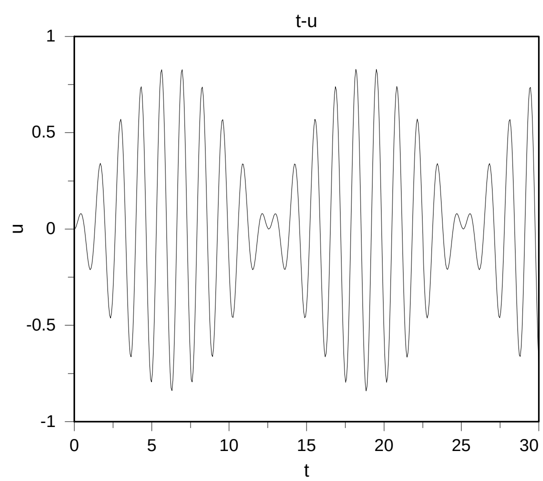
CHAPTER 4

Second-Order Linear Equations



The phenomenon of beats.

Overview Second-order linear differential equations, especially those with constant coefficients, have a host of important applications. In this chapter we explore some phenomena involving mechanical and electrical oscillations. The first submodule deals with some basic features common to oscillations of all sorts. The second submodule applies some of these results to seismographs, which are instruments used for recording earthquake data.

Key words Oscillation; period; frequency; amplitude; phase; simple harmonic motion; viscous damping; underdamping; overdamping; critical damping; transient; steady-state solution; forced oscillation; seismograph; Kirchhoff's laws

See also Chapter 5 for more on vectors and damping, Chapters 6 and 10 for more on oscillations, and Chapter 12 for more on forced oscillations.

◆ Second-Order ODEs and the Architect

ODE Architect will accept only first-order ODEs, so how can we use it to solve a second-order ODE? There is a neat trick that does the job, and an example will show how. Suppose we want to use ODE Architect to study the behavior of the *initial value problem* (or IVP):

$$u'' + 3u' + 10u = 5\cos(2t), \quad u(0) = 1, \quad u'(0) = 0 \tag{1}$$

Let's write $v = u'$, then

$$v' = \frac{d}{dt}(v) = \frac{d}{dt}(u') = u''$$

but

$$u'' = -10u - 3u' + 5\cos(2t)$$

so IVP (1) becomes

$$\begin{aligned} u' &= v, \quad u(0) = 1 \\ v' &= -10u - 3v + 5\cos(2t), \quad v(0) = 0 \end{aligned} \tag{2}$$

☞ ODE Architect only accepts ODEs in *normal form*; for example, write $2x' + x = 6$ as $x' = x/2 + 3$ with the x' term alone on the left.

ODE Architect won't accept IVP (1), but it will accept the equivalent IVP (2). The components u and v give the solution of IVP (1) and its first derivative $u' = v$. Therefore, if we use ODE Architect to solve and plot the component curve $u(t)$ of system (2), we are simultaneously plotting the solution $u(t)$ of IVP (1).

✓ "Check" your understanding by converting the IVP

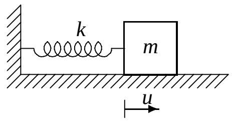
$$2u'' - 2u' + 3u = -\sin(4t), \quad u(0) = -1, \quad u'(0) = 2$$

to an equivalent IVP involving a system of two normalized first-order ODEs.

◆ Undamped Oscillations

Second-order differential equations arise naturally in physical situations; for example, the motion of an object is described by Newton's second law, $F = ma$. Here, a is the acceleration, which is the second derivative of the object's position. Many of these differential equations lead to *oscillations* or *vibrations*. Many oscillating systems can be modeled by a system consisting of a mass attached to a spring where the motion takes place in a horizontal direction on a table. This simplifies the derivation of the equation of motion, but the same equation also describes the up-and-down motion of a mass suspended by a vertical spring.

