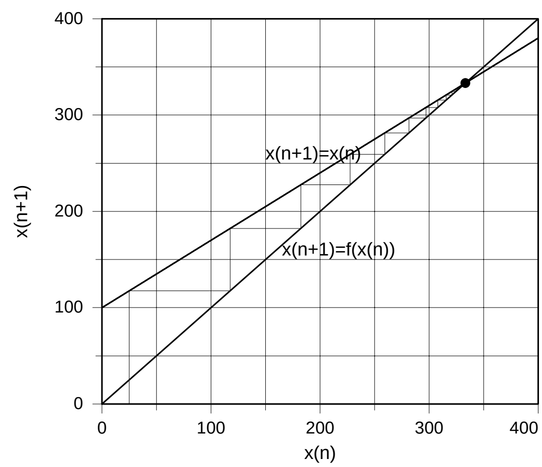
13 CHAPTER Discrete Dynamical Systems



Supply and demand converge to a stable equilibrium.

Overview Processes such as population dynamics that evolve in discrete time steps are best modeled using discrete dynamical systems. These take the form $x_{n+1} = f(x_n)$, where the variable x_n is the state of the system at "time" n and x_{n+1} is the state of the system at time $n+1$. Discrete dynamical systems are widely used in ecology, economics, physics and many other disciplines. In this section we present the basic techniques and phenomena associated with discrete dynamical systems.

Key words Iteration; fixed point; periodic point; cobweb and stairstep diagrams; stability; sinks; sources; bifurcation diagrams; logisitic maps; chaos; sensitive dependence on initial conditions; Julia sets; Mandelbrot sets

See also Chapter 6 for more on sinks and sources in differential equations; Chapter 12 for Poincaré sections.

Discrete dynamical systems arise in a large variety of applications. For example, the population of a species that reproduces on an annual basis is best modeled using discrete systems. Discrete systems also play an important role in understanding many *continuous* dynamical systems. For example, points calculated by a numerical ODE solver form a discrete dynamical system that approximates the solution of an initial value problem for an ODE. The Poincaré section described in Chapter 12 is another example of a discrete dynamical system that gives information about a system of ODEs.

A *discrete* dynamical system is defined by the *iteration* of a function f, and takes the form

$$x_{n+1} = f(x_n), \quad n \geq 0, \quad x_0 \text{ given} \tag{1}$$

Here are another two examples. In population dynamics, some populations are modeled using a *proportional growth* model

☞ The function $f(x) = \lambda x$ is denoted L_λ, and so $L_\lambda(x) = \lambda x$.

$$x_{n+1} = L_\lambda(x_n) = \lambda x_n, \quad n \geq 0, \quad x_0 \text{ given} \tag{2}$$

where x_n is the population density at generation n and λ is a positive number that measures population growth from generation to generation. Another common model is the *logistic growth* model:

☞ The function $\lambda x(1-x)$ is denoted by $g_\lambda(x)$.

$$x_{n+1} = g_\lambda(x_n) = \lambda x_n(1 - x_n), \quad n \geq 0, \quad x_0 \text{ given}$$

Let's return to the general discrete system (1). Starting with an initial condition x_0, we can generate a *sequence* using this rule for iteration: Given x_0, we get $x_1 = f(x_0)$ by evaluating the function f at x_0. We then compute $x_2 = f(x_1)$, $x_3 = f(x_2)$, and so on, generating a sequence of points $\{x_n\}$. Each x_n is the *n-fold composition* of f at x_0 since

☞ The superscript $^\circ$ reminds us that this is just the composition of f with itself; f is *not* being raised to a power.

$$\begin{aligned} x_2 &= f(f(x_0)) = f^{\circ 2}(x_0) \\ x_3 &= f(f(f(x_0))) = f^{\circ 3}(x_0) \\ &\vdots \\ x_n &= f^{\circ n}(x_0) \end{aligned}$$

(Some authors omit the superscript $^\circ$.)

