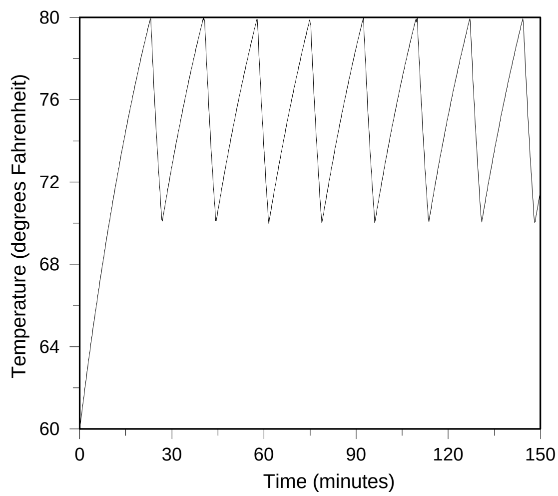
CHAPTER 3 Some Cool ODEs



A room heats up in the morning, and the air conditioner in the room starts its on-off cycles.

Overview In this chapter, we'll use Newton's law of cooling to build mathematical models of a number of situations that involve the variation of temperature in a body with time. Some of our models involve ODEs that can be solved analytically; others will be solved numerically by ODE Architect. We'll compare the analytical solutions and the numerical results and see how both can be used to verify predictions made by the models.

Key words Modeling; Newton's law of cooling (and warming); initial conditions; general solution; separation of variables; integrating factor; heat energy; melting; air conditioning

See also Chapter 1 for more on modeling and Chapter 2 for the technique of separation of variables.

◆ Newton's Law of Cooling

Have you ever gotten an order of piping hot French fries, only to find them ice cold in what seems like a matter of moments? Whenever an object (or substance) is warmer than its surroundings, it cools because it loses heat energy. The greater the temperature difference between the object and its surroundings, the faster the object cools. The temperature of a body rises if its surroundings are at a higher temperature than it is. What happens to the ice cream in a cone on a hot day?

Although it is an oversimplification, we will assume that the temperature is uniform at all points in the objects we wish to model, but the temperature may change with time. Let's assume that the rate of change of the object's temperature is proportional to the difference between its temperature and that of its surroundings. Stated mathematically, we have:

☞ This becomes a "law of warming" if the surroundings are hotter than the object.

Newton's law of cooling. If $T(t)$ is the temperature of an object at time t and $T_{out}(t)$ is the temperature of its surroundings, then

$$\frac{dT}{dt} = k(T_{out} - T) \tag{1}$$

where k is a positive constant called the *cooling coefficient*.

◆ Cooling an Egg

What happens to the temperature of a hard-boiled egg when you take it out of a pot of boiling water? At first, the egg is the same temperature as the boiling water. Once you take it out of the water the egg begins to cool, rapidly at first and then more slowly. The temperature of the egg, $T(t)$, drops at a rate proportional to the difference between the temperature of the air, T_{out}, and $T(t)$. Notice from ODE (1) that if $T_{out} < T(t)$, the rate of change of temperature, dT/dt, is negative, so $T(t)$ decreases and your egg cools.

✓ "Check" your understanding by answering this question: What happens to the temperature of an egg if it is boiled at 212°F and then transferred to an oven at 400°F?

◆ Finding a General Solution

Equation (1) is a first-order ODE and its general solution contains one arbitrary constant. We can see this as follows: If T_{out} is a constant, then ODE (1) is separable, and separating the variables we have

☞ See Chapter 2 for how to solve a separable ODE.

$$\int \frac{dT}{T_{out} - T} = \int k\,dt$$

Why are the absolute value signs needed?

Finding an antiderivative for each side we obtain

$$-\ln|T_{out} - T(t)| = kt + K$$

where K is an arbitrary constant. Multiplying through by -1 and exponentiating gives us

$$|T_{out} - T(t)| = e^{-K}e^{-kt}$$

or, after dropping the absolute value signs, we have that

$$T(t) = T_{out} + Ce^{-kt} \tag{2}$$

where $C = \pm e^{-K}$ is now the arbitrary constant. The solution formula (2) is called the *general solution* of ODE (1).

✓ How does the temperature $T(t)$ in (2) behave as $t \to +\infty$? Why can the constant C be positive or negative?

