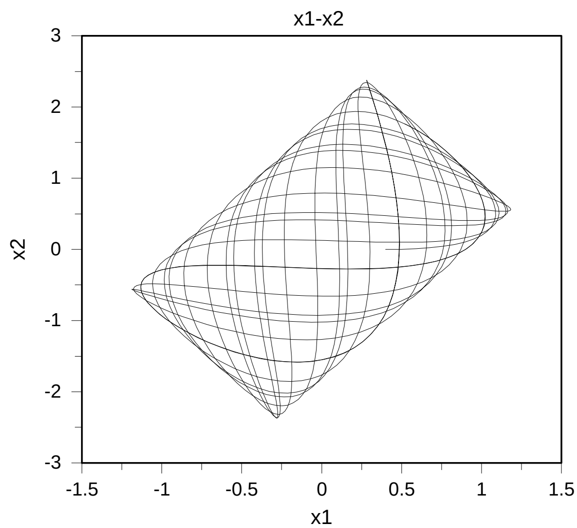
CHAPTER 6

First-Order Linear Systems



Oscillating displacements $x_1(t)$ and $x_2(t)$ of two coupled springs play off against each other.

Overview This chapter outlines some of the main facts concerning systems of first-order linear ODEs, especially those with constant coefficients. You'll have the opportunity to work with physical problems that have two or more dependent variables. Such problems can be modeled using systems of differential equations, which can always be written as systems of first-order equations, as can higher-order differential equations. The eigenvalues and eigenvectors of a matrix of coefficients help us understand the behavior of solutions of these systems.

Key words Linear systems; pizza and video; coupled springs; connected tanks; linearized double pendulum; matrix; component; component plot; phase space; phase plane; phase portrait; eigenvalue; eigenvector; saddle point; node; spiral; center; source; sink

See also Chapter 5 for definitions of vector mathematics.

◆ Background

Many applications involve a single independent variable (usually time) and two or more dependent variables. Some examples of dependent variables are:

- the concentrations of a chemical in organs of the body
- the voltage drops across the elements of an electrical network
- the populations of several interacting species
- the profits of businesses in a mall

Applications with more than one dependent variable lead naturally to *systems* of ordinary differential equations. Such systems, as well as higher-order ODEs, can be rewritten as systems of first-order ODEs.

☞ How to convert a second-order ODE to a system of first-order ODEs.

Here's how to reduce a second-order ODE to a system of first-order ODEs (see also Chapter 4). Let's look at the the second-order ODE

$$y'' = f(t, y, y') \tag{1}$$

Introduce the variables $x_1 = y$ and $x_2 = y'$. Then we get the first-order system

$$x_1' = x_2 \tag{2}$$

$$x_2' = f(t, x_1, x_2) \tag{3}$$

ODE (2) follows from the definition of x_1 and x_2, and ODE (3) is ODE (1) rewritten in terms of x_1 and x_2.

✓ "Check" your understanding now by reducing the second-order ODE $y'' + 5y' + 4y = 0$ to a system of first-order ODEs.

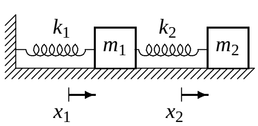
◆ Examples of Systems: Pizza and Video, Coupled Springs

Module 6 shows how to model the profits $x(t)$ and $y(t)$ of a pizza parlor and a video store by a system that looks like this:

$$x' = ax + by + c$$
$$y' = fx + gy + h$$

where a, b, c, f, g, and h are constants. Take another look at Screens 1.1–1.4 in Module 6 to see how ODE Architect handles these systems.

Module 6 also presents a model system of second-order ODEs for oscillating springs and masses. A pair of coupled springs with spring constants k_1 and k_2 are connected to masses m_1 and m_2 that glide back forth on a table. As shown in the "Coupled Springs" submodule, if damping is negligible then the second-order linear ODEs that model the displacements of the masses from equilibria are



$$m_1x_1'' = -(k_1 + k_2)x_1 + k_2x_2$$
$$m_2x_2'' = k_2x_1 - k_2x_2$$

☞ A system of first-order ODEs is *autonomous* if the terms on the right-hand sides of the equations do not explicitly depend on time.

Let's set $m_1 = 4,\ m_2 = 1,\ k_1 = 3$, and $k_2 = 1$. Then, setting $x_1' = v_1,\ x_2' = v_2$, the corresponding autonomous system of four first-order ODEs is

$$\begin{aligned} x_1' &= v_1 \\ v_1' &= -x_1 + \frac{1}{4}x_2 \\ x_2' &= v_2 \\ v_2' &= x_1 - x_2 \end{aligned}$$

☞ Trajectories of an autonomous system can't intersect because to do so would violate the uniqueness property that only one trajectory can pass through a given point.

The cover figure of this chapter shows how x_1 and x_2 play off against each other when $x_1(0) = 0.4,\ v_1(0) = 1,\ x_2(0) = 0$, and $v_2(0) = 0$. The trajectories for this IVP are defined in the 4-dimensional $x_1v_1x_2v_2$-space and cannot intersect themselves. However, the projections of the trajectories onto any plane *can* intersect, as we see in the cover figure.

◆ Linear Systems with Constant Coefficients

The model first-order systems of ODEs for pizza and video and for coupled springs have the special form of linear systems with constant coefficients. Now we shall see just what linearity means and how it allows us (sometimes) to construct solution formulas for linear systems.

☞ Dependent variables are also called *state variables*.

