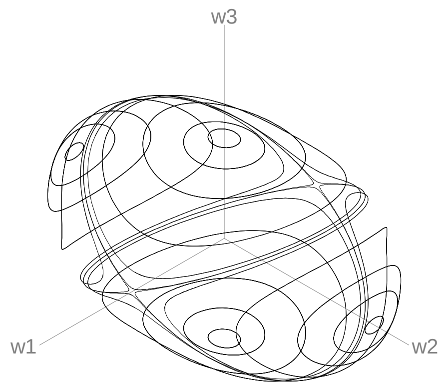
7 CHAPTER Nonlinear Systems



Angular velocity: twenty-four ways to spin a book.

Overview While many natural processes can be modeled by linear systems of ODEs, others require *nonlinear systems*. Fortunately, some of the ideas used to understand linear systems can be modified to apply to nonlinear systems. In particular, state (or phase) spaces and equilibrium solutions (as well as eigenvalues and eigenvectors) continue to play a key role in understanding the long-term behavior of solutions. You will also see some new phenomena that occur only in nonlinear systems. We restrict our attention to *autonomous* equations, that is, equations in which time does not explicitly appear in the rate functions.

Key words Nonlinear systems of differential equations; linearization; direction fields; state (phase) space; equilibrium points; Jacobian matrices; eigenvalues; separatrices; bifurcations; limit cycles; predator-prey; van der Pol system; saxophone; spinning bodies; conservative systems; integrals; angular velocity; nonlinear double pendulum

See also Chapter 6 for background on linear systems and Chapters 8–10 and 12 for more examples of nonlinear systems.

◆ Linear vs. Nonlinear

In modleing a dynamical process with ODEs we aim for a model that is both reasonably accurate and solvable. By the latter we mean that there are either explicit solution formulas that reveal how solutions behave, or reliable numerical solvers for approximating solutions. Constant-coefficient linear ODEs and linear systems have explicit solution formulas (see Chapters 4 and 6), and that is one reason linearity is widely assumed in modeling. However, nonlinearity is an essential feature of many dynamical processes, but explicit solution formulas for nonlinear ODEs are rare. So for nonlinear systems we turn to the alternative approaches and that's what this chapter is about.

◆ The Geometry of Nonlinear Systems

Let's start with the *linear* system of ODEs that models the motion of a certain viscously damped spring-mass system that obeys Hooke's Law for the displacement x of a unit mass from equilibrium:

$$x' = y, \quad y' = -x - 0.1y \tag{1}$$

In Chapter 4 we saw that the equivalent linear second-order ODE, $x'' + 0.1x' + x = 0$ has an explicit solution formula, which we can use to determine the behavior of solutions and of trajectories in the xy-phase plane.

Now let's suppose that the Hooke's-law spring is replaced by a stiffening spring, which can be modeled by replacing the Hooke's-law restoring force $-x$ in system (1) with the nonlinear restoring force $-x - x^3$. We obtain the system

$$x' = y, \quad y' = -x - x^3 - 0.1y \tag{2}$$

As in the linear system (1), the nonlinear system (2) defines a *vector* (or *direction*) *field* in the *xy-state* (or *phase*) *plane*. The field lines are tangent to the *trajectories* (or *orbits*) and point in the direction of increasing time.

There are no solution formulas for system (2), so we turn to direction fields and ODE Architect for visual clues to solution behavior. As you can see from Figure 7.1, the graphs generated by ODE Architect tell us that the trajectories of both systems spiral into the *equilibrium point* at the origin as $t \to +\infty$, even though the shapes of the trajectories differ. The origin corresponds to the constant solution $x = 0$, $y = 0$, which is called a *spiral sink* for each system because of the spiraling nature of the trajectories and because the trajectories, like water in a draining sink, are "pulled" into the origin with the advance of time. This is an indication of *long-term* or *asymptotic* behavior. Note that in this case the nonlinearity does not affect long-term behavior, but clearly does affect short-term behavior.

☞ The equilibrium points of a system correspond to the constant solutions, that is, to the points where all the rate functions of the system are zero.

✓ "Check" your understanding by answering these questions: Do the systems (1) and (2) have any equilibrium points other than the origin? How do the corresponding springs and masses behave as time increases? Why does the $-x^3$ term seem to push orbits toward the y-axis if $|x| \geq 1$, but not have much effect if $|x|$ is close to zero?

◆ Linearization

If we start with a nonlinear system such as (2), we can often use *linear approximations* to help us understand some features of its solutions. Our approximations will give us a corresponding linear system and we can apply what we know about that linear system to try to understand the nonlinear system. In particular, we will be able to verify our earlier conclusions about the long-term behavior of the nonlinear spring-mass system (2).

The nonlinearity of system (2) comes from the $-x^3$ term in the rate function $g(x, y) = -x - x^3 - 0.1y$. In calculus you may have seen the following formula for the linear approximation of the function $g(x, y)$ near the point (x_0, y_0):

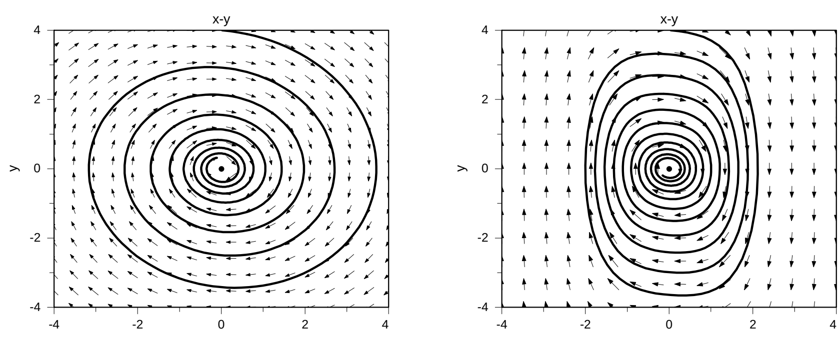
☞ This is a *finite Taylor series* approximation to $g(x, y)$.