Given an initial condition, we can determine C uniquely and identify a single solution from the general solution (2). If we take $T(0) = T_0$, then since $T(0) = T_{out} + C$ we see that $C = T_0 - T_{out}$ and we get the unique solution

$$T(t) = T_{out} + (T_0 - T_{out})e^{-kt} \tag{3}$$

The constant of proportionality, k, in ODE (1) determines the rate at which the body cools. It can be evaluated in a number of ways, for example, by measuring the body's temperature at two different times and using formula (3) to solve for T_0 and k. Figure 3.1 shows temperature curves corresponding to $T_{out} = 70°$F, $T_0 = 212°$F, and five values of k.

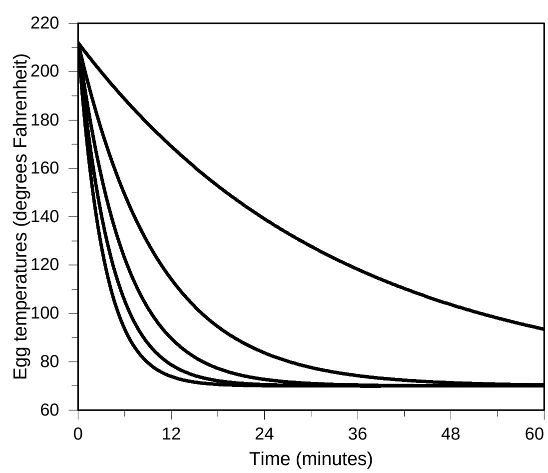


Figure 3.1: The cooling coefficient k ranges from 0.03 to 0.3 min^{-1} for eggs of different sizes. Which is the $k = 0.03$ egg?

✓ An object is initially at 212°F and cools to 190°F after 5 minutes in a room that is at 72°F. Find the coefficient of cooling, k, and determine how long it will take to cool to 100°F.

Finding the general solution formula (2) for ODE (1) was straightforward. However, the vast majority of ODEs are not so simple to solve and we have to use numerical methods. To demonstrate the accuracy of such methods, you can compare the numerical solutions from ODE Architect with a known solution formula.

☞ ODE Architect helps out again.

✓ How long will it take for a 212°F egg to cool to 190°F in a 72°F room if $k = 0.03419\ \text{min}^{-1}$? Use ODE Architect and formula (3) and compare the results.

◆ Time-Dependent Outside Temperature

When considering the cooling of an egg, ODE (1) is separable because T_{out} is constant in this instance. Let's consider what happens when the outside temperature changes with time.

We can still use Newton's law of cooling, so that if $T(t)$ is the egg's temperature and $T_{out}(t)$ is the room's temperature, then

$$\frac{dT}{dt} = k(T_{out}(t) - T) \tag{4}$$

Note that ODE (4) is not separable (because T_{out} varies with time) but it is linear, so we can find its general solution as follows. Rearrange the terms to give the linear ODE in standard form:

$$\frac{dT}{dt} + kT = kT_{out}(t)$$

Multiply both sides by e^{kt}, so that

$$e^{kt}\left(\frac{dT}{dt} + kT\right) = kT_{out}(t)e^{kt} \tag{5}$$

Since the left-hand side of ODE (5) is $(d/dt)(e^{kt}T(t))$, it can be rewritten:

$$\frac{d}{dt}\left(e^{kt}T\right) = kT_{out}(t)e^{kt} \tag{6}$$

Integrate both sides of ODE (6) we have that

$$e^{kt}T = \int kT_{out}(t)e^{kt}dt + C$$

where C is an arbitrary constant. The magic factor $\mu(t) = e^{kt}$ that enabled us to do this is called an *integrating factor*. So ODE (4) has the general solution

☞ Every ODE text discusses integrating factors and first-order linear ODEs.

$$T(t) = e^{-kt}\left(\int kT_{out}(t)e^{kt}dt + C\right)$$

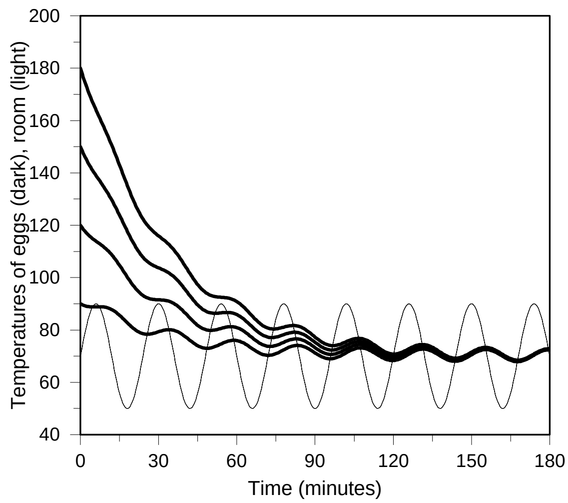


Figure 3.2: Eggs at initial temperatures of 180, 150, 120, and 90°F cool in a room whose temperature oscillates sinusoidally about 70°F for $k = 0.03$ min^{-1}. Do the initial temperatures matter in the long term?