☞ *Homogeneous* means that there are no free terms, that is, terms that don't involve any x_i.

Let t (time) be the independent variable and let $x_1, x_2, \ldots, x_n$ denote the dependent variables. Then a general system of first-order linear *homogeneous* ODEs with constant coefficients has the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \tag{4}$$

where $a_{11},\ a_{12},\ \ldots, a_{nn}$ are given constants. To find a unique solution, we need a set of initial conditions, one for each dependent variable:

$$x_1(t_0) = \alpha_1, \quad \ldots, \quad x_n(t_0) = \alpha_n \tag{5}$$

☞ An equilibrium point of an autonomous system of ODEs is a point where all the rates are zero; it corresponds to a constant solution.

where t_0 is a specific time and $\alpha_1, \ldots, \alpha_n$ are given constants. The system (4) and the initial conditions (5) together constitute an *initial value problem* (IVP) for $x_1, \ldots, x_n$ as functions of t. Note that $x_1 = \cdots = x_n = 0$ is an equilibrium point of system (4).

The model on Screen 1.4 of Module 6 for the profits of the pizza and video stores is the system

☞ If $n = 2$, we often use x and y for the dependent variables.

$$\begin{aligned} x' &= 0.06x + 0.01y - 0.013 \\ y' &= 0.04x + 0.05y - 0.013 \end{aligned} \tag{6}$$

with the initial conditions

$$x(0) = 0.30, \quad y(0) = 0.20 \tag{7}$$

The ODEs (6) are nonhomogeneous due to the presence of the free term -0.013 in each equation. The coordinates of an equilibrium point of a system are values of the dependent variables for which all of the derivatives $x_1', \ldots, x_n'$ are zero. For the system (6) the only equilibrium point is $(0.2, 0.1)$. The translation $X = x - 0.2$, $Y = y - 0.1$ transforms the system (6) into the system

☞ A change of variables puts the equilibrium point at the origin.

$$\begin{aligned} X' &= 0.06X + 0.01Y \\ Y' &= 0.04X + 0.05Y \end{aligned} \tag{8}$$

which is homogeneous and has the same coefficients as the system (6). In terms of X and Y, the initial conditions (7) become

$$X(0) = 0.1, \quad Y(0) = 0.1 \tag{9}$$

Although we have converted a nonhomogeneous system to a homogeneous system in this particular case, it isn't always possible to do so.

☞ Vectors and matrices appear as bold letters.

☞ $\mathbf{A}$ is called the *linear system matrix*, or the *Jacobian matrix*.

It is useful here to introduce matrix notation: it saves space and it expresses system (4) in the form of a single equation. Let $\mathbf{x}$ be the vector with components $x_1, x_2, \ldots, x_n$ and let $\mathbf{A}$ be the matrix of the coefficients, where a_{ij} is the element in the ith row and jth column of $\mathbf{A}$. The derivative of the vector $\mathbf{x}$, written $d\mathbf{x}/dt$, or $\mathbf{x}'$ is defined to be the vector with the components $dx_1/dt, \ldots, dx_n/dt$. Therefore we can write the system (4) in the compact form

☞ The vector $\mathbf{x}(t)$ is called the *state* of system (10) at time t.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{x}' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{10}$$

In vector notation, the initial conditions (5) become

$$\mathbf{x}(t_0) = \boldsymbol{\alpha} \tag{11}$$

where $\boldsymbol{\alpha}$ is the vector with components $\alpha_1, \ldots, \alpha_n$.

✓ Find the linear system matrix for system (8).

A solution of the initial value problem (10) and (11) is a set of functions

$$\begin{aligned} x_1 &= x_1(t) \\ &\vdots \\ x_n &= x_n(t) \end{aligned} \tag{12}$$

that satisfy the differential equations and initial conditions. Using our new notation, if $\mathbf{x}(t)$ is the vector whose components are $x_1(t), \ldots, x_n(t)$, then $\mathbf{x} = \mathbf{x}(t)$ is a solution of the corresponding vector IVP, (10) and (11). The system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is homogeneous, while a nonhomogeneous system would have the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{F}$, where $\mathbf{F}$ is a vector function of t or else a constant vector.

◆ Solution Formulas: Eigenvalues and Eigenvectors

To find a solution formula for system (10) let's look for an exponential solution of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t} \tag{13}$$

where λ is a constant and $\mathbf{v}$ is a constant vector to be determined. Substituting $\mathbf{x}$ as given by (13) into the ODE (10), we find that $\mathbf{v}$ and λ must satisfy the algebraic equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{14}$$

Equation (14) can also be written in the form

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \tag{15}$$

where $\mathbf{I}$ is the *identity matrix* and $\mathbf{0}$ is the *zero vector* with zero for each component. Equation (15) has nonzero solutions if and only if λ is a root of the nth-degree polynomial equation

☞ The determinant of a matrix is denoted by det.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{16}$$

called the *characteristic equation* for the system (10). Such a root is called an *eigenvalue* of the matrix $\mathbf{A}$. We will denote the eigenvalues by $\lambda_1, \ldots, \lambda_n$. For each eigenvalue λ_i there is a corresponding nonzero solution $\mathbf{v}^{(i)}$, called an *eigenvector*. The eigenvectors are not determined uniquely but only up to an arbitrary multiplicative constant.

☞ The keys to finding a solution formula for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ are the eigenvalues and eigenvectors of $\mathbf{A}$.