$$g(x, y) \approx g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \qquad (3)$$

However, $g(x_0, y_0)$ will always be zero at an equilibrium point (do you see why?), so formula (3) simplifies in this case to

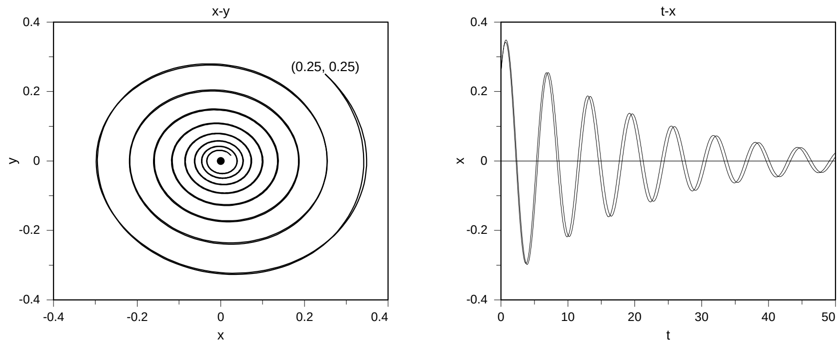
$$g(x, y) \approx \frac{\partial g}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial g}{\partial y}(x_0, y_0)(y - y_0) \qquad (4)$$

Since we're interested in long-term behavior and the trajectories of system (2) seem to be heading toward the origin, we want to use the equilibrium point



Linear System $x' = y,\ y' = -x - 0.1y$ — Nonlinear System $x' = y,\ y' = -x - x^3 - 0.1y$

Figure 7.1: Trajectories of both systems have the same long-term, spiral-sink behavior, but behavior differs in the short-term.



Trajectories $\qquad$ *t*-*x* curves

Figure 7.2: Near the equilibrium at the origin trajectories and tx-component curves of nonlinear system (2) and its linearization (1) are nearly look-alikes.

$(x_0, y_0) = (0, 0)$ in formula (4). Near the origin, the rate function for our nonlinear spring can be approximated by

$$g(x, y) \approx -x - 0.1y$$

since $\partial g/\partial x = -1$ and $\partial g/\partial y = -0.1$ at $x_0 = 0,\ y_0 = 0$. Therefore the nonlinear system (2) reduces to the *linearized system* (1). You can see the approximation when the phase portraits are overlaid. The trajectories and tx-component curves of both systems, issuing from a common initial point close to the origin, are shown in Figure 7.2. The linear approximation is pretty good because the nonlinearity $-x^3$ is small near $x = 0$. Take another look at Figure 7.1; the linear approximation is *not* very good when $|x| > 1$.

☞ Linear and nonlinear look-alikes.

✓ How good an approximation to system (2) is the linearized system (1) if the initial point of a trajectory is far away from the origin? Explain what you mean by "good" and "far away."

In matrix notation, linear system (1) takes the form

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{5}$$

so the characteristic equation of the system matrix is $\lambda^2 + 0.1\lambda + 1 = 0$. The matrix has eigenvalues $\lambda = (-0.1 \pm i\sqrt{3.99})/2$, making $(0, 0)$ a spiral sink (due to the negative real part of both eigenvalues). This supports our earlier conclusion that was based on the computer-generated pictures in Figure 7.2. The addition of a nonlinear term to a linear system (in this example, a cubic nonlinearity) does not change the stability of the equilibrium point (a sink in this case) or the spiraling nature of the trajectories (suggested by the complex eigenvalues).

☞ Look back at Chapter 6 for more on complex eigenvalues and spiral sinks.

The linear and nonlinear trajectories and the tx-components shown in Figure 7.2 look pretty much alike. This is often the case for a system

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \tag{6}$$

☞ The point $\mathbf{x}_0$ is an equilibrium point of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ if $\mathbf{F}(\mathbf{x}_0) = 0$.

and its *linearization*

$$\mathbf{x}' = \mathbf{A}(\mathbf{x} - \mathbf{x}_0) \tag{7}$$

at an equilibrium point $\mathbf{x}_0$. Let's assume that the dependent vector variable $\mathbf{x}$ has n components $x_1, \ldots x_n$, that $F_1(\mathbf{x}), \ldots, F_n(\mathbf{x})$ are the components of $\mathbf{F}(\mathbf{x})$, and that these components are at least twice continuously differentiable functions. Then the $n \times n$ constant matrix $\mathbf{A}$ in system (7) is the matrix of the first partial derivatives of the components of $\mathbf{F}(\mathbf{x})$ with respect to the components of $\mathbf{x}$, all evaluated at $\mathbf{x}_0$:

$$\mathbf{A} = \begin{bmatrix} \dfrac{\partial \mathbf{F}_1}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{F}_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \mathbf{F}_n}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{F}_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_0}$$

$\mathbf{A}$ is called the *Jacobian matrix* of $\mathbf{F}$ at $\mathbf{x}_0$, and is often denoted by $\mathbf{J}$ or $\mathbf{J}(\mathbf{x}_0)$. As an example, look back at system (1) and its linearization, system (2) or system (5).

☞ Here's why linearization is so widely used.

It is known that if none of the eigenvalues of the Jacobian matrix at an equilibrium point is zero or pure imaginary, then close to the equilibrium point the trajectories and component curves of systems (6) and (7) look alike. We can use ODE Architect to find equilibrium points, calculate Jacobian matrices and their eigenvalues, and so, check out whether the eigenvalues meet the conditions just stated. If $n = 2$, we can apply the vocabulary of planar linear systems from Chapter 6 to nonlinear systems. We can talk about a *spiral sink*, a *nodal source*, a *saddle point*, etc. ODE Architect uses a solid dot for a sink, an open dot for a source, a plus sign for a center, and an open square for a saddle.