Finally, letting $T(0) = T_0$ and integrating from 0 to t, we get the solution

$$T(t) = e^{-kt}\left(\int_0^t kT_{out}(s)e^{ks}\,ds + T_0\right) \tag{7}$$

It may be possible to evaluate the integral (7) analytically, but it is easier to use ODE Architect right from the start. See Figure 3.2 for egg temperatures in a room whose temperature oscillates between hot and cold.

✓ Show that if T_{out} is a constant, then formula (7) reduces to formula (3).

☞ You may find a computer algebra system or a table of integrals helpful!

✓ Use equation (7) to find a formula for $T(t)$ if

$$T_{out}(t) = 82 - 10\sin\left(\frac{2\pi(t+3)}{24}\right)$$

Use a table of integrals to carry out the integration.

◆ Air Conditioning a Room

Now let's build a model that describes a room cooled by an air conditioner. Without air conditioning, we can model the change in temperature using ODE (1). When the air conditioner is running, its coils remove heat energy at a rate proportional to the difference between T_r, the room temperature, and the temperature T_{ac} of the coils. So, using Newton's law of cooling for the temperature change due to both the air outside the room and the air conditioner coils, our model ODE is

☞ Newton's law of cooling (twice)!

$$\frac{dT_r}{dt} = k(T_{out} - T_r) + k_{ac}(T_{ac} - T_r)$$

where T_{out} is the temperature of the outside air and k and k_{ac} are the appropriate cooling coefficients. If the unit is turned off, then $k_{ac} = 0$ and this equation reduces to ODE (1).

Let's assume that the initial temperature of the room is 60°F and the outside temperature is a constant 100°F. The air conditioner operates with a coil temperature of 40°F, switches on when the room reaches 80°F, and switches off at 70°F. Initially, the unit is off and the change in the room temperature is modeled by

☞ Time t is measured in minutes.

$$\frac{dT_r}{dt} = 0.03(100 - T_r), \quad T_r(0) = 60 \tag{8}$$

where we have taken the cooling coefficient $k = 0.03$ min^{-1}. As we expect, the temperature in the room will rise as time passes. At some time t_{on} the room's temperature will reach 80°F and the air conditioner will switch on. If $k_{ac} = 0.1$ min^{-1}, then for $t > t_{on}$ the temperature is modeled by the IVP

$$\frac{dT_r}{dt} = 0.03(100 - T_r) + 0.1(40 - T_r), \quad T_r(t_{on}) = 80 \tag{9}$$

which is valid until the room cools to 70°F at some time t_{off}. Then for $t > t_{off}$ the room temperature satisfies the IVP (8) but with the new initial condition $T_r(t_{off}) = 70$. Each time the unit turns on or off the ODE alternates between the two forms given in (8) and (9).

☞ The modeling here is more advanced than you have seen up to this point. You may want to just use the equations and skim the modeling.

Solving the problem by hand in the manner just described is very tedious. However, we can use ODE Architect to change the ODE automatically and without having to find t_{on} and t_{off}. The key is to define k_{ac} to be a function of temperature by using a step function; here's how we do it. In the equation quadrant of ODE Architect write the ODE as

☞ The Step function is one of the engineering functions. You can find them by going to ODE Architect and clicking on Help, Topic Search, and Engineering Functions.

$$Tr' = 0.03 * (100 - Tr) + kac * (40 - Tr)$$

Now define k_{ac} as follows:

$$kac = 0.1 * \text{Step}(Tr, Tc)$$

where

$$Tc = 75 + 5 * B$$

Here T_c is the control temperature and

$$B = 2 * \text{Step}(Tr', 0) - 1$$

Note that $B = +1$ when $T_r' > 0$ (the room is warming) and $B = -1$ when $T_r' < 0$ (the room is cooling). This causes T_c to change from 80°F to 70°F (or