Let's assume an ideal situation: there is no friction between the mass and the table, there is no air resistance, and there is no dissipation of energy in the



spring or anywhere else in the system. The differential equation describing the motion of the mass is

$$m\frac{d^2u}{dt^2} = -ku \tag{3}$$

where $u(t)$ is the position of the mass m relative to its equilibrium and k is the spring constant. The natural tendency of the spring to return to its equilibrium position is represented by the restoring force $-ku$. Two initial conditions,

☞ This is also called *Hooke's law restoring force*.

$$u(0) = u_0, \qquad u'(0) = v_0 \tag{4}$$

where u_0 and v_0 are the initial position and velocity of the mass, respectively, determine the position of the mass uniquely. ODE (3) together with the initial conditions (4) constitute a well-formulated initial value problem whose solution predicts the position of the mass at any future time.

☞ See the first two references for derivations of formula (5).

The general solution of ODE (3) is

$$u(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) \tag{5}$$

where C_1 and C_2 are arbitrary constants and $\omega_0^2 = k/m$. Applying the initial conditions (4), we find that $C_1 = u_0$ and $C_2 = v_0/\omega_0$. Thus the solution of the IVP (3), (4) is

$$u(t) = u_0\cos(\omega_0 t) + (v_0/\omega_0)\sin(\omega_0 t) \tag{6}$$

The corresponding motion of the mass is *periodic*, which means that it repeats itself after the passage of a time interval T called the *period*. If we measure time in seconds, then the quantity ω_0 is the *natural (circular) frequency* in radians/sec, and T is given by

☞ The term "circular frequency" is only used with trigonometric functions.

$$T = 2\pi/\omega_0 \tag{7}$$

The reciprocal of T, or $\omega_0/2\pi$, is the *frequency* of the oscillations measured in cycles per second, or *hertz*. Notice that since $\omega_0 = \sqrt{k/m}$, the frequency and the period depend only on the mass and the spring constant and not on the initial data u_0 and v_0.

By using a trigonometric identity, the solution (6) can be rewritten in the *amplitude-phase form* as a single cosine term:

☞ This motion is called *simple harmonic motion*. See Screeen 1.3 of Module 4 for graphs.

$$u(t) = A\cos(\omega_0 t - \delta) \tag{8}$$

where A and δ are expressed in terms of u_0 and v_0/ω_0 by the equations

$$A = \sqrt{u_0^2 + (v_0/\omega_0)^2}, \quad \tan\delta = \frac{v_0}{u_0\omega_0} \tag{9}$$

The quantity A determines the magnitude or *amplitude* of the oscillation (8), and δ, called the *phase* (or *phase shift*), measures the time translation from a standard cosine curve.

✓ Show that (8) is equivalent to (6) when A and δ are defined by (9).

◆ The Effect of Damping

☞ The viscous damping force is $-c\,du/dt$.

Equation (8) predicts that the periodic oscillation will continue indefinitely. A more realistic model of an oscillating spring must include damping. A simple, useful model results if we represent the damping force by a single term that is proportional to the velocity of the mass. This model is known as the *viscous damping* model; it leads to the differential equation

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0 \tag{10}$$

where the positive constant c is the viscous damping coefficient.

The behavior of the solutions of ODE (10) is determined by the roots r_1 and r_2 of the *characteristic polynomial equation*,

$$mr^2 + cr + k = 0$$

Using the quadratic formula, we find that the *characteristic roots* r_1 and r_2 are

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}, \quad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \tag{11}$$

☞ Check that this equation gives a solution of ODE (10).

The nature of the solutions of ODE (10) depends on the sign of the *discriminant* $c^2 - 4mk$. If $c^2 \neq 4mk$, then $r_1 \neq r_2$ and the general solution of ODE (10) is

$$u = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{12}$$

where C_1 and C_2 are arbitrary constants.

The most important case is *underdamping* and occurs when $c^2 - 4mk < 0$, which means that the damping is relatively small. In the underdamped case, the characteristic roots r_1 and r_2 in formula (11) are the complex numbers

$$r_1 = -\frac{c}{2m} + i\mu, \quad r_2 = -\frac{c}{2m} - i\mu, \quad \text{where } \mu = \frac{\sqrt{4mk - c^2}}{2m} \neq 0 \tag{13}$$

Euler's formula implies that

$$e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

for any real numbers α and β, so

$$e^{r_1 t} = e^{-ct/2m}(\cos\mu t + i\sin\mu t), \quad e^{r_2 t} = e^{-ct/2m}(\cos\mu t - i\sin\mu t) \tag{14}$$

Now, using the initial conditions together with equations (12) and (14), we find that the solution of the IVP