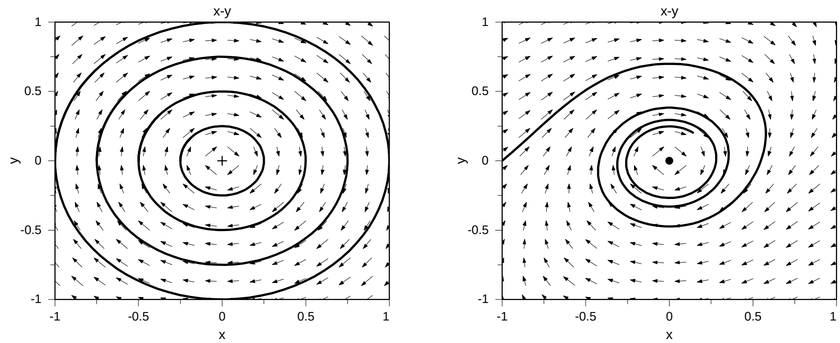
What happens when, say, the matrix $\mathbf{A}$ *does* have pure imaginary eigenvalues? Then all bets are off, as the following example shows.

Start with the linear system

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

The system matrix has the pure imaginary eigenvalues $\pm i$, making the origin a center. Now give the system a nonlinear perturbation to get

$$\begin{aligned} x' &= y - x^3 \\ y' &= -x \end{aligned}$$



Linear system Perturbed system

Figure 7.3: Nonlinear terms convert a linear center to a nonlinear sink.

☞ But linearity can be misleading!

By picturing the direction field defined by this system, we can see that each vector has been nudged slightly inward, toward the origin. This causes solutions to spiral inward, making $(0, 0)$ a spiral sink. Figure 7.3 shows trajectories from the original linear system on the left, and a trajectory of the nonlinear system on the right, spiraling inward. Now it should be clear why we had to exclude pure imaginary eigenvalues!

✓ What happens if you perturb the linear system by adding the x^3 term, instead of subtracting? What about the system $x' = y - x^3,\ \ y' = -x + y^3$?

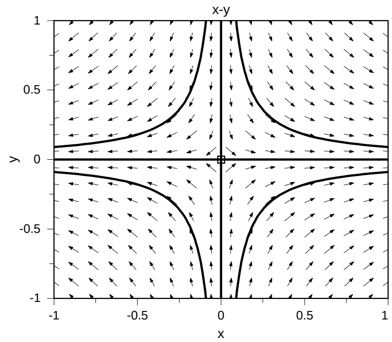
◆ Separatrices and Saddle Points

A linear saddle point has two trajectories that leave the point (as time increases from $-\infty$) along a straight line in the direction of an eigenvector. Another two trajectories approach the point as time increases to $+\infty$) along a straight line in the direction of an eigenvector. These four trajectories are called *saddle separatrices* because they divide the neighborhood of the saddle point into regions of quite different long-term trajectory behavior. The left plot in Figure 7.4 shows the four separatrices along the x and y axes for the linear system

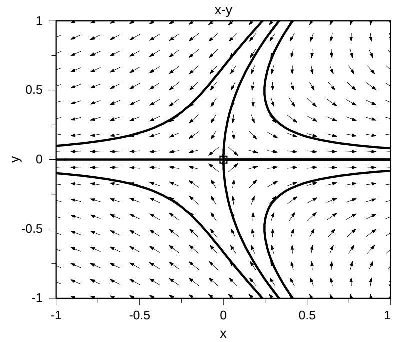
$$x' = x, \quad y' = -y \tag{8}$$

with a saddle point at the origin. The two that leave the origin as t increases are the *unstable separatrices*, the two that enter the origin are the *stable separatrices*.

If we add some higher-order nonlinear terms to a linear saddle-point system, the separatrices persist but their shapes may change. They still divide a neighborhood of the equilibrium point into regions of differing long-term behavior. And, most important, they still leave or approach the equilibrium point



(a) Trajectories of System (8)



(b) Trajectories of System (9)

Figure 7.4: Saddle separatrices lie along the axes in (a); two of the spearatrices are bent to the rightt by a nonlinearity in (b).

tangent to eigenvectors of the linearized system. The right plot in Figure 7.4 suggests all this for the system (8) with a nonlinear term tacked on:

$$x' = x - y^2, \quad y' = -y \tag{9}$$

Note how the nonlinearity bends two of the separatrices.

◆ Behavior of Solutions Away from Equilibrium Points

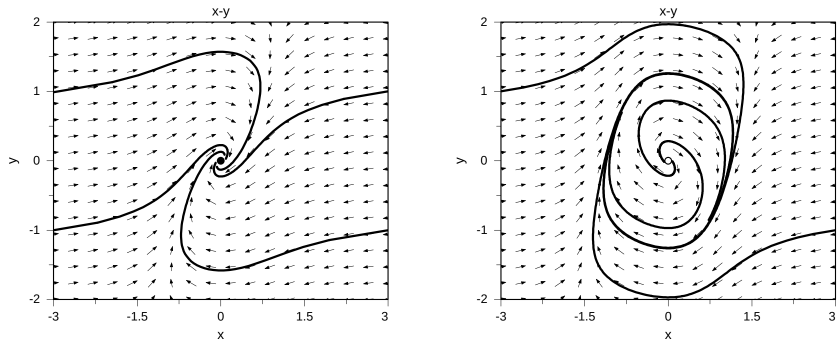
While we can use linearization in most cases to determine the long-term behavior of solutions near an equilibrium point, it may not be a good method for studying the behavior of solutions "far away" from the equilibrium point. Consider, for example, the spider-fly system of Module 7:

$$S' = -4S + 2SF, \quad F' = 3\left(1 - \frac{F}{5}\right)F - 2SF$$

where S is a population of spiders preying on F, a population of flies (all measured in thousands). This nonlinear system has several equilibrium points, one of which is at $p^* = (0.9, 2)$.