For each eigenvalue-eigenvector pair $(\lambda_i, \mathbf{v}^{(i)})$ there is a corresponding vector solution $\mathbf{v}^{(i)}e^{\lambda_i t}$ of the ODE (10). If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are all different, then there are n such solutions,

$$\mathbf{v}^{(1)}e^{\lambda_1 t}, \ldots, \mathbf{v}^{(n)}e^{\lambda_n t}$$

In this case the *general solution* of system (10) is the linear combination

☞ Formula (17) is called the general solution formula of system (10) because every solution has the form of (17) for some choice of the constants C_j. The other way around, every choice of the constants yields a solution of system (10).

$$\mathbf{x} = C_1\mathbf{v}^{(1)}e^{\lambda_1 t} + \cdots + C_n\mathbf{v}^{(n)}e^{\lambda_n t} \tag{17}$$

The arbitrary constants $C_1, \ldots, C_n$ can always be chosen to satisfy the initial conditions (11). If the eigenvalues are not distinct, then the general solution takes on a slightly different (but similar) form. The texts listed in the references give the formulas for this case. If some of the eigenvalues are complex, then the solution given by formula (17) is complex-valued. However, if all of the coefficients a_{ij} are real, then the complex eigenvalues and eigenvectors occur in complex conjugate pairs, and it is always possible to express the solution formula (17) in terms of real-valued functions. Look ahead to formulas (20) and (21) for a way to accomplish this feat.

◆ Calculating Eigenvalues and Eigenvectors

Here's how to find the eigenvalues and eigenvectors of a 2×2 real matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

First define the *trace* of $\mathbf{A}$ (denoted by $\operatorname{tr}\mathbf{A}$) to be the sum $a+d$ of the diagonal entries, and the *determinant* of $\mathbf{A}$ (denoted by $\det\mathbf{A}$) to be the number $ad-bc$. Then the characteristic equation for $\mathbf{A}$ is

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \\ &= \lambda^2 - (a+d)\lambda + ad - bc \\ &= \lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} \\ &= 0 \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots λ_1 and λ_2 of this quadratic equation. We assume $\lambda_1 \neq \lambda_2$. For the eigenvalue λ_1 we can find a corresponding eigenvector $\mathbf{v}^{(1)}$ by solving the vector equation

$$\mathbf{A}\mathbf{v}^{(1)} = \lambda_1\mathbf{v}^{(1)}$$

for $\mathbf{v}^{(1)}$. In a similar fashion we can find an eigenvector $\mathbf{v}^{(2)}$ corresponding to the eigenvalue λ_2.

Example: Take a look at the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{18}$$

Since

$$\operatorname{tr}\mathbf{A} = 0 + 3 = 3 \quad \text{and} \quad \det\mathbf{A} = 0 \cdot 3 - 1 \cdot (-2) = 2$$

the characteristic equation is

$$\lambda^2 - (\operatorname{tr}\mathbf{A})\lambda + \det\mathbf{A} = \lambda^2 - 3\lambda + 2 = 0$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. To find an eigenvector $\mathbf{v}^{(1)}$ for λ_1, let's solve

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\mathbf{v}^{(1)} = \lambda_1\mathbf{v}^{(1)} = \mathbf{v}^{(1)}$$

for $\mathbf{v}^{(1)}$. Denoting the components of $\mathbf{v}^{(1)}$ by α and β, we have

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ -2\alpha + 3\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

This gives two equations for α and β:

$$\beta = \alpha, \quad -2\alpha + 3\beta = \beta$$

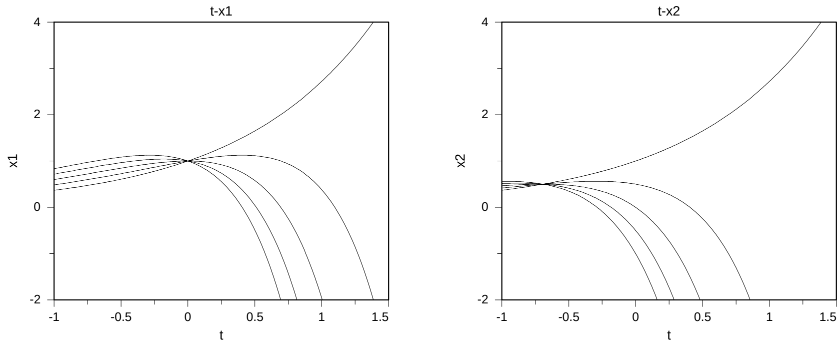


Figure 6.1: Graphs of five solutions $x_1(t)$ (left), $x_2(t)$ (right) of system (18).

The second equation is equivalent to the first, so we may as well set $\alpha = \beta = 1$, which gives us an eigenvector $\mathbf{v}^{(1)}$. In a similar way for the eigenvalue λ_2, we can find an eigenvector $\mathbf{v}^{(2)}$ with components $\alpha = 1,\ \beta = 2$. So the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in this case is

$$\begin{aligned}\mathbf{x} &= C_1\mathbf{v}^{(1)}e^{\lambda_1 t} + C_2\mathbf{v}^{(2)}e^{\lambda_2 t}\\ &= C_1\begin{bmatrix}1\\1\end{bmatrix}e^t + C_2\begin{bmatrix}1\\2\end{bmatrix}e^{2t}\end{aligned}$$

or in component form

$$\begin{aligned}x_1 &= C_1e^t + C_2e^{2t}\\ x_2 &= C_1e^t + 2C_2e^{2t}\end{aligned}$$

where C_1 and C_2 are arbitrary constants.