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0, \quad u(0) = u_0, \quad u'(0) = v_0 \tag{15}$$

☞ Solutions of an underdamped ODE oscillate with circular frequency μ and an exponentially decaying amplitude.

is given by

$$u = e^{-ct/2m}\left\{u_0\cos(\mu t) + \left[\frac{v_0}{\mu} + \frac{cu_0}{2m\mu}\right]\sin(\mu t)\right\} \tag{16}$$

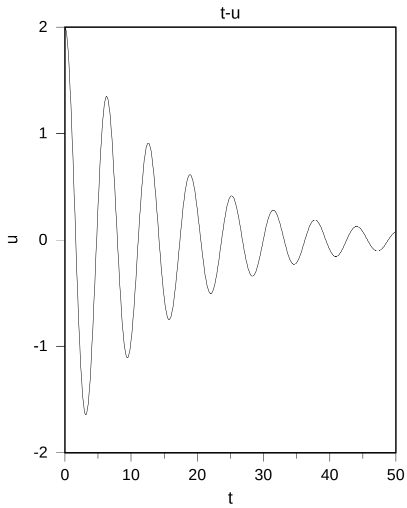


Figure 4.1: A solution curve of the underdamped spring-mass ODE, $u'' + 0.125u' + u = 0$.

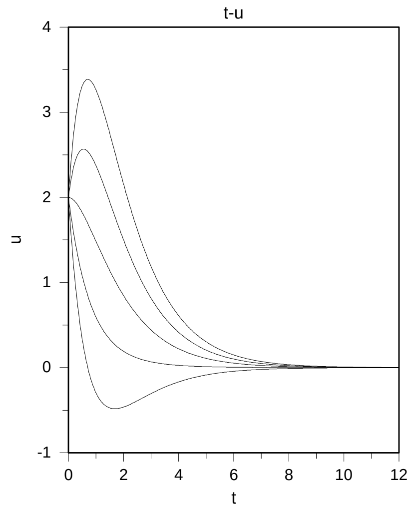


Figure 4.2: Solution curves of the overdamped spring-mass ODE, $u'' + 2.1u' + u = 0$.

✓ Verify that $u(t)$ defined in formula (16) is a solution of IVP (15).

The solution (16) represents an oscillation with circular frequency μ and an exponentially decaying amplitude (see Figure 4.1). From the formula in (13) we see that $\mu < \omega_0$, where $\omega_0 = \sqrt{k/m}$, but the difference is small for small c.

☞ Take a look at Screen 1.6 of Module 4.

If the damping is large enough so that $c^2 - 4mk > 0$, then we have *over-damping* and the solution of IVP (15) decays exponentially to the equilibrium position but does not oscillate (see Figure 4.2). The transition from oscillatory to nonoscillatory motion occurs when $c^2 - 4mk = 0$. The corresponding value of c, given by $c_0 = 2\sqrt{mk}$, is called *critical damping*.

◆ Forced Oscillations

☞ $F(t)$ is also called the *input*, or *driving term*; solutions $u(t)$ are the *responses* to the input and the initial data.

Now let's see what happens when an external force is applied to the oscillating mass described by ODE (10). If $F(t)$ represents the external force, then ODE (10) becomes

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = F(t) \tag{17}$$

Some interesting things happen if $F(t)$ is periodic, so we will look at the ODE

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = F_0\cos(\omega t) \tag{18}$$

☞ Check that this formula gives solutions of ODE (18).

where F_0 and ω are the amplitude and circular frequency, respectively, of the external force F. Then, in the underdamped case, the general solution of ODE (18) has the form

$$u(t) = e^{-ct/2m}[C_1 \cos(\mu t) + C_2 \sin(\mu t)] + a\cos(\omega t) + b\sin(\omega t) \tag{19}$$

where a and b are constants determined so that $a\cos(\omega t) + b\sin(\omega t)$ is a solution of ODE (18). The constants a and b depend on m, c, k, F_0, and ω of ODE (18), but not on the initial data. The constants C_1 and C_2 can be chosen so that $u(t)$ given by formula (19) satisfies given initial conditions.