Take a look at the graphics windows in Experiment 2 of "The Spider and Fly" (Screen 1.5). The trajectories of the linearized system that are close to p^* approximate well those of the nonlinear system. However, trajectories of the linearized system that are not near the equilibrium point diverge substantially from those of the nonlinear system, and may even venture into a region of the state space where the population of spiders is negative!

✓ Look at the Library file "Mutualism: Symbiotic Interactions" in the "Population Models" folder and investigate the long-term behavior of solution curves by using linear approximations near equilibrium points.



An attracting spiral sink ($a = -1$).

A repelling spiral source and attracting limit cycle ($a = 1$).

Figure 7.5: The system $x' = y + ax - x^3, \ y' = -x$, undergoes a hopf bifurcation to an attracting limit cycle as the parameter transists the value $a = 0$.

◆ Bifurcation to a Limit Cycle

☞ If you are not "into" nonlinear electrical circuits, ignore the modeling here and just consider ODE (10) as a particular nonlinear system.

☞ Current, voltages and time are scaled to dimensionless quantities in system (10).

The model equations for an electrical circuit (the *van der Pol circuit*) containing a nonlinear resistor, an inductor, and a capacitor, all in series, are

$$x' = y + ax - x^3, \quad y' = -x \tag{10}$$

where x is the current in the circuit and y is the voltage drop across the capacitor. The voltage drop across the nonlinear resistor is $ax - x^3$, where a is a parameter. The characteristics of the resistor, and thus the performance of the circuit, changes when we change the value of this parameter. Let's look at the phase portrait and the corresponding eigenvalues of the linearization of this system at the equilibrium point $(0, 0)$ for three different values of a.

As a increases from -1 to 1, the eigenvalues of the Jacobian matrix of system (10) at the origin change from complex numbers with negative real parts to complex numbers with positive real parts, but at $a = 0$ they are pure imaginary. The circuit's behavior changes as a increases, and it changes in a qualitative way at $a = 0$. The phase portrait shows a spiral sink at $(0, 0)$ for $a \leq 0$, then a spiral source for $a > 0$. Further, the trajectories near the source spiral out to a closed curve that is itself a solution. Our electrical circuit has gone from one where current and voltage die out to one that achieves a continuing oscillation described by a periodic steady state. A change like this in the behavior of a model at a particular value of a parameter is called a *Hopf bifurcation*. Figure 7.5 shows the changes in a trajectory of system (10) due to the bifurcation that occurs when a is increased through zero.

✓ Find the Jacobian matrix of system (10) at the origin and calculate its eigenvalues in terms of the parameter a. Write out the linearized version of system (10). Check your work by using ODE Architect's equilibrium, Jacobian, and eigenvalue capabilities.

☞ A limit cycle is exclusively a nonlinear phenomenon. Any cycle in a linear autonomous system is always part of a family of cycles, none of which are limit cycles.

The closed solution curve in Figure 7.5 that represents a periodic steady state is called an *attracting limit cycle* because nearby trajectories spiral into it as time increases. As the parameter value changes in a Hopf bifurcation, you can observe an equilibrium point that is a spiral sink changing into a source with nearby orbits spiralling onto the limit cycle. You'll investigate this kind of phenomenon when you use ODE Architect to investigate the model system in the "Saxophone" submodule of Module 7.

◆ Higher Dimensions

So far we have looked at systems of nonlinear ODEs involving only two state variables. However it is not uncommon for a model to have a system with more than two state variables. Fortunately our ideas extend in a natural way to cover these situations. Analysis by linear approximation may still work in these cases, and ODE Architect can always be used to find equilibrium points, Jacobian matrices, and eigenvalues in any dimension. See for example Problem 4 in Exploration 7.3.

The chapter cover figure shows trajectories of a system with three state variables; this system describes the angular velocity of a spinning body. The "Spinning Bodies" submodule of Module 7 and Problem 1 in Exploration 7.3 model the rotational motion of an object thrown into space; this model is described below.

✓ How could you visualize the trajectories of a system of four equations?

◆ Spinning Bodies: Stability of Steady Rotations

☞ As we shall see, not all axes **L** will support steady rotations.

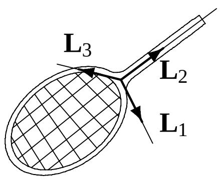
Suppose that a rigid body is undergoing a steady rotation about an axis **L** through its center of mass. In a plane perpendicular to **L** let θ be the angle swept out by a point in the body, but not on the axis. Steady rotations about **L** are characterized by the fact that $\theta' = d\theta/dt =$ constant, for all time. In mechanics, it is useful to describe such steady rotations by a vector $\boldsymbol{\omega}$ parallel to **L** whose magnitude $|\boldsymbol{\omega}| = d\theta/dt$ is constant. Notice that $-\boldsymbol{\omega}$ in this case also corresponds to a steady spin about **L**, but in the opposite direction. The vector $\boldsymbol{\omega}$ is called the *angular velocity*, and for steady rotations we see that $\boldsymbol{\omega}$ is a constant vector. The angular velocity vector $\boldsymbol{\omega}$ can also be defined for an unsteady rotation of the body, but in this case $\boldsymbol{\omega}(t)$ is not a constant vector.