✓ Find a formula for the solution of system (18) if $x_1(0) = 1,\ x_2(0) = -1$. Figure 6.1 shows graphs of $x_1(t)$ and $x_2(t)$ where $x_1(0) = 1,\ x_2(0) = 0, \pm 0.5, \pm 1$. Which graphs correspond to $x_1(0) = 1,\ x_2(0) = -1$? What happens as $t \to +\infty$? As $t \to -\infty$?

◆ Phase Portraits

☞ Another term for phase space is *state space*.

We can view solutions graphically in several ways. For example, we can draw plots of $x_1(t)$ vs. t, $x_2(t)$ vs. t, and so on. These plots are called *component plots* (see Figure 6.1). Alternatively, we can interpret equations (12) as a set of parametric equations with t as the parameter. Then each specific value of t corresponds to a set of values for $x_1, \ldots, x_n$. We can view this set of values as coordinates of a point in $x_1x_2\cdots x_n$-space, called the *phase space*. (If $n = 2$ it's called the *phase plane*.) For an interval of t-values, the corresponding points form a curve in phase space. This curve is called a *phase plot*, a *trajectory*, or an *orbit*.

Phase plots are particularly useful if $n = 2$. In this case it is often worthwhile to draw several trajectories starting at different initial points on the same set of axes. This produces a *phase portrait*, which gives us the best possible overall view of the behavior of solutions. Whatever the value of n, the trajectories of system (10) can never intersect because system (10) is autonomous.

If $\mathbf{A}$ in system (10) is a 2×2 matrix, then it is useful to examine and classify the various cases that can arise. There aren't many cases when $n = 2$, but even so these cases give important information about higher-dimensional linear systems, as well as nonlinear systems (see Chapter 7). We won't consider here the cases where the two eigenvalues are equal, or where one or both of them are zero.

A *direction field* (or *vector field*) for an autonomous system when $n = 2$ is a field of line segments. The slope of the segment at the point (x_1, x_2) is x_2'/x_1'. The trajectory through (x_1, x_2) is tangent to the segment. An arrowhead on the segment shows the direction of the flow. See Figures 6.2–6.5 for examples.

Real Eigenvalues

If the eigenvalues λ_1 and λ_2 are real, the general solution is

$$\mathbf{x} = C_1 \mathbf{v}^{(1)} e^{\lambda_1 t} + C_2 \mathbf{v}^{(2)} e^{\lambda_2 t} \qquad (19)$$

where C_1 and C_2 are arbitrary real constants.

☞ Trajectories starting on either line at $t = 0$ stay on the line.

Let's first look at the case where λ_1 and λ_2 have opposite signs, with $\lambda_1 > 0$ and $\lambda_2 < 0$. The term in formula (19) involving λ_1 dominates as $t \to +\infty$, and the term involving λ_2 dominates as $t \to -\infty$. Thus as $t \to +\infty$ the trajectories approach the line that goes through the origin and has the same slope as $\mathbf{v}^{(1)}$, and as $t \to -\infty$, they approach the line that goes through the origin and has the same slope as $\mathbf{v}^{(2)}$. A typical phase portrait for this case is shown in Figure 6.2. The origin is called a *saddle point*, and it is *unstable*, since most solutions move away from the point.

☞ Eigenvalues of opposite signs imply a *saddle*.

Now suppose that λ_1 and λ_2 are both negative, with $\lambda_2 < \lambda_1 < 0$. The solution is again given by formula (19), but in this case both terms approach

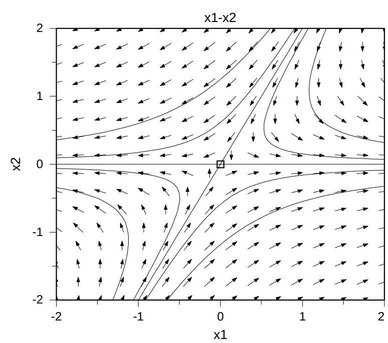


Figure 6.2: Phase portrait of a saddle: $x_1' = x_1 - x_2,\ x_2' = -x_2$.

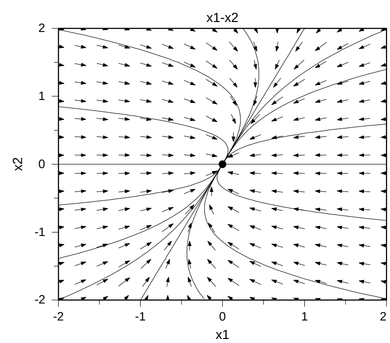


Figure 6.3: Phase portrait of a nodal sink: $x_1 = -3x_1 + x_2,\ x_2' = -x_2$.

zero as $t \to +\infty$. However, for large positive t, the factor $e^{\lambda_2 t}$ is much smaller than $e^{\lambda_1 t}$, so for $C_1 \neq 0$ the trajectories approach the origin tangent to the line with the same slope as $\mathbf{v}^{(1)}$, and if $C_1 = 0$ the trajectory lies on the line with the same slope as $\mathbf{v}^{(2)}$. For large negative t, the term involving r_2 is the dominant one and the trajectories approach asymptotes that have the same slope as $\mathbf{v}^{(2)}$. A typical phase portrait for this case is shown in Figure 6.3. The origin attracts all solutions and is called an *asymptotically stable node*. It is also called a *sink* because all nearby orbits get pulled in as $t \to +\infty$.