The first term on the right side of the solution (19) approaches zero as $t \to +\infty$; this is called the *transient* term. The remaining two terms do not diminish as t increases, and their sum is called the *steady-state solution* (or the *forced oscillation*), here denoted by $u_s(t)$. Since the steady-state solution persists forever with constant amplitude, it is frequently the most interesting solution. Notice that it oscillates with the circular frequency ω of the driving force F. It can be written in the amplitude-phase form (8) as

$$u_s(t) = A\cos(\omega t - \delta) \tag{20}$$

where A and δ are now given by

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}, \qquad \tan\delta = \frac{c\omega}{m(\omega_0^2 - \omega^2)} \tag{21}$$

Figure 4.3 shows a graphical example of solutions that tend to a forced oscillation.

For an underdamped system with fixed c, k, and m, the amplitude A of the steady-state solution depends upon the frequency of the driving force. It

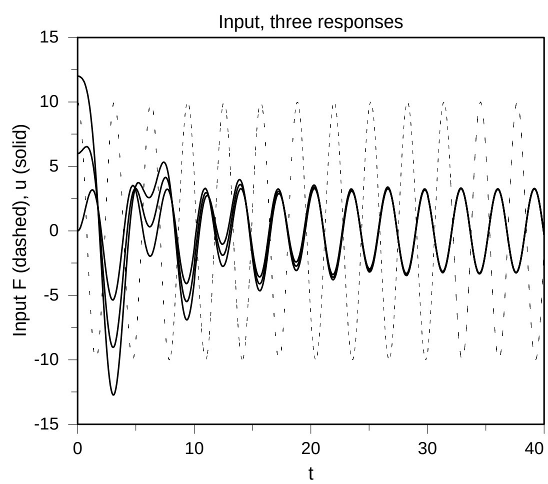


Figure 4.3: Solutions of $u'' + 0.3u' + u = 10\cos 2t$ approach a unique forced oscillation with the circular frequency 2 of the input.

is important to know whether there is a value $\omega = \omega_r$ for which the amplitude is maximized. If so, then driving the system at the circular frequency ω_r produces the greatest response. Using methods of calculus, it can be shown that if $c^2 < 2mk$ then ω_r is given by

☞ Recall that the *natural circular frequency* ω_0 is the value $\omega_0 = \sqrt{k/m}$.

$$\omega_r^2 = \omega_0^2 \left(1 - \frac{c^2}{2mk}\right) \tag{22}$$

The corresponding maximum value A_r of the amplitude when $\omega = \omega_r$ is

$$A_r = \frac{F_0}{c\omega_0\sqrt{1-(c^2/4mk)}} \tag{23}$$

✓ Does A have a maximum value when $2mk < c^2 < 4mk$?

✓ Find the forced oscillation for the ODE of Figure 4.3.

◆ Beats

Let's polish the table and streamline the mass so that damping is negligible. Then we apply a forcing function whose frequency is close to the natural frequency of the spring-mass system, and watch the response. We can model this by the IVP

$$u'' + \omega_0^2 u = \frac{F_0}{m}\cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0 \tag{24}$$

where $|\omega_0 - \omega|$ is small (but not zero). The solution is

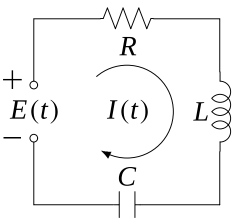
$$\begin{aligned} u(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)}[\cos(\omega t) - \cos(\omega_0 t)] \\ &= \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)}\sin\left(\frac{\omega_0 - \omega}{2}t\right)\right]\sin\left(\frac{\omega_0 + \omega}{2}t\right) \end{aligned} \tag{25}$$

where trigonometric identities have been used to get from the first form of the solution to the second. The term in square brackets in formula (25) can be viewed as a varying amplitude for the sinusoid term $\sin[(\omega_0 + \omega)/2]t$. Since $|\omega_0 - \omega|$ is small, the circular frequency $(\omega_0 + \omega)/2$ is much higher than the low circular frequency $(\omega_0 - \omega)/2$ of the varying amplitude. Therefore we have a rapid oscillation with a slowly varying amplitude. This is the *beat phenomenon* illustrated on the chapter cover figure for the IVP

$$u'' + 25u = 2\cos(4.5t), \quad u(0) = 0, \quad u'(0) = 0$$

If you try this out with a driven mass on a spring you will see rapid oscillations whose amplitude slowly grows and then diminishes in a repeating pattern. This phenomenon can actually be heard when a pair of tuning forks which have nearly equal frequencies are struck simultaneously. We hear the "beats" as each acts as a driving force for the other.