The infinite sequence of iterates $O(x_0) = \{x_n\}_{n=0}^{\infty}$ is called the *orbit of* x_0 *under* f, and the function f is often referred to as a *map*. For example, if we take $\lambda = 1/2$ and the initial condition $x_0 = 1$ in the proportional growth model (2), we get the orbit for the map L:

$$x_0 = 1, \quad x_1 = 1/2, \quad x_2 = 1/4, \ldots$$

Refer to Screen 1.2 of Module 13 for four representations of the orbit of an iteration: as a *sequence* $\{x_0, x_1 = f(x_0), x_2 = f(x_1), \ldots\}$; a *numerical list* whose columns are labeled n, x_n, $f(x_n)$; a *time series* where x_n is plotted against "time" n; and a *stairstep/cobweb diagram* for graphical analysis.

The chapter cover figure shows a stairstep diagram for the model $x_{n+1} = 0.7x_n + 100$. Figures 13.1 and 13.2 show cobweb diagrams for the logistic model $x_{n+1} = \lambda x_n(1 - x_n)$, with $\lambda = 3.51$ and 3.9, respectively. In all of these figures, the diagonal line $x_{n+1} = x_n$ is also plotted. The stairstep and cobweb diagrams are constructed by selecting a value for x_0 on the horizontal axis, moving up to the graph of the iterated function to obtain x_1, horizontally over to the diagonal then up (or down) to the graph of the function to obtain x_2, and so on. These diagrams are used to guide the eye in moving from x_n to x_{n+1}.

◆ Equilibrium States

As with autonomous ODEs, it is useful to determine the equilibrium states for a discrete dynamical system. First we need some definitions:

☞ A fixed point of a discrete dynamical system is the analogue of an equilibrium point for a system of ODEs.

- A point x^* is a *fixed point* of f if $f(x^*) = x^*$. A fixed point is easy to spot in a stairstep or cobweb diagram even before the steps and webs are plotted: the fixed points of f are where the graph of f intersects the diagonal.
- A point x^* is a *periodic point of period n* of f if $f^{\circ n}(x^*) = x^*$ and $f^{\circ k}(x^*) \neq x^*$ for $k < n$. A fixed point is a periodic point of period 1.

Both the proportional and logistic growth models have the fixed point $x = 0$. For certain values of λ the logistic model has periodic points; Figure 13.1 suggests that the model has a period-4 orbit if $\lambda = 3.51$.

✓ "Check" your understanding by showing that the logistic model has a second fixed point $x^* = (\lambda - 1)/\lambda$. Does the proportional growth model for $\lambda > 0$ have any periodic points that are not fixed?

☞ This use of the words "stable" and "unstable" for points and orbits of a discrete system differs from the way the words are used for equilibrium points of an ODE. For example, a saddle point of an ODE is unstable, but a saddle point of a discrete system is neither stable or unstable.

A fixed point x^* of f is said to be *stable* (or a *sink*, or an *attractor*) if every point p in some neighborhood of x^* approaches x^* under iteration by f, that is, if $f^{\circ n}(p) \to x^*$ as $n \to +\infty$. The set of *all* points such that $f^{\circ n}(p) \to x^*$ as $n \to +\infty$ is the *basin of attraction* of p. A fixed point x^* is *unstable* (or a *source* or *repeller*) if every point in some neighborhood of x^* moves out of the neighborhood under iteration by f. If x^* is a period-n point of f, then the orbit of x^* is said to be *stable* if x^* is stable as a fixed point of the map $f^{\circ n}$. The orbit is *unstable* if x^* is unstable as a fixed point of $f^{\circ n}$. Stability is determined by the *derivative* of the map f, as the following tests show:

- A fixed point x^* is stable if $|f'(x^*)| < 1$, and unstable if $|f'(x^*)| > 1$.
- The orbit of a periodic point x^* of period n is stable if $|(f^{\circ n})'(x^*)| < 1$, and unstable if $|(f^{\circ n})'(x^*)| > 1$.

Stable periodic orbits are *attracting* because nearby orbits approach them, while unstable periodic orbits are *repelling* because nearby orbits move away from them.