☞ Both eigenvalues negative imply a *nodal sink*.

If both eigenvalues are positive, the situation is similar to when both eigenvalues are negative, but in this case the direction of motion on the trajectories is reversed. For example, suppose that $0 < \lambda_1 < \lambda_2$: then the trajectories are unbounded as $t \to +\infty$ and asymptotic to lines parallel to $\mathbf{v}^{(2)}$. As $t \to -\infty$ the trajectories approach the origin either tangent to the line through the origin with the same slope as $\mathbf{v}^{(1)}$ or lying on the line through the origin with the same slope as $\mathbf{v}^{(2)}$. A typical phase portrait for this case looks like Figure 6.3 but with the arrows reversed. The origin is an *unstable node*. It is also called a *source* because all orbits (except $\mathbf{x} = \mathbf{0}$ itself) flow out and away from the origin as t increases from $-\infty$.

☞ Both eigenvalues positive imply a *nodal source*.

✓ Find the eigenvalues and eigenvectors of the systems of Figures 6.2 and 6.3 and interpret them in terms of the phase plane portraits.

Complex Eigenvalues

Now suppose that the eigenvalues are complex conjugates $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. The exponential form (13) of a solution remains valid, but usually it is preferable to use Euler's formula:

$$e^{i\beta t} = \cos(\beta t) + i\sin(\beta t) \tag{20}$$

This allows us to write the solution in terms of real-valued functions. The result is

$$\mathbf{x} = C_1 e^{\alpha t}[\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)] + C_2 e^{\alpha t}[\mathbf{b}\cos(\beta t) + \mathbf{a}\sin(\beta t)] \tag{21}$$

where $\mathbf{a}$ and $\mathbf{b}$ are the real and imaginary parts of the eigenvector $\mathbf{v}^{(1)}$ associated with λ_1, and C_1 and C_2 are constants. The trajectories are spirals about the origin. If $\alpha > 0$, then the spirals grow in magnitude and the origin is called a *spiral source* or an *unstable spiral point*. A typical phase portrait in this case looks like Figure 6.4. If $\alpha < 0$, then the spirals approach the origin as $t \to +\infty$, and the origin is called a *spiral sink* or an *asymptotically stable spiral point*. In both cases the spirals encircle the origin and may be directed in either the clockwise or counterclockwise direction (but not both directions in the same system).

☞ Complex eigenvalues with nonzero real parts imply a *spiral sink* or a *spiral source*.

Finally, consider the case $\lambda = \pm i\beta$, where β is real and positive. Now the exponential factors in solution formula (21) are absent so the trajectory is bounded as $t \to \pm\infty$, but it does not approach the origin. In fact, the trajectories are ellipses centered on the origin (see Figure 6.5), and the origin is called a *center*. It is *stable*, but not asymptotically stable.

☞ Pure imaginary eigenvalues imply a *center*.

✓ Find the eigenvalues of the systems of Figures 6.4 and 6.5, and interpret them in terms of the phase plane portraits. Why can't you "see" the eigenvectors in these portraits?

There is one other graphing technique that is often useful. If $n = 2$, ODE Architect can draw a plot of the solution in tx_1x_2-space. If we project this curve onto each of the coordinate planes, we obtain the two component plots and the phase plot (Figure 6.6).

◆ Using ODE Architect to Find Eigenvalues and Eigenvectors

ODE Architect will find equilibrium points of a system and the eigenvalues and eigenvectors of the Jacobian matrix of an autonomous system at an equilibrium point. Here are the steps:

- Enter an autonomous system of first-order ODEs.
- Click on the lower left Equilibrium tab; enter a guess for the coordinates of an equilibrium point.
- The Equil. tab at the lower right will bring up a window with calculated coordinates of an equilibrium point close to your guess.
- Double click anywhere on the boxed coordinates of an equilibrium in the window (or click on the window's editing icon) to see the eigenvalues, eigenvectors, and the Jacobian matrix.

☞ Use this Architect feature to calculate the eigenvalues, eigenvectors.

If you complete the above steps for a system of two first-order, autonomous ODEs, ODE Architect will insert a symbol at the equilibrium point in the phase plane: An open square for a saddle, a solid dot for a sink, an open dot for a source, and a plus sign for a center (Figures 6.2–6.5). The symbols can be edited using the Equilibrium tab on the edit window.

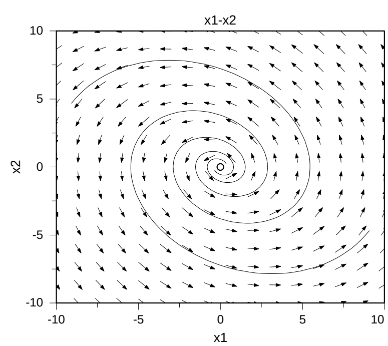


Figure 6.4: Phase portrait of a spiral source: $x_1' = x_2,\ x_2' = x_1 + 0.4x_2$.

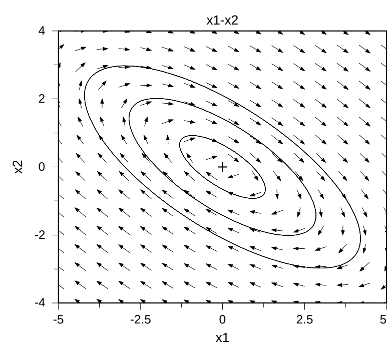


Figure 6.5: Phase portrait of a center: $x_1' = x_1 + 2x_2,\ x_2' = -x_1 - x_2$.