◆ Electrical Oscillations: An Analogy



Linear differential equations with constant coefficients are important because they arise in so many different physical contexts. For example, an ODE similar to ODE (17) can be used to model charge oscillations in an electrical circuit. Suppose an electrical circuit contains a resistor, an inductor, and a capacitor connected in series. The current I in the circuit and the charge Q on the capacitor are functions of time t. Let's assume we know the resistance R, the inductance L, and the capacitance C. By Kirchhoff's voltage law for a closed circuit, the applied voltage $E(t)$ is equal to the sum of the voltage drops through the various elements of the circuit. Observations of circuits suggests that these voltage drops are as follows:

- The voltage drop through the resistor is RI (Ohm's law);
- The voltage drop through the inductor is $L(dI/dt)$ (Faraday's law);
- The voltage drop through the capacitor is Q/C (Coulomb's law).

Thus, by Kirchhoff's law, we obtain the differential equation

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t) \tag{26}$$

Since $I = dQ/dt$, we can write ODE (26) entirely in terms of Q,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t) \tag{27}$$

ODE (27) models the charge $Q(t)$ on the capacitor of what is called the *simple RLC circuit* with voltage source $E(t)$. ODE (27) is equivalent to ODE (17), except for the symbols and their interpretations. Therefore we can also apply conclusions about our spring-mass system to electrical circuits. For example, we can interpret the ODE $u'' + 0.3u + u = 10\cos 2t$, whose solutions are graphed in Figure 4.3, as a model either for the oscillations of a damped and driven spring-mass system, or the charge on the capacitor of a driven RLC circuit. We see that a mathematical model can have many interpretations, and any mathematical conclusions about the model apply to every interpretation.

✓ What substitutions of parameters and variables would you have to make in ODE (27) to transform it to ODE (17)?

◆ Seismographs

☞ Look at "Earthquakes and the Richter Scale" in Module 4.

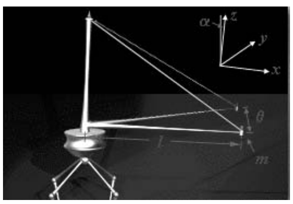
Seismographs are instruments that record the displacement of the ground as a function of time, and a seismometer is the part of a seismograph that responds to the motion. Seismographs come in two generic types. Matt's friend Seismo is a horizontal-component seismograph, which records one of the horizontal components of the earth's local motion. Of course, two horizontal components are required to specify fully horizontal motion, usually by means of north-south and east-west components. The other type of seismograph records the vertical component of motion. Both of these instruments are based on pendulums that respond to the motion of the ground relative to the seismograph.

☞ If you're queasy about cross products or approximating functions (as we do in formula (29)) you may prefer to skip directly to ODE (33) or ODE (34).

Since Seismo is an animation of a horizontal-component seismograph, we'll outline the derivation of the ODEs that govern the motion of his arm. The starting point is the angular form of Newton's second law of motion, also known as the *angular momentum law*:

$$\frac{d}{dt}\mathbf{L} = \mathbf{R} \times \mathbf{F} \tag{28}$$

where $\mathbf{L}$ is the angular momentum of a mass (Seismo's arm and hand) about a fixed axis, $\mathbf{F}$ is the force acting on the mass, $\mathbf{R}$ is the position vector from the center of mass of Seismo's arm and hand to the axis, and $\times$ is the vector cross product.



We'll apply this law using an orthogonal xyz-coordinate system which is illustrated on Screen 2.2 of Module 4. In this system the y-axis is horizontal. Seismo's body is parallel to the z-axis and the rest position of Seismo's arm is parallel to the x-axis. The z-axis is not parallel to the local vertical, but instead is the axis which results from rotating the local vertical through a small angle α about the y-axis. Because of this small tilt, the x-axis points slightly downward and the arm is in a stable equilibrium position when it is parallel to the x-axis. The seismic disturbance is assumed to be in the direction of the y-axis. The xz-plane is called Seismo's *rest plane*.

