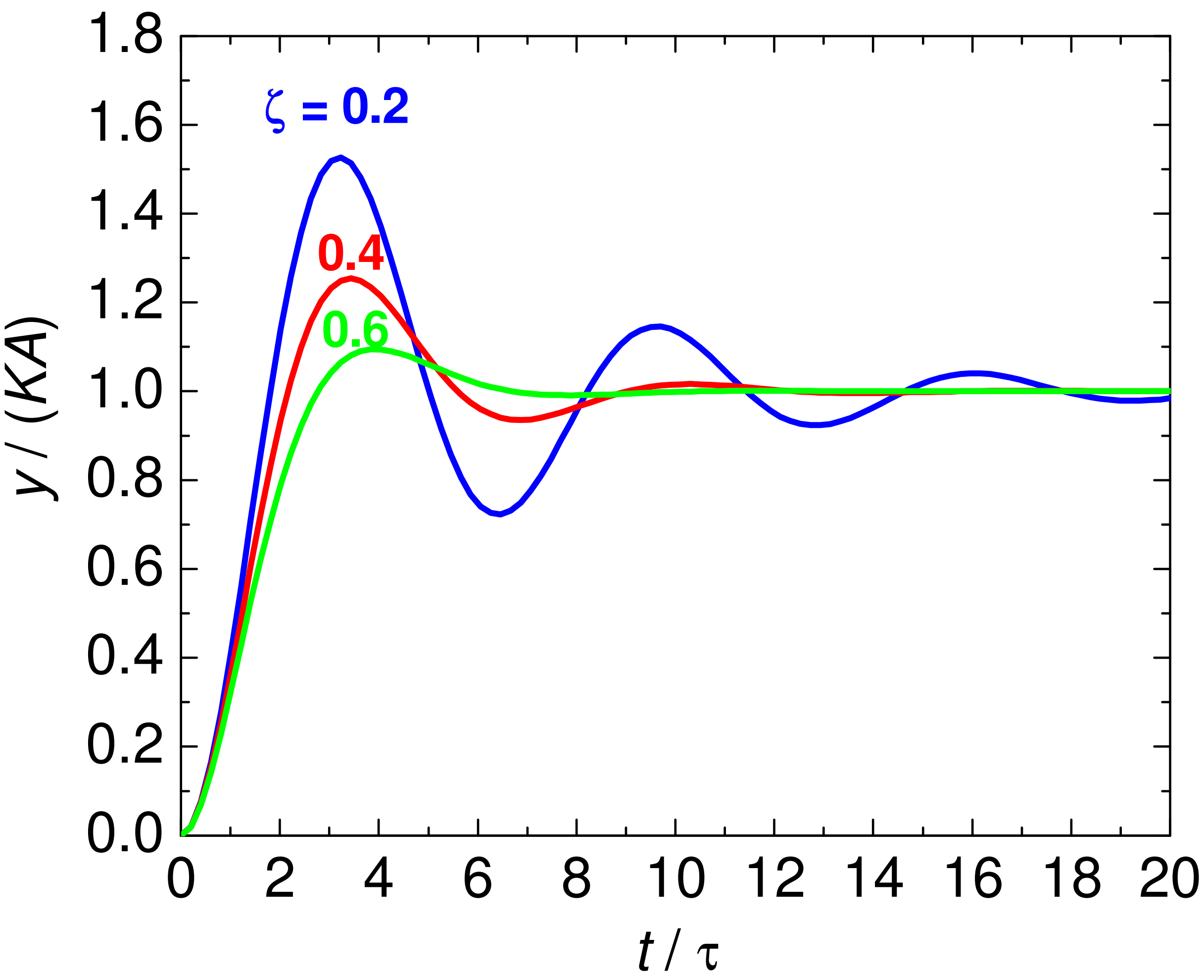
Case 3) $\xi<1$: Two complex roots
System has an oscillatory behavior

Overdamped systems (ξ>1)



Open-loop response to a input step disturbance

Underdamped systems (ξ<1)