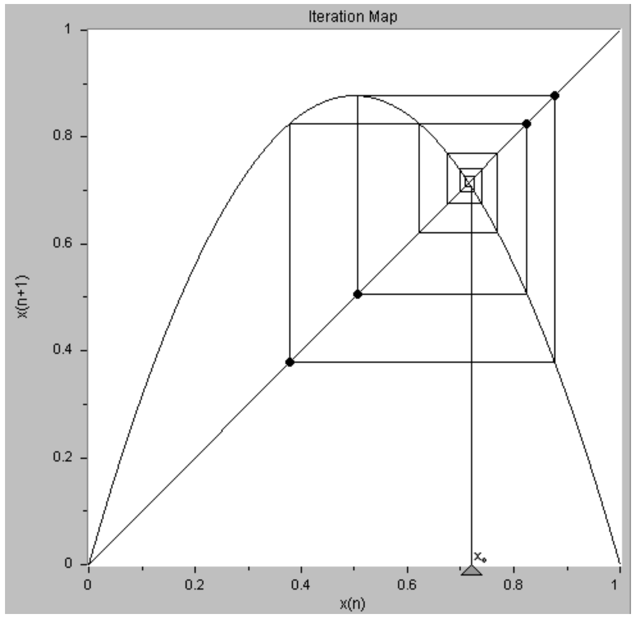


Figure 13.1: The cobweb diagram for the logistic map, $x_{n+1} = 3.51x_n(1 - x_n)$, suggests that iterates of $x_0 = 0.72$ approach a stable orbit of period 4.

✓ Is the fixed point in the chapter cover figure stable? Is the period-4 orbit in Figure 13.1 stable? How about the two fixed points in that figure? [*Suggestion:* Use the Discrete Tool as an aid in answering these questions.]

◆ Linear versus Nonlinear Dynamics

The solutions of linear and of nonlinear ODEs are compared and contrasted in Chapter/Modules 6 and 7. Now we will do the same comparison for linear and nonlinear maps of the real line into itself.

☞ Refer to the first submodule of Module 13 for examples.

Let's look at the iteration of linear functions such as the proportional growth model $x_{n+1} = L_\lambda(x_n)$, which has a fixed point at $x^* = 0$. This fixed point is stable if $|\lambda| < 1$, so the orbit of every initial population tends to 0 as $n \to \infty$. If $\lambda = 1$, then $x_{n+1} = x_n$, and hence every point is a fixed point. The fixed point at $x^* = 0$ is unstable if $|\lambda| > 1$, and all initial populations tend to ∞ as $n \to \infty$. If $\lambda = -1$ then $x^* = 0$ is the only fixed point and every other point is of period 2 since $x_{n+1} = -x_n$.

The iteration of any linear function $f(x) = ax + b$ (with slope $a \neq 1$) behaves much like the proportional growth model. Fixed points are found by solving $ax^* + b = x^*$, and their stability is governed by the magnitude of a.

The iteration of nonlinear functions can be much more complex than that of linear functions. In particular, nonlinear functions can exhibit chaotic behavior, as well as periodic behavior. To illustrate the types of behavior typical

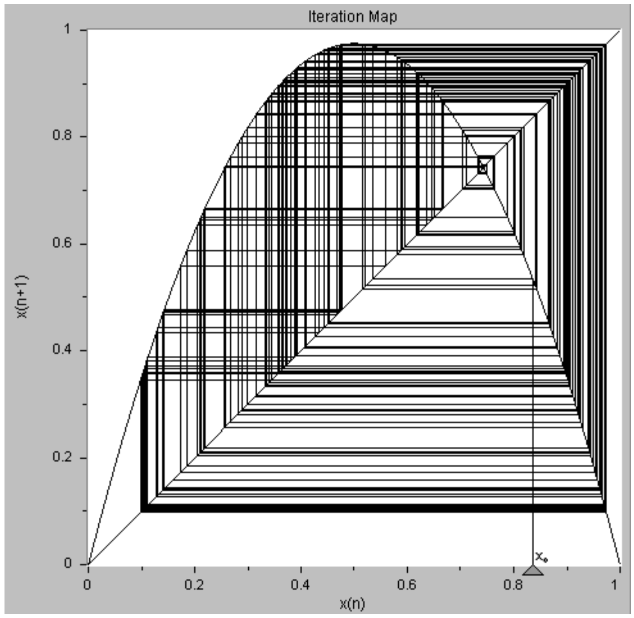


Figure 13.2: The cobweb diagram of the logistic map $x_{n+1} = 3.9x_n(1 - x_n)$ suggests that iterates of $x_0 = 0.8361$ either approach an attracting periodic orbit of very high period, or else wander chaotically.

to nonlinear functions we consider; in the second submodule of Module 13, the one-parameter family of *logistic* functions

$$g_\lambda(x) = \lambda x(1 - x)$$

Figure 13.2 shows how complex an orbit of a logistic map may be for certain values of λ.