It turns out that in a uniform force field (such as the gravitational field near the earth's surface), the differential equations for the rotational motion of the body about its center of mass decouple from the ODEs for the translational motion of the center of mass. How shall we track the rotational motion of the body? For each rigid body there is a natural triple of orthogonal axes $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$ (called *body axes*) which, as it turns out, makes it relatively easy to model the rotational motion by a system of ODEs. To define the body axes we need the *inertia tensor* I of the body. Given a triple of orthogonal axes through the body's center of mass, put an orientation on each axis and label them to form a *right-handed frame* (i.e., it follow the right-hand rule). In that frame, I is represented by a 3×3 *positive definite matrix*. Body axes are just the frame for which the representation of I is a diagonal matrix with the positive entries I_1, I_2, and I_3 along the diagonal. These values I_1, I_2, and I_3 are called the *principal moments of inertia* of the body. Note that I_k is the moment of inertia about the principal axis $\mathbf{L}_k$, for $k = 1, 2, 3$. If a body has uniform density and an axis $\mathbf{L}$ such that turning the body 180° about that axis brings the body into coincidence with itself again, then that axis $\mathbf{L}$ is a principal axis.

☞ A matrix A is positive definite if it is symmetric (i.e., $A^T = A$) and all of its eigenvalues are positive.

Let's say that a book has uniform density (not quite true, but nearly so). Then the three axes of rotational symmetry through the center of mass are the principal axes: $\mathbf{L}_3$, the *short axis* through the center of the book's front and back covers; $\mathbf{L}_2$, the *long axis* parallel to the book's spine; and $\mathbf{L}_1$, the *intermediate axis* which is perpendicular to $\mathbf{L}_2$ and $\mathbf{L}_3$. For a tennis racket, the body axis $\mathbf{L}_2$ is obvious on geometrical grounds. The other axes $\mathbf{L}_1$ and $\mathbf{L}_3$ are a bit more difficult to discern, but they are given in the margin sketch.

☞ Described graphically in Module 7.



Throw a tennis racket up into the air and watch its gyrations. Wrap a rubber band around a book, toss it into the air, and look at its spinning behavior. Now try to get the racket or the book to spin steadily about each of three perpendicular body axes L_1, L_2, and L_3. Not so hard to do about two of the axes—but nearly impossible about the third. Why is that? Let's construct a model for the rotation of the body and answer this question.

Let's confine our attention to the body's angular motion while aloft, not its vertical motion. Let's ignore air resistance. The key parameters that influence the angular motion are the principal inertias I_1, I_2, I_3 about the respective body axes $\mathbf{L}_1$, $\mathbf{L}_2$, $\mathbf{L}_3$. Let ω_1, ω_2, and ω_3 be the components of the vector $\boldsymbol{\omega}$ along the body axes $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_3$. There is an analogue of Newton's Second Law applied to the body which involves the angular velocity vector $\boldsymbol{\omega}$. The components of the rotational equation of motion in the body axes frame are given by $I_1\omega_1' = (I_2 - I_3)\omega_2\omega_3$, $I_2\omega_2' = (I_3 - I_1)\omega_1\omega_3$, $I_3\omega_3' = (I_1 - I_2)\omega_1\omega_2$.

☞ A complete derivation of the model ODEs can be found in the first of the listed references.

Dividing by the principal inertias, we have the nonlinear system

$$\begin{aligned}
\omega_1' &= \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \\
\omega_2' &= \frac{I_3 - I_1}{I_2}\omega_1\omega_3 \\
\omega_3' &= \frac{I_1 - I_2}{I_3}\omega_1\omega_2
\end{aligned} \tag{11}$$

Let's measure angles in radians and time in seconds, so that each ω_i has units of radians per second.

First, we note that for any constant $\alpha \neq 0$, the equilibrium point $\boldsymbol{\omega} = (\alpha, 0, 0)$ of system (11) represents a pure steady rotation (or spinning motion)

☞ Pure steady rotations are possible about any body axis.

about the first body axis $\mathbf{L}_1$ with angular velocity α. The equilibrium point $(-\alpha, 0, 0)$ represents steady rotation about $\mathbf{L}_1$ in the opposite direction. Similar statements are true for the equilibrium points $\boldsymbol{\omega} = (0, \alpha, 0)$ and $(0, 0, \alpha)$.

Now the *kinetic energy of angular rotation* is given by

$$KE(\omega_1, \omega_2, \omega_3) = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

The valueof KE stays fixed n an orbit of system (11) since

$$\begin{aligned}\frac{d(KE)}{dt} &= I_1\omega_1\omega_1' + I_2\omega_2\omega_2' + I_3\omega_3\omega_3' \\ &= (I_2 - I_3)\omega_1\omega_2\omega_3 + (I_3 - I_1)\omega_1\omega_2\omega_3 + (I_1 - I_2)\omega_1\omega_2\omega_3 = 0\end{aligned}$$

☞ A system of autonomous ODEs is *conservative* if there is a function F of the dependent variables whose value is constant along each orbit (i.e., trajectory), but varies from one orbit to another. F is said to be an *integralof motion* of the system.

So system (11) is *conservative* and KE is an *integral*. The ellipsoidal integral surface $KE = C$, where C is a positive constant, is called an *inertial ellipsoid* for system (11). Note that any orbit of (11) that starts on one of the ellipsoids stays on the ellipsoid, and orbits on that ellipsoid share the same value of KE.

✓ Show that the functions

$$K = \frac{I_3 - I_1}{I_2}\omega_1^2 - \frac{I_2 - I_3}{I_1}\omega_2^2$$

and

$$M = \frac{I_1 - I_2}{I_3}\omega_1^2 - \frac{I_2 - I_3}{I_1}\omega_3^2$$

are also integrals for system (11). Describe the surfaces $K =$ const., $M =$ const.

Let's put in some numbers for I_1, I_2, and I_3 and see what happens. Set $I_1 = 2,\ I_2 = 1,\ I_3 = 3$. Then system (11) becomes

$$\begin{aligned}\omega_1' &= -\omega_2\omega_3 \\ \omega_2' &= \omega_1\omega_3 \\ \omega_3' &= \frac{1}{3}\omega_1\omega_2\end{aligned} \tag{12}$$

With the given values for I_1, I_2, I_3 we have the integral

$$KE = \frac{1}{2}\left(2\omega_1^2 + \omega_2^2 + 3\omega_3^2\right) \tag{13}$$

☞ Body axis $\mathbf{L}_1$ is parallel to the ω_1-axis, $\mathbf{L}_2$ to the ω_2-axis, and $\mathbf{L}_3$ to the ω_3-axis in Figure 7.6.