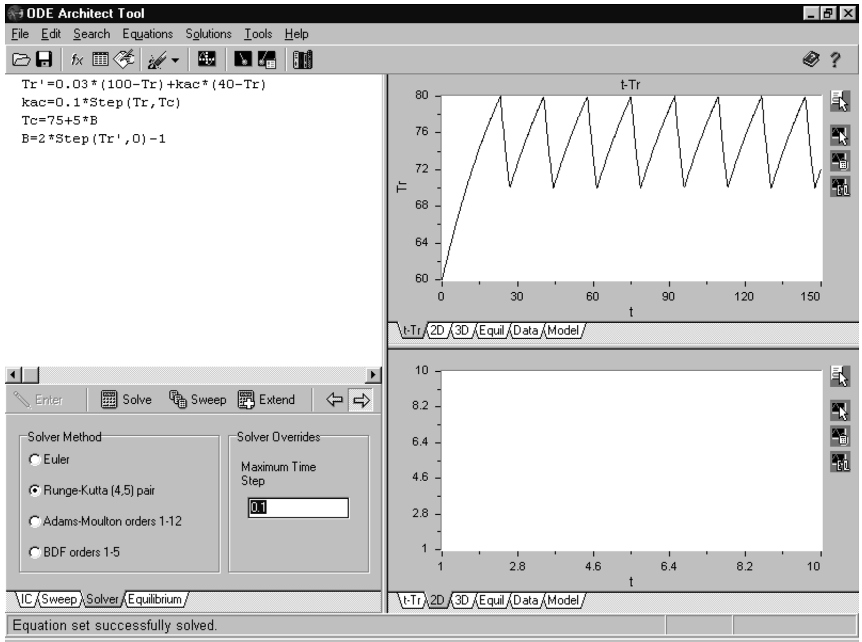


Figure 3.3: Air conditioning keeps the room temperature in the comfort zone, $70°\text{F} \leq T_r \leq 80°\text{F}$.

the reverse) depending on whether the room is warming or cooling. Finally, k_{ac} is zero (the air conditioner is off) when $T_r < T_c$, and $k_{ac} = 0.1$ (the air conditioner is on) when $T_r > T_c$.

The overall effect is that the air conditioner switches on only if the room temperature is above 80°F, then it runs until the room is cooled to 70°F, and then it switches off. The room temperature rises again to 80°F, and the process repeats. The temperature-vs.-time plot is shown in Figure 3.3. The accompanying screen image shows that we have set the maximum time step to 0.1 (under the Solver tab). If the internal time steps are not kept small, the Architect will not correctly notice when the step functions turn on and off.

◆ The Case of the Melting Snowman

It is difficult to model the melting of a snowman because of its complicated geometry: a large roundish ball of snow with another smaller mound on top. So let's simplify the model by treating the snowman as a single spherical ball of snow. The rate at which the snowman melts is proportional to the rate at which it gains thermal energy from the surrounding air, and it is given by

$$\frac{dV}{dt} = -h\frac{dE}{dt} \tag{10}$$

where V is the ball's volume, E is thermal energy, and h is a positive constant.

Our snowman will gain thermal energy only at its surface, where it is exposed to the warm air. So, it is reasonable to assume that the energy gain is proportional to both the surface area of the snowman and the temperature difference between the air and the snow:

☞ Remember that the snowman's temperature is always 32°F.

$$\frac{dE}{dt} = \kappa A(V)(T_{out} - 32) \tag{11}$$

where κ is a positive constant, and $A(V)$ is the surface area of a sphere of volume V.

☞ This is the snowman's law of melting.

If we combine equations (10) and (11) and take $k = \kappa h$, we obtain

$$\frac{dV}{dt} = -kA(V)(T_{out} - 32) \tag{12}$$

✓ The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$ and its surface area is $A = 4\pi r^2$. Eliminate r between these two formulas to express A as a function of V. (You will need this soon.)

Note that ODE (12) is separable even when the outside temperature T_{out} is a function of time. Separating the variables and integrating we find the formula

$$\int \frac{1}{A(V)}\, dV = -\int k(T_{out}(t) - 32)\, dt + C \tag{13}$$

which defines V implicitly as a function of t. We can find the constant of integration C from the volume of the snowman at a specific time. However, expressions for the integrals in formula (13) may be hard to find. Once again ODE Architect comes to the rescue and solves ODE (12) numerically, given formulas for $A(V)$, $T_{out}(t)$, and the initial volume.