◆ Stability of a Discrete Dynamical System

Now we turn our attention to the stability of an entire dynamical *system* rather than just that of a fixed point. One of the most important ideas of dynamical systems (discrete or continuous) is that of *hyperbolicity*. Hyperbolic points are stable to small changes in the parameters of a dynamical system. This does not mean that a perturbation (a small change) of the function leaves the fixed or periodic point unchanged. It simply means that the perturbed function will also have a fixed point or periodic point "nearby," and that this point has the stability properties of the corresponding point of the unperturbed function. For example, at $\lambda = 2$ the function $g_2(x)$ has an attracting fixed point $x^* = 0.5$. For values of λ near 2, the function $g_\lambda(x)$ also has an attracting fixed point $x^* = (\lambda - 1)/\lambda$. For example, if $\lambda = 2.1$ then the attracting fixed point is $x^* = 0.524$. Even though the fixed point moved a little as λ increased, the fixed point still exists and it is still attracting. The following theorem provides a way of determining whether fixed points and periodic orbits are hyperbolic.

THEOREM 13.1 Given a discrete dynamical system $x_{n+1} = f_\lambda(x_n)$, a fixed point x^* of $f_\lambda(x)$ is hyperbolic if $|f'_\lambda(x^*)| \neq 1$. Similarly, a periodic point x^* of period n (and its orbit) is hyperbolic if $|(f_\lambda^{\circ n})'(x^*)| \neq 1$.

Because the number and type of periodic points do not change at parameter values where $f_\lambda(x)$ has hyperbolic points, we say that the qualitative structure of the dynamical system remains unchanged. On the other hand, this theorem also implies that changes in the qualitative structure of a family of discrete dynamical systems can occur only when a fixed or periodic point is *not* hyperbolic. We see this in the proportional growth model $x_{n+1} = \lambda x_n$ when $\lambda = 1$ and $\lambda = -1$. For $\lambda = 1 - \varepsilon$ [and hence $L'_{1-\varepsilon}(1) = \lambda = 1 - \varepsilon$] the fixed point $x = 0$ is attracting. But for $\lambda = 1 + \varepsilon$ the fixed point is repelling. Thus, as λ passes through the value 1, the stability of the fixed point changes from attracting to repelling and the qualitative structure of the dynamical system changes.

◆ Bifurcations

A change in the qualitative structure of a discrete dynamical system, such as a change in the stability of a fixed point, is known as a *bifurcation*. Two other types of bifurcations can also occur when f_λ is nonlinear.

The first, known as a *saddle-node* bifurcation, occurs when x^* is a periodic point of period n and $(f_\lambda^{\circ n})'(x^*) = 1$. In a saddle-node bifurcation, the periodic point x^* splits into a pair of periodic points, both of period n, one of which is attracting and the other repelling. A saddle-node bifurcation occurs in the logistic growth family $g_\lambda(x)$ when $\lambda = 1$. At this value the fixed point $x^* = 0$ (which is attracting for $\lambda < 1$) splits into a pair of fixed points, $x^* = 0$ (repelling for $\lambda > 1$), and $x^* = (\lambda - 1)/\lambda$ (attracting for $\lambda > 1$). This type of bifurcation is sometimes called an *exchange of stability* bifurcation.

The second important type of bifurcation is called *period-doubling* and occurs when x^* is a periodic point of period-n and $(f_\lambda^{\circ n})'(x^*) = -1$. In this bifurcation the attracting period n point becomes repelling and an attracting period-$2n$ orbit is spawned. (Note that the stability can be reversed.) This occurs in the logistic family $g_\lambda(x)$ when $\lambda = 3$. At this parameter value, the attracting fixed point $x^* = (\lambda - 1)/\lambda$ becomes repelling and a stable period-2 orbit emerges with one point on each side of $x^* = (\lambda - 1)/\lambda$. Since the logistic equations model population growth, this says that the population converges to an equilibrium for growth rate constants λ less than 3. However, for values of λ greater than 3, the population oscillates through a sequence of values.