The left graph in Figure 7.6, which is also the chapter cover figure, shows the inertial ellipsoid $KE = 12$ and twenty-four orbits on the surface. The geometry of the orbits indicates that if the body is started spinning about an axis very near the body axes $\mathbf{L}_2$ or $\mathbf{L}_3$, then the body continues to spin almost steadily about those body axes. Attempting to spin the body about the intermediate body axis $\mathbf{L}_1$ is another matter. Any attempt to spin the body about the $\mathbf{L}_1$ body axis leads to strange gyrations. Note in Figure 7.6 that each of the four trajectories that starts near the equilibrium point $(\sqrt{12}, 0, 0)$ where the

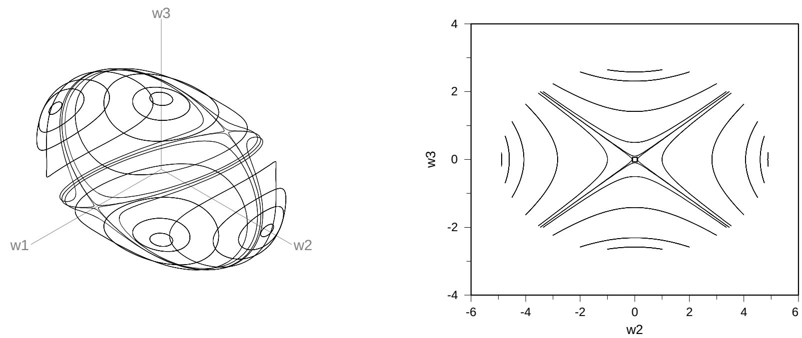


Figure 7.6: Twenty-four trajectories on the inertia ellipsoid $KE = 12$ (left); head-on view from the ω_1-axis (right) shows a saddle point on the ellipsoid.

ω_1-axis pierces the ellipsoid goes back near the antipodal point (and reverses its direction of rotation) then returns in an endlessly repeating periodic path. This corresponds to unstable gyrations near the ω_1-axis.

✓ Match up the trajectories in Figure 7.6 with actual book rotations. Put a rubber band around a book, flip the book into the air, and check out the rotations. Do the projected trajectories in the right graph of Figure 7.6 really terminate, or is something else going on?

◆ The Planar Double Pendulum

☞ This is pretty advanced material here, so feel free to skip the text and go directly to the "Double Pendulum Movies". Just click on the ODE Architect library, open the "Physical Models" folder and the "Double Pendulum Animator" file, and create chaos!

The planar double pendulum is an interesting physical system with two degrees of freedom. It consists of two rods, of lengths l_1 and l_2, and two masses, specified by m_1 and m_2, attached together so that the rods are constrained to oscillate in a vertical plane. We'll neglect effects of damping in this system.

The governing equations are most conveniently written in terms of the angles $\theta_1(t)$ and $\theta_2(t)$ shown in Figure 7.7. One way to obtain the equations of motion is by applying Newton's Law to the motions of the masses. First we'll consider mass m_2 and the component in the direction shown by the unit vector $\mathbf{u}_3$ in Figure 7.7. Define a coordinate system centered at mass m_1 and rotating with angular velocity $\mathbf{\Omega} = (d\theta_1/dt)\mathbf{k}$, where $\mathbf{k}$ is the unit vector normal to the plane of motion. If $\hat{\mathbf{a}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{r}}$ denote the acceleration, velocity, and position of m_2 with respect to the rotating coordinate system, then the acceleration $\mathbf{a}$ with respect to a coordinate system at rest is known to be

$$\mathbf{a} = \hat{\mathbf{a}} + \frac{d\Omega}{dt} \times \hat{\mathbf{r}} + 2\Omega \times \hat{\mathbf{v}} + \Omega \times (\Omega \times \hat{\mathbf{r}}) \tag{14}$$

For our configuration it follows that

$$\hat{\mathbf{r}} = -[l_1 \sin(\theta_2 - \theta_1)]\mathbf{u}_3 + [l_2 + l_1 \cos(\theta_2 - \theta_1)]\mathbf{u}_4 \tag{15}$$

with the unit vector $\mathbf{u}_4$ in the direction shown in Figure 7.7. Since the only forces acting are gravity and the tensile forces in the rods, the $\mathbf{u}_3$-component of $\mathbf{F} = m_2\mathbf{a}$ in combination with Eqs. (14) and (15) gives

$$m_2 l_2 (\theta_2 - \theta_1)'' + m_2[l_2 + l_1\cos(\theta_2 - \theta_1)]\theta_1'' + m_2 l_1 (\theta_1')^2 \sin(\theta_2 - \theta_1) = -m_2 g \sin\theta_2 \quad (16)$$

Similarly, the component of Newton's Law in the direction of the unit vector $\mathbf{u}_1$ is given by

$$m_2 l_1 \theta_1'' + m_2 l_2 \cos(\theta_2 - \theta_1)\theta_2'' - m_2 l_2 (\theta_2')^2 \sin(\theta_2 - \theta_1) = -m_2 g \sin\theta_1 - f_2 \sin(\theta_2 - \theta_1) \quad (17)$$

where f_2 is the magnitude of the tensile force in the rod l_2. Equations (16) and (17) will provide the system governing the motion, once the quantity f_2 is determined. An equation for f_2 is found from the $\mathbf{u}_1$-component of Newton's Law applied to the mass m_1:

$$m_1 l_1 \theta_1'' = -m_1 g \sin\theta_1 + f_2 \sin(\theta_2 - \theta_1) \quad (18)$$