Seismo's hand writes on the paper in the xy-plane, and the angle θ measures the angular displacement of his arm from its rest position. Consider an axis pointing in the z-direction and through the center of mass of Seismo's arm and hand, and let m represent the mass of the arm and hand. The z-component of the angular momentum about that axis is $mr^2(d\theta/dt)$ where r is the radius of gyration of the arm.

To compute the right-hand side of ODE (28), we need to know $\mathbf{R}$, the position vector from the center of mass of Seismo's arm and hand to the origin. Note that

$$\mathbf{R} = -l\cos\theta\hat{\mathbf{x}} - l\sin\theta\hat{\mathbf{y}}$$

where l is the distance from the center of mass to Seismo's body, and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors along the positive x- and y-axes. For small θ, we have the approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$, so

$$\mathbf{R} = -l\hat{\mathbf{x}} - l\theta\hat{\mathbf{y}} \tag{29}$$

Using equation (29) in ODE (28) and computing the cross product, we obtain

$$mr^2\frac{d^2\theta}{dt^2} = -lF^{(y)} + l\theta F^{(x)} \tag{30}$$

indicating by superscripts the components of the net force **F** exerted on the arm and hand.

Now we need expressions for the two components of **F** in ODE (30). If the x-component of friction is assumed negligible then the two force components acting in the x-direction are the x-component of the gravitational force and the x-component of the force due to the seismic disturbance. Because the arm displacement angle θ and the body inclination angle α are both assumed small, the x-component of the force due to the seismic disturbance can be shown to be negligible also. Therefore the x-component of the net force, $F^(x)$ is given by the simple form

$$F^{(x)} \approx mg\alpha \tag{31}$$

The right side of equation (31) is the gravity component $mg \sin \alpha$ approximated by $mg\alpha$.

☞ This is *viscous friction*.

In the y-direction, the forces acting are the force due to the seismic disturbance and to friction, the latter assumed to be proportional to the angular velocity $d\theta/dt$. The force due to the seismic disturbance can be computed as follows: Let h be a small ground displacement in the y-direction. Then the y-coordinate of the center of mass is approximately $h + l\theta$. Therefore the force due to the earthquake is approximated by

$$m\frac{d^2}{dt^2}(h + l\theta) = m\frac{d^2h}{dt^2} + ml\frac{d^2\theta}{dt^2}$$

and the net force in the y-direction is

$$F^{(y)} \approx m\frac{d^2h}{dt^2} + ml\frac{d^2\theta}{dt^2} - k\frac{d\theta}{dt} \tag{32}$$

where k is a positive constant characterizing the effect of friction.

Combining ODE (30) with formulas (31) and (32), we find that the motions of Seismo's arm are governed by the ODE

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \omega_0^2\theta = -\frac{1}{L}\frac{d^2h}{dt^2} \tag{33}$$

In ODE (33) the quantities ω_0^2, L, and c are given by $\omega_0^2 = g\alpha/L$, where $L = (r^2 + l^2)/l$, and $c = k/(mL)$. We can interpret the terms in (33) as follows. The first term on the left arises from the inertia of Seismo's hand and arm. The second term models the frictional force due to the angular motion of the arm. The third term, arising from gravity and the tilt of the arm, is the restoring force for the oscillations of the arm and hand. Finally, the term on the right arises from the effective force of the seismic displacement.

To simplify ODE (33) a little more, we let $h(t) = Hf(t)$, where H is the maximum ground displacement, which means that the maximum value of the dimensionless ground displacement $f(t)$ is one. Then ODE (33) becomes the following equation for the dimensionless arm displacement $y(t) = L\theta(t)/H$:

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2f}{dt^2} \tag{34}$$

This is the ODE used in Screen 2.3 of Module 4.

☞ A given ODE can model a variety of phenomena.

It's an important result that ODE (34) is the same type as ODE (18), the only differences being in the definitions of the parameters that multiply the individual terms, and in the choice of variables. A second striking result is that this same ODE (34) applies as well to the motions of a vertical component seismograph. All that is necessary are other modifications in the meanings of the parameters and functions. Details are available in the book by Bullen and Bolt in the references.