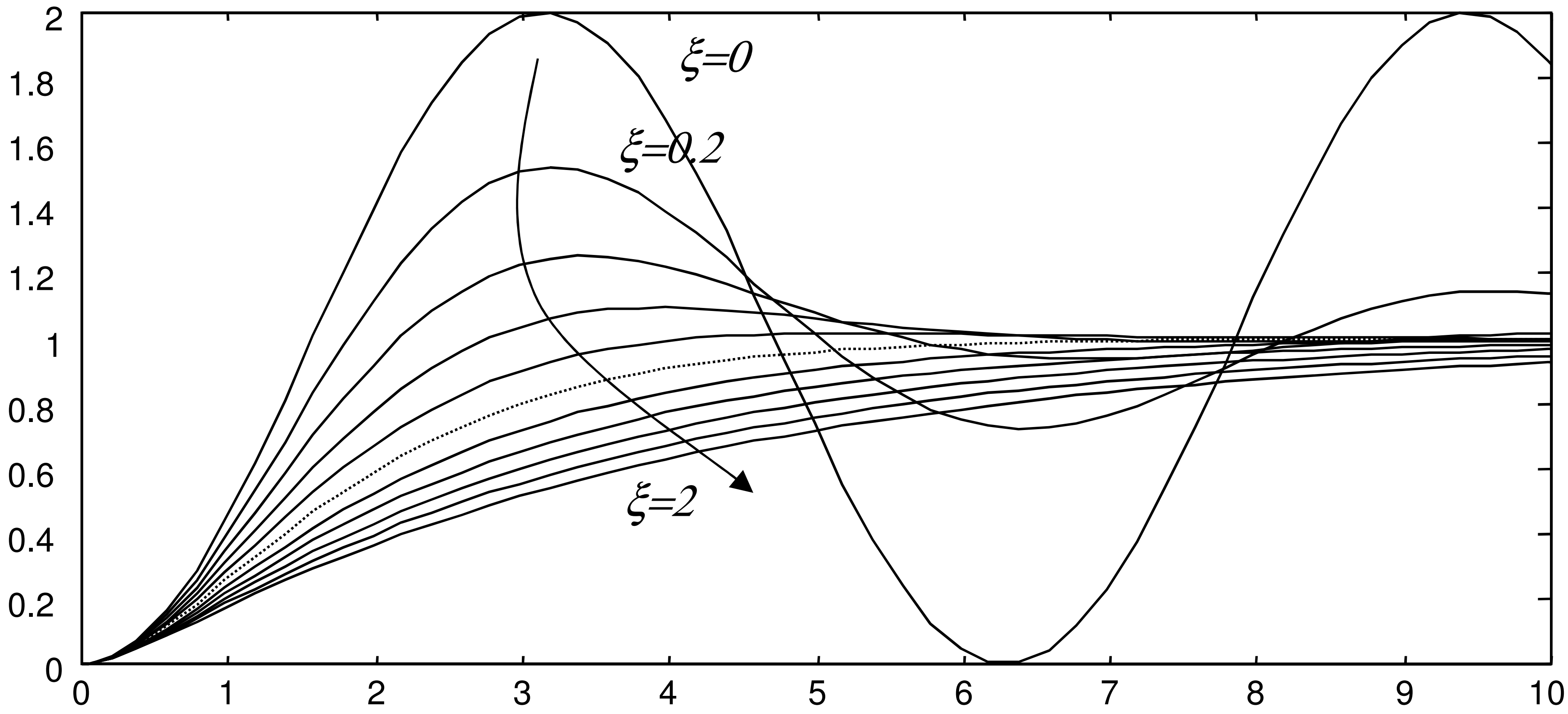


Open-loop response to a input step disturbance

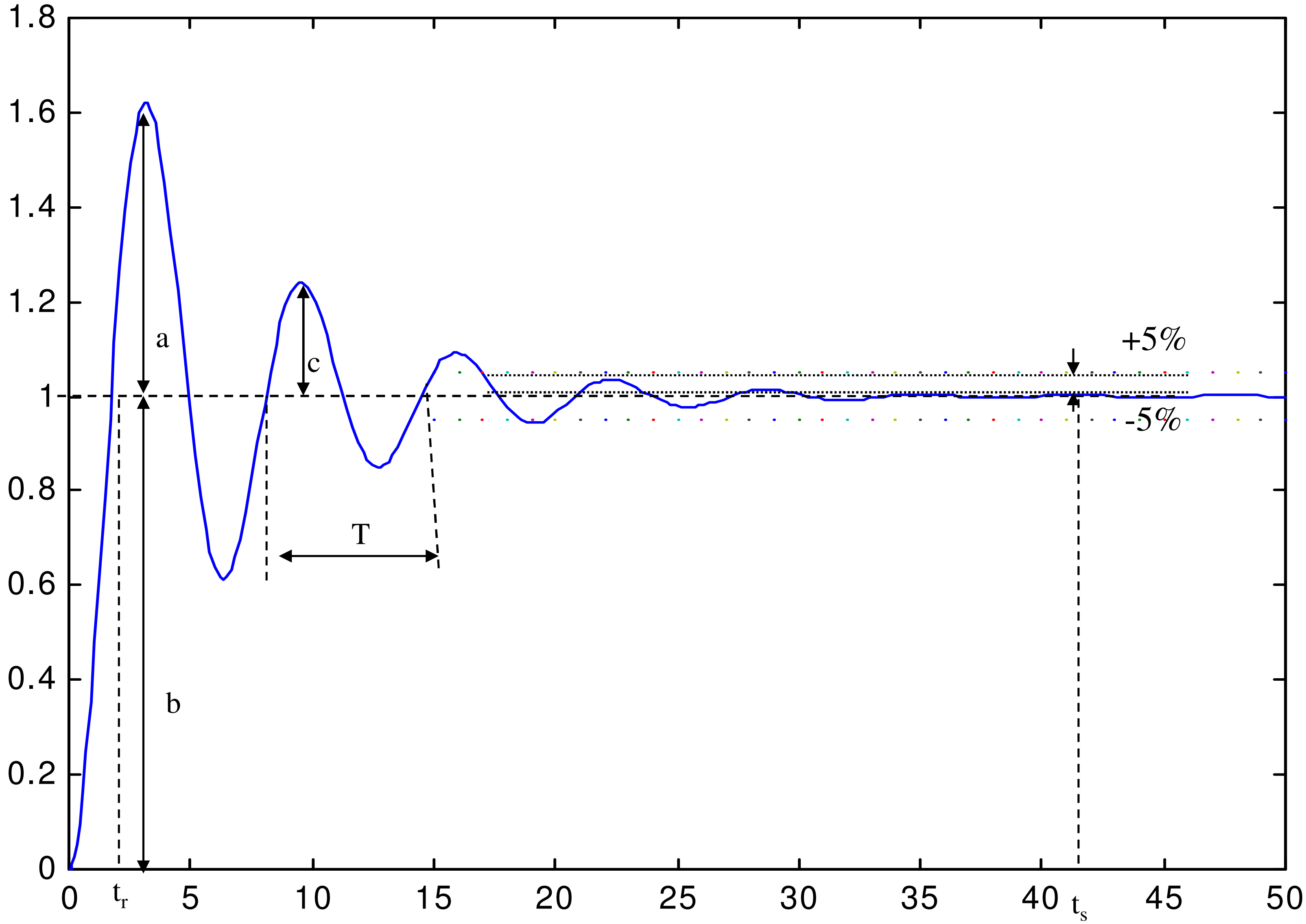
Second order Processes

Step response of magnitude M

$$Y(S) = \frac{K_P}{\tau^2 s^2 + 2\xi\tau s + 1}\frac{M}{s}$$



Characteristics of underdamped second order process



Characteristics of underdamped second order process

1. Rise time, t_r
2. Settling time (response time), t_s
3. Overshoot:

$$OS = \frac{a}{b} = \exp\left(-\frac{\xi}{\sqrt{1-\xi^2}}\pi\right)$$

5. Decay ratio:

$$DR = \frac{c}{b} = \exp\left(-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\right)$$

Period of oscillation (T)

/

$\mathbf{T = 2\pi\tau \; / \; (1-\xi^2)^{1/2}}$ $\qquad \omega = \textit{radian frequency} = (1-\xi^2)^{1/2} / \tau$