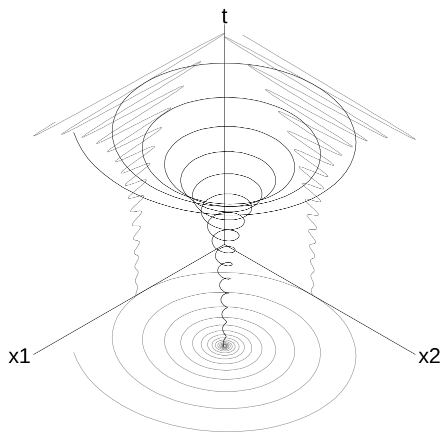


Figure 6.6: Solution curve of $x_1' = x_2$, $x_2' = -100.25x + x_2$, $x_1(0) = 1$, $x_2(0) = 1$, the two component curves, and the trajectory in the x_1x_2-phase plane.

✓ Use ODE Architect to find the eigenvalues and eigenvectors of the system in Figure 6.2.

◆ Separatrices

A trajectory Γ of a planar autonomous system is a *separatrix* if the long-term behavior of trajectories on one side of Γ is quite different from the behavior of those on the other side. Take a look at the four *saddle separatrices* in Figure 6.2, each of which is parallel to an eigenvector of the system matrix. The two separatrices that approach the saddle point as t increases are the *stable separatrices*, and the two that leave are the *unstable separatrices*.

◆ Parameter Movies

The eigenvalues of a 2×2 matrix $\mathbf{A}$ depend on the values of $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$, and the behavior of the trajectories of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ depends very much on the eigenvalues. So it makes sense to see what happens to trajectories as we vary the values of $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$. When we do this varying, we can make the eigenvalues change sign, or move into the complex plane, or become equal. As the changes occur the behavior of the trajectories has to change as well. Take a look at the "Parameter Movies" part of Module 6 for some surprising views of the changing phase plane portraits as we follow along a path in the parameter plane of $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$.

References

Borrelli, R.L., and Coleman, C.S., *Differential Equations: A Modeling Perspective* (1998: John Wiley & Sons, Inc.)

Boyce, W.E., and DiPrima, R.C., *Elementary Differential Equations and Boundary Value Problems*, 6th ed., (1997: John Wiley & Sons, Inc.)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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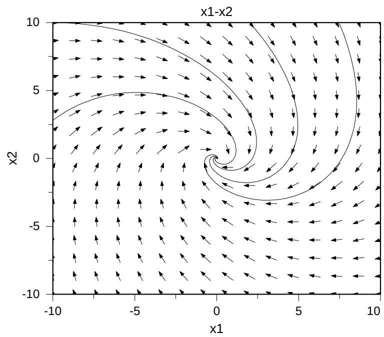
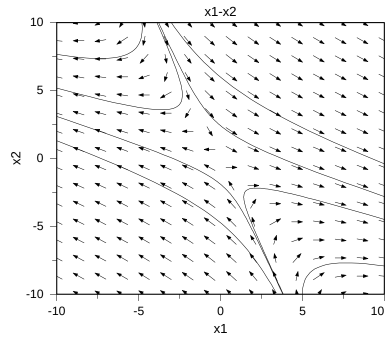
Exploration 6.1. Eigenvalues, Eigenvectors, and Graphs

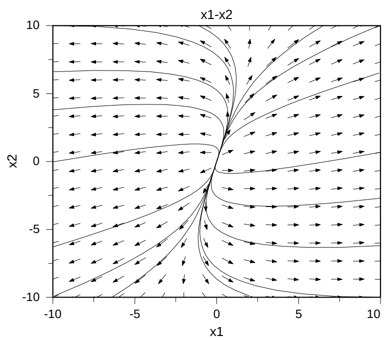
1. Each of the phase portraits in the graphs below is associated with a planar autonomous linear system with equilibrium point at the origin. What can you say about the eigenvalues of the system matrix **A** (e.g., are they real, complex, positive)? Sketch by hand any straight line trajectories. What can you say about the eigenvectors?



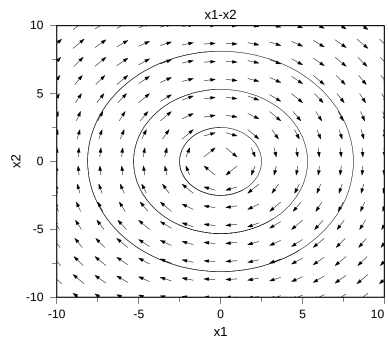
(a)



(b)



(c)



(d)

2. What does the phase portrait of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ look like if $\mathbf{A}$ is a 2×2 matrix with one eigenvalue zero and the other nonzero? How many equilibrium points are there? Include portraits of specific examples.

3. Using Figure 6.6 as a guide, make your own gallery of 2D and 3D graphs to illustrate solution curves, component curves, trajectories, and phase-plane portraits of the systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a 2×2 matrix of constants. List eigenvalues and eigenvectors of $\mathbf{A}$. Include examples of the following types of equilibrium points:

- Saddle
- Nodal sink
- Nodal source
- Spiral sink
- Spiral source
- Center
- Eigenvalues of $\mathbf{A}$ are equal and negative

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 6.2. Pizza and Video

Sometimes business enterprises are strongly affected by periodic (e.g., seasonal) influences. We can illustrate this in the case of Diffey and Cue.