Eliminating f_2 between Eqs. (17) and (18) and simplifying Eq. (16) slightly, we obtain the governing nonlinear system of second-order ODEs for the planar double pendulum:

$$(m_1 + m_2) l_1 \theta_1'' + m_2 l_2 \cos(\theta_2 - \theta_1)\theta_2'' - m_2 l_2 (\theta_2')^2 \sin(\theta_2 - \theta_1) + (m_1 + m_2) g \sin\theta_1 = 0 \quad (19)$$

$$m_2 l_2 \theta_2'' + m_2 l_1 \cos(\theta_2 - \theta_1)\theta_1'' + m_2 l_1 (\theta_1')^2 \sin(\theta_2 - \theta_1) + m_2 g \sin\theta_2 = 0 \quad (20)$$

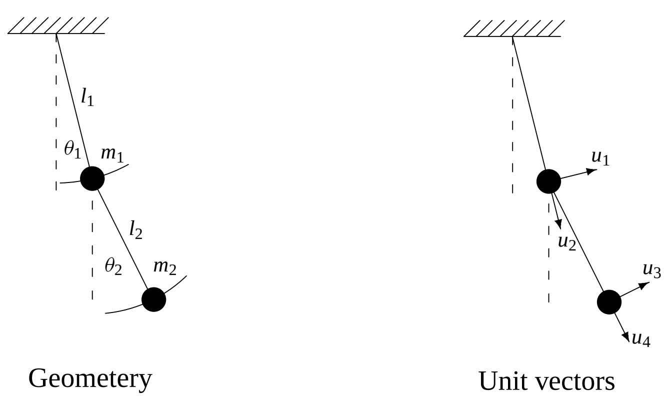


Figure 7.7: Geometry and unit vectors for the double pendulum.

Another way to derive the equations of motion of the double pendulum system is to use Lagrange's equations. These are

$$\frac{d}{dt}\left[\frac{\partial}{\partial\theta_1'}(T-V)\right]-\frac{\partial}{\partial\theta_1}(T-V)=0 \tag{21}$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial\theta_2'}(T-V)\right]-\frac{\partial}{\partial\theta_2}(T-V)=0 \tag{22}$$

where T is the kinetic energy of the system and V is its potential energy. The respective kinetic energies of the masses m_1 and m_2 are

$$T_1=\frac{1}{2}m_1 l_1^2(\theta_1')^2$$

$$T_2=\frac{1}{2}m_2(l_1\theta_1'\sin\theta_1+l_2\theta_2'\sin\theta_2)^2+\frac{1}{2}m_2(l_1\theta_1'\cos\theta_1+l_2\theta_2'\cos\theta_2)^2$$

The corresponding potential energies of m_1 and m_2 are

$$V_1=m_1 g l_1(1-\cos\theta_1)$$

$$V_2=m_2 g l_1(1-\cos\theta_1)+m_2 g l_2(1-\cos\theta_2)$$

Then, we have $T=T_1+T_2$ and $V=V_1+V_2$. Inserting the expressions for T and V into Eqs. (21) and (22), we find the equations of motion of the double pendlum. These equations are equivalent to the ones obtained previously using Newton's Law. The formalism of Lagrange pays the dividend of producing the equations with "relatively" shorter calculations.

References

Borrelli, R.L., Coleman, C.S., and Boyce, W.E., *Differential Equations Laboratory Workbook*, (1992: John Wiley & Sons, Inc.), Experiment 6.9

Hirsch, M., and Smale, S., *Differential Equations, Dynamical Systems, and Linear Algebra* (1974: Academic Press), Chapter 10

Hubbard, J., and West, B., *Differential Equations: A Dynamical Systems Approach*, Vol. 18 of Texts in Appl. Math. (1995: Springer-Verlag), Sections 8.2, 8.3

Marion, J.B., and Thornton, S.T., *Classical Dynamics of Particles and Systems*, 3rd ed. (1988: Harcourt Brace Jovanovich)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 7.1. Predator and Prey: Linearization and Stability

☞ Take a look as the "Spider and Fly" submodule of Module 7.

1. Let F represent the number of flies and S the number of spiders (both in 1000s). Assume that the model for their interaction is given by:

$$S' = -4S + 2SF, \quad F' = 3F - 2SF \tag{23}$$

where the SF-term is a measure of the interaction between the two species.

(a) Why is the SF-term negative in the first ODE and positive in the second when (S, F) is inside the population quadrant?

(b) Show that the system has an equilibrium point at $(2, 1.5)$.

(c) Show that the system matrix of the linearization of system (23) about $(2, 1.5)$ has pure imaginary eigenvalues.

☞ This makes the point $(2, 1.5)$ a center for the linearized system.

(d) Now plot phase portraits for system (23) and for its linearization about $(2, 1.5)$. What do you see?

2. Suppose that an insecticide reduces the spider population at a rate proportional to the size of the population.

(a) Modify the predator-prey model of system (23) to account for this.

(b) Model how insecticide can be made more or less effective.

(c) Use the model to predict the long-term behavior of the populations.

3. In a predator-prey system that models spider-fly interaction

$$S' = -4S + 2SF, \quad F' = 3\left(1 - \frac{F}{N}\right)F - 2SF$$

the number N represents the maximum fly population (in 1000s). Investigate the effect of changing the value of N. What's the largest the spider population can get? The fly population?

4. Suppose the spider-fly model is modified so that there are two predators, spiders and lizards, competing to eat the flies. One model for just the two predator populations is

$$S' = 4\left(1 - \frac{S}{5}\right)S - SL, \quad L' = 3\left(1 - \frac{L}{2}\right)L - SL$$

(a) What do the numbers 2, 3, 4, and 5 represent?