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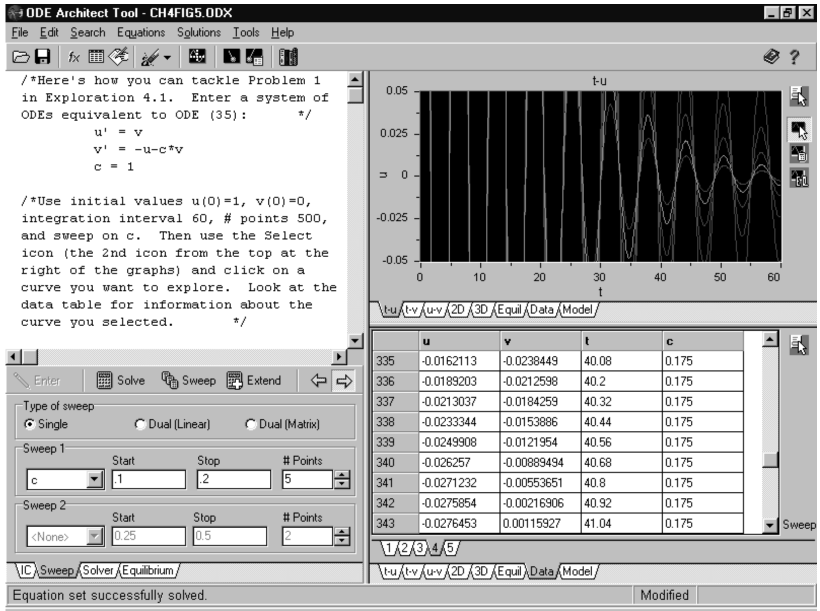

Figure 4.4: The sweep on c generated five solution curves. The selected curve is highlighted, and the corresponding solution $u(t)$ satisfies the condition $|u(t)| < 0.05$ for $t \geq 40$. The data tells us that $c = 0.175$ for the curve.

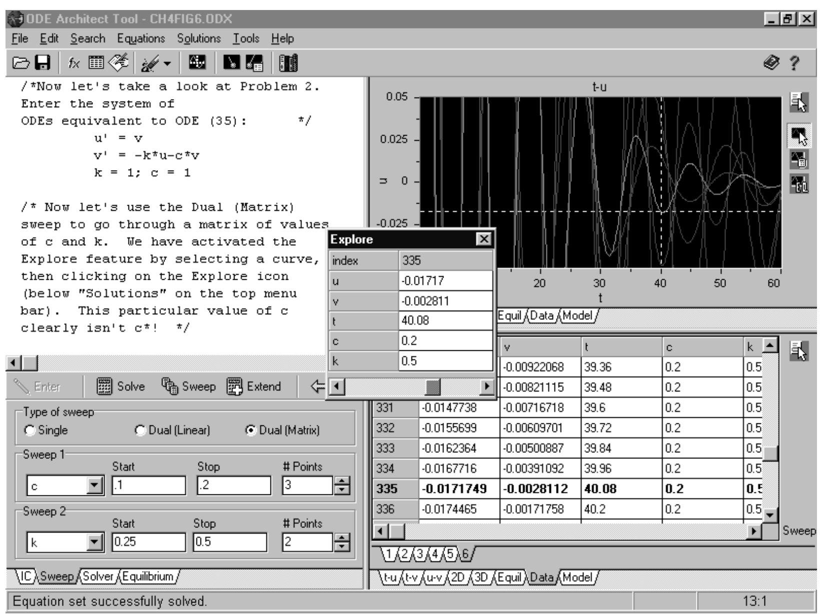

Figure 4.5: The Dual (Matrix) feature produces six solutions for various values of c and k. We have selected one of them (the highlighted curve) and used the Explore option to get additional information.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date ______________________

Course/Section ______________________

Exploration 4.1. The Damping Coefficient

Assume that Dogmatic's oscillations satisfy the IVP

$$u'' + cu' + ku = 0, \quad u(0) = 1, \quad u'(0) = 0 \tag{35}$$

1. Let $k = 1$ and use ODE Architect to estimate the smallest value c^* of the damping coefficient c so that $|u(t)| \le 0.05$ for all $t \ge 40$. [*Suggestion:* Figure 4.4 illustrates one way to estimate c^* by using the Select feature and the Data table.]