The model describing Diffey's and Cue's profits on Screen 1.4 in Module 6 is

$$\begin{aligned} x' &= 0.06x + 0.01y - 0.013 \\ y' &= 0.04x + 0.05y - 0.013 \end{aligned} \tag{22}$$

Let's introduce a periodic fluctuation in the coefficient of x in the first ODE and in the coefficient of y in the second ODE.

Sine and cosine functions are often used to model periodic phenomena. We'll use $\sin(2\pi t)$ so that the fluctuations have a period of one time unit. We will also include a variable amplitude parameter a so that the intensity of the fluctuations can be easily controlled. We have the modified system

$$\begin{aligned} x' &= 0.06\left(1 + \frac{1}{2}a\sin(2\pi t)\right)x + 0.01y - 0.013 \\ y' &= 0.04x + 0.05\left(1 + \frac{3}{10}a\sin(2\pi t)\right)y - 0.013 \end{aligned} \tag{23}$$

Note that if $a = 0$, we recover system (22), and that as a increases the amplitude of the fluctuations in the coefficients also increases.

1. Interpret the terms involving $\sin(2\pi t)$ in the context of Diffey's and Cue's businesses. Use ODE Architect to solve the system (23) subject to the initial conditions $x(0) = 0.3,\ y(0) = 0.2$ for $a = 1$. Use the time interval $0 \le t \le 10$, or an even longer interval. Plot x vs. t, y vs. t, and y vs. x. Compare the plots with the corresponding plots for the system (22). What is the effect of the fluctuating coefficients on the solution? Repeat with the same initial data, but sweeping a from 0 to 5 in 11 steps. What is the effect of increasing a on the solution?

2. Use ODE Architect to solve the system (23) subject to the initial conditions $x(0) = 0.25,\ y(0) = 0$ for $a = 3$. Draw a plot of y vs. x only. Be sure to use a sufficiently large t-interval to make clear the ultimate behavior of the solution. Repeat using the initial conditions $x(0) = 0.2,\ y(0) = -0.2$. Explain what you see.

3. For the two initial conditions in Problem 2 you should have found solutions that behave quite differently. Consider initial points on the line joining $(0.25, 0)$ and $(0.2, -0.2)$. For $a = 3$, estimate the coordinates of the point where the solution changes from one type of behavior to the other.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 6.3. Control of Interconnected Water Tanks

☞ Take a look at Chapter 8 for a way to diagram this "compartment" model.

Consider two interconnected tanks containing salt water. Initially Tank 1 contains 5 gal of water and 3 oz of salt while Tank 2 contains 4 gal of water and 5 oz of salt.

Water containing p_1 oz of salt per gal flows into Tank 1 at a rate of 2 gal/min. The mixture in Tank 1 flows out at a rate of 6 gal/min, of which half goes into Tank 2 and half leaves the system.

Water containing p_2 oz of salt per gal flows into Tank 2 at a rate of 3 gal/min. The mixture in Tank 2 flows out at a rate of 6 gal/min: 4 gal/min goes to Tank 1, and the rest leaves the system.

1. Draw a diagram showing the tank system. Does the amount of water in each tank remain the same during this flow process? Explain. If $q_1(t)$ and $q_2(t)$ are the amounts of salt (in oz) in the respective tanks at time t, show that they satisfy the system of differential equations:

$$q_1' = 2p_1 - \tfrac{6}{5}q_1 + q_2$$
$$q_2' = 3p_2 + \tfrac{3}{5}q_1 - \tfrac{3}{2}q_2$$

What are the initial conditions associated with this system of ODEs?

2. Suppose that $p_1 = 1$ oz/gal and $p_2 = 1$ oz/gal. Solve the IVP, plot $q_1(t)$ vs. t, and estimate the limiting value q_1^* that $q_1(t)$ approaches after a long time. In a similar way estimate the limiting value q_2^* for $q_2(t)$. Repeat for your own initial conditions, but remember that $q_1(0)$ and $q_2(0)$ must be nonnegative. How are q_1^* and q_2^* affected by changes in the initial conditions? Now use ODE Architect to find q_1^* and q_2^*. [*Hint:* Use the Equilibrium tab.] Is the equilibrium point a source or a sink? A node, saddle, spiral, or center?

3. The operator of this system (you) can control it by adjusting the input parameters p_1 and p_2. Note that q_1^* and q_2^* depend on p_1 and p_2. Find values of p_1 and p_2 so that $q_1^* = q_2^*$. Can you find values of p_1 and p_2 so that $q_1^* = 1.5q_2^*$? So that $q_2^* = 1.5q_1^*$?

4. Let c_1^* and c_2^* be the limiting concentrations of salt in each tank. Express c_1^* and c_2^* in terms of q_1^* and q_2^*, respectively. Find p_1 and p_2, if possible, so as to achieve each of the following results:

(a) $c_1^* = c_2^*$ **(b)** $c_1^* = 1.5c_2^*$ **(c)** $c_2^* = 1.5c_1^*$

Finally, consider all possible (nonnegative) values of p_1 and p_2. Describe the set of limiting concentrations c_1^* and c_2^* that can be obtained by adjusting p_1 and p_2.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 6.4. Three Interconnected Tanks

☞ Take a look at Chapter 8 for a way to diagram this "compartment" model.

Consider three interconnected tanks containing salt water. Initially Tanks 1 and 2 contain 10 gal of water while Tank 3 contains 15 gal. Each tank initially contains 6 oz of salt.