(b) What does the term SL represent? Why is it negative?

(c) What will become of the predator populations in the long run?

5. Take a look at the library file "A Predator-Prey System with Resource Limitation" in the "Biological Models" folder. Compare and contrast the system you see in that file with that given in Problem 2. Create a system where both the predator and the prey are subject to resource limitations, and analyze the behavior of the trajectories.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 7.2. Bifurcations and Limit Cycles

1. Alter the model in the "Saxophone" submodule of Module 10 by adding a parameter c:

$$u' = v, \quad v' = -su + cv - \frac{1}{b}v^3$$

(a) What part of the model does this affect?

(b) How do solutions behave for values of c between 0 and 2, taking $s = b = 1$?

(c) As c increases, what happens to the pitch and amplitude?

2. Suppose the model for a *simple harmonic oscillator* (a linear model),

$$x' = y, \quad y' = -x$$

is modified by adding a parameter c:

$$x' = cx + y, \quad y' = -x + cy$$

(a) What happens to the equilibrium point as c goes from -1 to 1?

(b) What happens to the eigenvalues of the matrix of coefficients as c changes from -1 to 1?

3. Suppose we further modify the system of Problem 2:

$$x' = cx + y - x(x^2 + y^2), \qquad y' = -x + cy - y(x^2 + y^2)$$

where $-1 \le c \le 1$. Analyze the behavior of the equilibrium point at $(0, 0)$ as c increases from -1 to 1. How does it compare with the behavior you observed in Problem 2?

4. You can modify the system for a simple, undamped nonlinear pendulum (see Chapter 10) to produce a *torqued pendulum*:

$$x' = y, \qquad y' = -\sin(x) + a$$

☞ Use the model-based pendulum animation in ODE Architect and watch the pendulum gyrate.

Here a represents a torque applied about the axis of rotation of the pendulum arm. Investigate the behavior of this torqued pendulum for the values of a between 0 and 2 by building the model and animating the phase space as a increases. Explain what kind of behavior the pendulum exhibits as a increases; explain the behavior of any equilibrium points you see.

5. The motion of a thin, flexible steel beam, affixed to a rigid support over two magnets, can be modeled by *Duffing's equation*:

$$x' = y, \quad y' = ax - x^3$$

☞ Sweep on the parameter a, and then animate. To animate a graph with multiple trajectories corresponding to different values of a, click on the animate icon below the word "Tools" at the top left of the tools screen.

where x represents the horizontal displacement of the beam from the rest position and a is a parameter that is related to the strength of the magnets. Investigate the behavior of this model for $-1 \le a \le 1$. In particular:

(a) Find all equilibrium points and classify them as to type (e.g., center, saddle point), verifying your phase plots with eigenvalue calculations (use ODE Architect for the eigenvalue calculations). Some of your answers will depend on a.

(b) Give a physical interpretation of your answers to Question (a).

(c) What happens to the equilibrium points as the magnets change from weak ($a \le 0$) to strong ($a > 0$)?

(d) What happens if you add a linear damping term to the model? (Say, $y' = ax - x^3 - \nu y$.)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 7.3. Higher Dimensions

1. *Spinning Bodies.*
Use ODE Architect to draw several distinct trajectories on the ellipsoid of inertia, $0.5(2\omega_1^2 + \omega_2^2 + 3\omega_3^2) = 6$, for system (12).

☞ For example, $\omega_1 = 0$, $\omega_2 = 3$, $\omega_3 = 1$ can be taken as an initial point and so can $\omega_1 = 1$, $\omega_2 = 0$, $\omega_3 = (10/3)^{0.5}$.

Choose initial data on the ellipsoid so that the trajectories become the "visible skeleton" of the invisible ellipsoid. What do the trajectories look like? What kind of motion does each represent? You should be able to get a picture that resembles the chapter cover figure and Figure 7.6. Project your 3D graphs onto the $\omega_1\omega_2$-, $\omega_2\omega_3$-, and $\omega_1\omega_3$-planes, and describe what you see. Now apply the equilibrium/eigenvalue/eigenvector calculations from ODE Architect to equilibrium points on each of the ω_1, ω_2, and ω_3 axes. Describe the results and their correlation with what you saw on the coordinate planes. Now go to the Library file "A Conservative System: The Momentum Ellipsoid" in the folder "Physical Models" and explain what you see in terms of the previous questions in this problem.

2. Exploration 7.1 (Problem 4) gives a predator-prey model where two species, spiders and lizards, prey on flies. Construct a system of three differential equations that includes the prey in the model. You'll need to represent growth rates and interactions, and you may want to limit population sizes. Make some reasonable assumptions about these parameters. What long-term behavior does your model predict?

3. Take another look at the ODEs of the coupled springs model in Module 6. Use ODE Architect for the system of four ODEs given in Experiment 1 of that section. Make 3D plots of any three of the five variables x_1, x_1', x_2, x_2', and t. What do the plots tell you about the corresponding motions of the springs?

4. Modify the coupled springs model from Module 6 (where coupled linear springs move on a frictionless horizontal surface) by making one of the springs hard or soft: add a term like $\pm x^3$ to the restoring force. Does this change the long-term behavior of the system? Make and interpret graphs as in Problem 3.

5. *Chaos in three dimensions*
Some nonlinear 3D systems seem to behave chaotically. Orbits stay bounded as time advances, but the slightest change in the initial data leads to an orbit that eventually seems to be completely uncorrelated with the original orbit. This is thought to be one feature of chaotic dynamics. Choose one of the following three Library files located in the folder "Higher Dimensional Systems":

- "The Scroll Circuit: Organized Chaos"
- "The Lorenz System: Chaos and Sensitivity"
- "The Roessler System: A Strange Attractor"

Change parameters until you see an example of this kind of chaos. You may want to look at Chapter 12 for additional insight into the meaning of chaos.