The bifurcations that occur in a one parameter family of discrete dynamical systems can be summarized in a *bifurcation diagram*. For each value of the parameter (on the horizontal axis) the diagram shows the long-term behavior under iteration of a “typical” initial point. For example, if you see

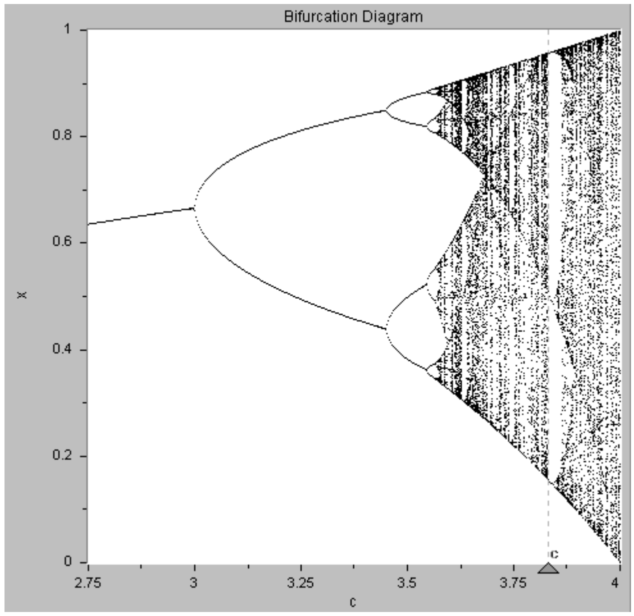


Figure 13.3: Part of the bifurcation diagram for the logistic map.

a single point in the diagram above a particular parameter value, that point corresponds to an attracting fixed point. The spot in the diagram where you see an arc of attracting fixed points split into two arcs corresponds to a bifurcation from an attracting fixed point for an attracting orbit of period 2 (i.e., period doubling). If the diagram shows a multitude of points above a given parameter value, then either you are seeing an attracting periodic orbit of a very high period, or else you are seeing chaotic wandering. It should be noted that when constructing the bifurcation diagram for each parameter value and initial point, the first 50 or so iterates are omitted so that only the long-term behavior is visible in the diagram. See Figure 13.3 and Screen 2.4 in Module 13 for the bifurcation diagram of the logistic map.

The stable arcs in these diagrams are usually straightforward to generate numerically. We constructed a bifurcation diagram on an interval $[\lambda_{\min}, \lambda_{\max}]$ for the logistic population model $x_{n+1} = g_\lambda(x_n)$ using the following procedure.

1. Fix $\lambda_{\min}$, $\lambda_{\max}$, λ_{inc}, $n_{\min}$, $n_{\max}$. Here λ_{inc} is the step size between successive values of λ while $n_{\min}$ and $n_{\max}$ are bounds on the number of iterates used to construct the diagram; they control the accuracy of the diagram. Typical values are $n_{\min} = 50$ and $n_{\max} = 150$.
2. Let $\lambda = \lambda_{\min}$.
3. Taking $x_0 = 0.5$ for example, compute the first $n_{\min}$ iterates of g_λ without plotting anything. This eliminates transient behavior.

4. For $n_{\min} \leq n \leq n_{\max}$, plot the points $(\lambda, g_\lambda^{\circ n}(0.5))$. If the orbit of 0.5 converges to a periodic orbit, only points near this orbit are plotted. If the orbit of 0.5 isn't periodic, then the points above λ seem to be almost randomly distributed.
5. Let $\lambda = \lambda + \lambda_{\text{inc}}$.
6. If $\lambda < \lambda_{\max}$, go back to Step 3 and repeat the process.

✓ Go to the one-dimensional tab of the Discrete Tool. Use the default values, but set the value of c at 1 (c in the tool plays the role of λ in Chapter/Module 13). Click on the bifurcation diagram. Keep your finger on the up-arrow for c and describe what is happening. Any attracting periodic orbits? For what values of c do these orbits occur? What are the periods?