Water containing 2 oz of salt per gal flows into Tank 1 at a rate of 1 gal/min. The mixture in Tank 1 flows into Tank 2 at a rate of r gal/min. Furthermore, the mixture in Tank 1 is discharged into the drain at a rate of 2 gal/min. Water containing 1 oz of salt per gal flows into Tank 2 at a rate of 2 gal/min. The mixture in Tank 2 flows into Tank 3 at a rate of $r+1$ gal/min and also flows back into Tank 1 at a rate of 1 gal/min. The mixture in Tank 3 flows into Tank 1 at a rate of r gal/min, and down the drain at a rate of 1 gal/min.

1. Draw a diagram that depicts the tank system. Does the amount of water in each tank remain constant during the process? Show that the flow process is modeled by the following system of equations, where $q_1(t)$, $q_2(t)$, and $q_3(t)$ are the amounts of salt (in oz) in the respective tanks at time t:

$$\begin{aligned} q_1' &= 2 - \frac{r+2}{10}q_1 + \frac{1}{10}q_2 + \frac{r}{15}q_3 \\ q_2' &= 2 + \frac{r}{10}q_1 - \frac{r+2}{10}q_2 \\ q_3' &= \frac{r+1}{10}q_2 - \frac{r+1}{15}q_3 \end{aligned}$$

What are the corresponding initial conditions?

2. Let $r = 1$, and use ODE Architect to plot q_1 vs. t, q_2 vs. t, and q_3 vs. t for the IVP in Problem 1. Estimate the limiting value of the amount of salt in each tank after a long time. Now suppose that the flow rate r is increased to 4 gal/min. What effect do you think this will have on the limiting values for q_1, q_2, and q_3? Check your intuition with ODE Architect. What do you think will happen to the limiting values if r is increased further? For each value of r use ODE Architect to find the limiting values for q_1, q_2, and q_3.

☞ Use ODE Architect to find the eigenvalues.

3. Although the two sets of graphs in Problem 2 may look similar, they're actually slightly different. Calculate the eigenvalues of the coefficient matrix when $r = 1$ and when $r = 4$. There is a certain "critical" value $r = r_0$ between 1 and 4 where complex eigenvalues first occur. Determine r_0 to two decimal places.

4. Complex eigenvalues lead to sinusoidal solutions. Explain why the oscillatory behavior characteristic of the sine and cosine functions is not apparent in your graphs from Problem 2 for $r = 4$. Devise a plan that will enable you to construct plots showing the oscillatory part of the solution for $r = 4$. Then execute your plan to make sure that it is effective.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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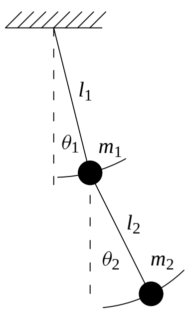
Exploration 6.5. Small Motions of a Double Pendulum; Coupled Springs



Another physical system with two degrees of freedom is the planar double pendulum. This consists of two rods of length l_1 and l_2 and two masses m_1 and m_2, all attached together so that motions are confined to a vertical plane. Here we'll investigate motions for which the pendulum system doesn't move too far from its stable equilibrium position in which both rods are hanging vertically downward. We'll assume the damping in this system is negligible.

A sketch of the double pendulum system is shown in the margin. A derivation of the nonlinear equations in terms of the angles $\theta_1(t)$ and $\theta_2(t)$ that govern the oscillations of the system is given in Chapter 7 (beginning on page 126). The equations of interest here are the linearized ODE in θ_1 and θ_2 where both of these angles are required to be small:

$$l_1\theta_1'' + \frac{m_2}{m_1+m_2} l_2\theta_2'' + g\theta_1 = 0$$

$$l_2\theta_2'' + l_1\theta_1'' + g\theta_2 = 0$$

For small values of θ_1, θ_1', θ_2, and θ_2' these ODEs are obtained by linearizing ODEs (19) and (20) on page 127.

1. Consider the special case where $m_1 = m_2 = m$ and $l_1 = l_2 = l$, and define $g/l = \omega_0^2$. Write the equations above as a system of four first-order equations. Use ODE Architect to generate motions for different values of ω_0. Experiment with different initial conditions. Try to visualize the motions of the pendulum system that correspond to your solutions. Then use the model-based animation tool in ODE Architect and watch the animated double pendulums gyrate as your initial value problems are solved.

2. Assume $\omega_0^2 = 10$ in Problem 1. Can you find in-phase and out-of-phase oscillations that are analogous to those of the coupled mass-spring system? Determine the relationships between the initial conditions $\theta_1(0)$ and $\theta_2(0)$ that are needed to produce these motions. Plot θ_2 against θ_1 for these motions. Then change $\theta_1(0)$ or $\theta_2(0)$ to get a motion which is neither in-phase nor out-of-phase. Overlay this graph on the first plot. Explain what you see. Use the model-based animation feature in ODE Architect to help you "see" the in-phase and out-of-phase motions, and those that are neither. Describe what you see.

3. Show that the linearized equations for the double pendulum in Problem 2 are equivalent to those for a particular coupled mass-spring system. Find the corresponding values of (or constraints on) the mass-spring parameters m_1, m_2, k_1, and k_2. Does this connection extend to other double-pendulum parameter values besides those in Problems 1 and 2? If so, find the relationships between the parameters of the corresponding systems. Use the model-based animation feature in ODE Architect and watch the springs vibrate and the double pendulum gyrate. Describe what you see.