2. Repeat Problem 1 for other values of k, including $k = \frac{1}{4}, \frac{1}{2}, 2$, and 4. How does c^* change as k changes? [*Suggestion:* Figure 4.5 shows the outcome of using a Dual (Matrix) sweep on the values of c and k, and then using the Explore feature.]

3. Let $k = 10$ in IVP (35).

(a) Find the value of c for which the ratio of successive maxima in the graph of u vs. t is 0.75.

(b) Why is the ratio between successive maxima always the same?
Note: Since the values of the maxima can be observed experimentally, this provides a practical way to determine the value of the damping coefficient c, which may be difficult to measure directly.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date ____________________

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Exploration 4.2. Response to the Forcing Frequency

1. Suppose that Dogmatic's oscillations satisfy the differential equation

$$2u'' + u' + 4u = 2\cos(\omega t)$$

Let $\omega = 1$. Select your own initial conditions and use ODE Architect to plot the solution over a long enough time interval that the transient part of the solution becomes negligible. From the graph, determine the amplitude A_s of Dogmatic's steady-state solution.

2. Repeat Problem 1 for other values of ω. Plot the corresponding pairs ω, A_s and sketch the graph of A_s vs. ω. Estimate the value of ω for which A_s is a maximum. *Note*: You may want to use the Lookup Table feature of ODE Architect (see Module 1 and Chapter 1 for details).

3. In Problems 1 and 2, the value of the damping coefficient c is 1. Repeat your calculations for $c = \frac{1}{2}$ and $c = \frac{1}{4}$. How does the maximum value of A_s change as the value of c changes? Compare your results with the predictions of formula (23).

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 4.3. Low- and High-Frequency Quakes

In experiments with Seismo, you used ODE (34) to find the response of his arm to different ground displacements of sinusoidal type, $f(t) = \cos \omega t$, when $1 \leq \omega \leq 5$. In this exploration you'll investigate what happens for ground displacements with frequencies that are lower or higher than these values.

1. Choose $c = 2$ and $\omega_0 = 3$ in ODE (34), and set the initial conditions $y(0)$ and $y'(0)$ to zero. Use $f(t) = \cos \omega t$ with $\omega = 0.5$ for the ground displacement. Use ODE Architect to plot the displacement $y(t)$ determined from ODE (34); also plot $f(t)$ on the same graph. How do the features of $y(t)$ compare with those of $f(t)$?

2. Repeat Problem 1 for values of ω smaller than 0.5. Be sure to plot for a long enough time interval to see the relevant time variations. What do you think is Seismo's arm response as ω approaches zero? How does this compare with the corresponding response of a mass on a spring from ODE (18)?

3. Repeat Problem 1 for values of ω larger than 5, such as $\omega = 10$ and $\omega = 20$. What do you think is Seismo's arm response as ω becomes very large?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 4.4. Different Ground Displacements

In explorations with Seismo, we assumed that the dimensionless ground displacements $f(t)$ are sinusoidal, with a single frequency. Real earthquakes however, are not so simple: you'll investigate other possibilities in the following problems. The ODE for Seismo's dimensionless arm displacement $y(t)$ is

$$\frac{d^2y}{dt^2} + c\frac{dy}{dt} + \omega_0^2 y = -\frac{d^2 f}{dt^2} \tag{36}$$

1. Suppose the ground displacement can be modeled by the function

$$f(t) = \begin{cases} (t/T)^2, & 0 \le t \le T \\ 1, & t > T \end{cases}$$

How do you interpret this motion? Choose $c = 2$ and $\omega_0 = 3$, and set $y(0) = y'(0) = 0$. Use ODE Architect to find $y(t)$ from ODE (36) for the case $T = 2$, and display both $y(t)$ and $f(t)$. *Note*: d^2f/dt^2 can be written using a step function. How do the features of $y(t)$ compare with those of $f(t)$? For example, what is the maximum magnitude of $y(t)$, and when does it occur?

2. Now suppose that the ground motion is given by the function $f = e^{-at}\sin(\pi t)$. Choose some values of a in the range $0 < a \leq 0.5$ and study how Seismo's arm displacements change with the parameter a.

3. How do you think the results of Problem 2 would change if the period of the sinusoidal oscillation were different from 2? Try a few cases to check your predictions.