◆ Periodic and Chaotic Dynamics

One of the most celebrated theorems of discrete dynamical systems is often paraphrased "Period 3 Implies Chaos." This theorem, originally proven by Šarkovskii and independently discovered by Li and Yorke[1], is a remarkable result in that it requires relatively little information about the dynamical system and yet it returns a treasure trove of information.

THEOREM 13.2 If f is a continuous function on the real line and if there exists a point of period 3, then there exist points of every period.

For the logistic population model there exists an attracting period-3 orbit at $\lambda = \sqrt{8} + 1 \approx 3.83$, and most initial conditions in the unit interval converge to this orbit (see Figure 13.4). In terms of our model, most populations tend to oscillate between the three different values of the period-3 orbit. Theorem 13.2 states that even more is going on at $\lambda = \sqrt{8} + 1$ than meets the eye. If we pick any positive integer n, there exists a point p such that n is the smallest positive integer allowing $g_\lambda^{\circ n}(p) = p$. Thus, for example, there exists a point that returns to itself in 963 iterates. The reason we don't "see" this periodic orbit (or, indeed, any periodic orbit, except that of period 3) is that it is unstable, so no iterate can approach it. But orbits of every period are indeed present if $\lambda = \sqrt{8} + 1$.

[1] James Yorke and T.Y. Li are contemporary mathematicians who published their result in 1975 (see References). They were the first to apply the word "chaos" to the strange behavior of the iterates of functions such as g_λ. A.N. Šarkovskii published a stronger result in 1964, in Russian, in the *Ukrainian Mathematical Journal*, but it remained unknown in the West until after the paper by Yorke and Li had appeared.

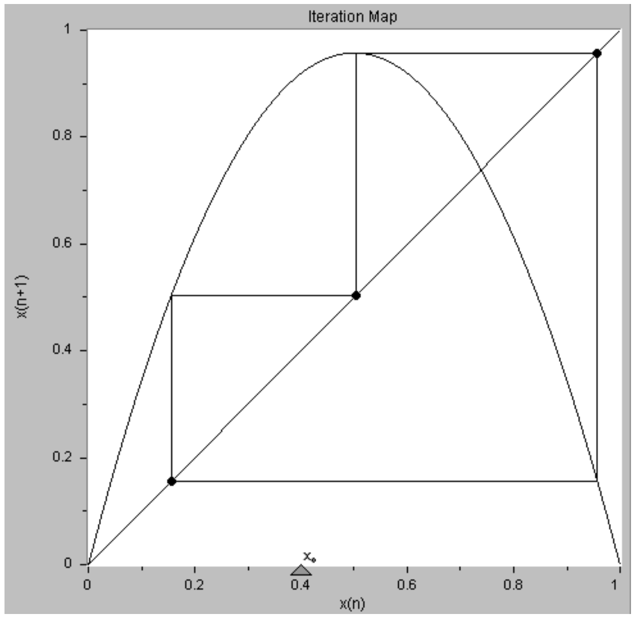


Figure 13.4: At $\lambda = \sqrt{8}+1 \approx 3.83$ the logistic map $g_\lambda = \lambda x(1-x)$ has an attracting orbit of period 3; the points $x_0, x_1, \ldots, x_{49}$ have been suppressed in this graph.

◆ What is Chaos?

So, you're probably asking, what is chaos? The definition of chaos is a bit slippery. In fact, mathematicians are still arguing about a proper definition. But to get the idea across we'll use one due to Devaney.

Let S be a set such that if x lies in S, then $f^{\circ n}(x)$ belongs to S for all positive integers n. The set S is called *invariant*. If you start in an invariant set, you can't get out! Now let's define what we mean by chaos in an invariant set S.

A map $f : S \to S$ is *chaotic* if:

1. periodic points are dense in S;
2. f displays sensitive dependence on initial conditions in S; and
3. f is topologically transitive in S.

☞ A set U of real numbers is *open* if every point p of U has the property that all points in some interval $(p-a, p+a)$ are also in U.