✓ If $k = 0.1451$ ft/(hr °F), the original volume of the snowman is 10 ft^3, and the outside temperature is 40°F, how many hours does it take the snowman's volume to shrink to 5 ft^3?

References

Nagle, R.K., and Saff, E.B., *Fundamentals of Differential Equations*, 3rd ed. (1993: Addison-Wesley)

Farlow, S.J., *An Introduction to Differential Equations and their Applications*, (1994: McGraw-Hill)

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date ____________________

Course/Section ____________________

Exploration 3.1. Cooling Bodies

1. *Too hot to handle.*
When eating an egg, you don't want it to be too hot! If an egg with an initial temperature of 15°C is boiled and reaches 95°C after 5 minutes, how long will you have to wait until it cools to 70°C?

2. *A dead body, methinks.*
In forensic science, it is important to be able to estimate the time of death if the circumstances are suspicious. Assume that a corpse cools according to Newton's law of cooling. Suppose the victim has a temperature of 72°F when it is found in a 40°F walk-in refrigerator. However, it has cooled to 66.8°F two hours later when the forensic pathologist arrives. Estimate the time of death.[1]

[1] From "Estimating the Time of Death" by T.K. Marshall and F.E. Hoare, *Journal of Forensic Sciences*, Jan. 1962.

3. *In hot water.*
Heat a pan of water to 120°F and measure its temperature at five-minute intervals as it cools. Plot a graph of temperature vs. time. For various values of the constant k in Newton's law of cooling, use ODE Architect to solve the rate equation for the water temperature. What value of k gives you a graph that most closely fits your experimental data?

4. *More hot water.*
In Problem 3 you may have found it difficult to find a suitable value of k. Here is the preferred way to determine k. The solution to ODE (1) is

$$T(t) = T_{out} + (T_0 - T_{out})e^{-kt}$$

where in this context T_{out} is the room temperature. We can measure T_{out} and the initial temperature, T_0. Rearranging and taking the natural logarithm of both sides gives

$$\ln|T(t) - T_{out}| = \ln|T_0 - T_{out}| - kt$$

Using the data of Problem 3, plot $\ln|T(t) - T_{out}|$ against t. What would you expect the graph to look like? Use your graph to estimate k, then use ODE Architect to check your results.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date ____________________

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Exploration 3.2. Keeping Your Cool

1. *On again, off again.*
When a room is cooled by an air conditioner, the unit switches on and off periodically, causing the temperature in the room to oscillate. How does the period of oscillation depend on the following factors?

- The upper and lower settings of the control temperature
- The outside temperature
- The coil temperature, T_{ac}

2. *Keeping your cool for less.*
The cost of operating an air conditioner depends on how much it runs. Which is the most economical way of cooling a room over a given time period?

- Set a small difference between the control temperatures, so that the temperature is always close to the average.
- Allow a large difference between the control temperatures so that the unit switches on and off less frequently.

Make sure the average of the control temperatures is the same in all your tests.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 3.3. The Return of the Melting Snowman

1. *The half-life of a snowman.*
Use ODE Architect to plot volume vs. time for several different initial snowman volumes between 5 and 25 ft^3, assuming that $k = 0.1451$ ft/(hr °F) and $T_{out} = 40$°F. For each initial volume Use the Explore feature of ODE Architect to find the time it takes the snowman to melt to half of its original size and make a plot of this "half-life" vs. initial volume. Any conclusions? [To access the Explore feature, click on Solutions on the menu bar and choose Explore. This will bring up a dialog box and a pair of crosshairs in the graphics window. Move the crosshairs to the appropriate point on the solution curve and read the coordinates of that point from the dialog box. Note that the Index entry gives the corresponding line in the Data table.]

2. *Sensitivity to outside temperature.*
Now fix the snowman's initial volume at 10 ft^3 and use ODE Architect to plot a graph of volume vs. time for several different outside temperatures between 35°F and 45°F, with $k = 0.1451$ ft/(hr °F). Find the time it takes the snowman to melt to 5 ft^3 for each outside temperature used and plot that time against temperature. Describe the shape of the graph.

3. *Other snowmen.*
In developing our snowman model, we assumed that the snowman could be modeled as a sphere. Sometimes snowmen are built by rolling the snow in a way that makes the body cylindrical. How would you model a cylindrical snowman? Which type of snowman melts faster, given the same initial volume and air temperature?