The first condition of this definition is relatively explained like this: A set A is *dense* in another set B if for every point x in B and every open set U containing x there exists points of A that are also in U. Therefore, condition 1 says that periodic points are almost everywhere in S. This means that S contains many periodic points; Theorem 13.1 gives a condition guaranteeing infinitely many of these points.

☞ Module 13 has examples.

In the second condition, *sensitive dependence on initial conditions* means that points that are initially close to one another eventually get moved far apart under iteration by f.

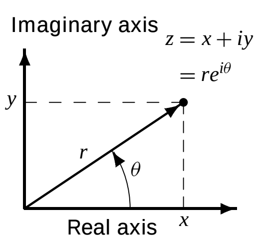
Finally, f is *topologically transitive* (or *mixing*) if given any pair of open sets U and V in S, some iterate of f takes one or more points of U into V. This means that points of open sets get spread throughout the set S.

The most significant item on this list for applied problems is sensitive dependence on initial conditions. Let's consider the logistic growth model at a parameter value where the dynamics are chaotic. Sensitive dependence implies that no matter how close two populations may be today, there will be a time in the future when the populations differ significantly. So environmental disturbances that cause small population changes will eventually lead to large changes, if chaotic dynamics exist.

Chaotic dynamics occur in a wide range of models. Although the definitions above are given in terms of a single scalar dynamical system, everything extends to higher dimensions, and many of the applications are two or three dimensional. In addition to models of population dynamics, chaos has been observed in models of the weather, electrical circuits, fluid dynamics, planetary motion, and many other phenomena. The relatively recent understanding of chaos has shed new light on the complexity and beauty of the world we inhabit.

◆ Complex Numbers and Functions

Probably the most popular type of discrete dynamical system is a *complex dynamical system* where the variables are complex numbers instead of real numbers. The intricate fractal structures common to images generated using complex dynamics have appeared everywhere from calendars to art shows and have inspired both artists and scientists alike. Many of the fundamental ideas of complex dynamics are identical to those of real dynamics and have been discussed in previous sections. In what follows, we will highlight both the similarities and differences between real and complex dynamics.



Recall that complex numbers arise when factoring quadratic polynomials with negative discriminant. Because the discriminant is negative we must take the square root of a negative real number, which we do by defining i to be $\sqrt{-1}$. We then write the *complex number* as $z = x + iy$. We say that x is the *real part of* z and y is the *imaginary part of* z. The complex number z is represented graphically on the complex plane by the point having coordinates (x, y). It is often useful to represent complex numbers in polar coordinates by letting $x = r\cos\theta$ and $y = r\sin\theta$ so that

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

☞ Euler's formula is also used in Chapter 4.

The remarkable relationship $\cos\theta + i\sin\theta = e^{i\theta}$ between polar coordinates and exponential functions is known as *Euler's Formula*. The number $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point z in the complex plane and is sometimes called the *modulus* of z, and it is denoted by $r = |z|$. The angle θ is called the *argument* of z. Note that the usual properties of exponential functions hold in the complex plane. Thus, given two complex numbers $z = re^{i\theta}$ and $w = se^{i\phi}$, their product is

$$zw = rse^{i(\theta+\phi)}$$

A *complex function* $f(z)$ takes a complex number z as its argument and returns a complex number $w = f(z)$. Differentiation proceeds as in the real case; for example, $(z^3)' = 3z^2$. Unlike functions of one real variable, we cannot graph a complex function since both the domain and range are two-dimensional.

◆ Iterating a Complex Function

Iteration of a complex function is identical to the iteration of a real function. Given an initial z-value z_0, iteration generates a sequence of complex numbers $z_1 = f(z_0)$, $z_2 = f(z_1)$, etc. Fixed and periodic points are defined in the same way as for real functions, as are stability and instability. Here are the previous criteria for stability, but now applied to complex functions.

- A fixed point z^* is stable if $|f'(z^*)| < 1$, and unstable if $|f'(z^*)| > 1$.
- A period-n point z^* (and its orbit) is stable if $|(f^{\circ n})'(z^*)| < 1$, and unstable if $|(f^{\circ n})'(z^*)| > 1$.

Let's consider a simple example to illustrate these ideas. Let $f(z) = z^2$. Then $z^* = 0$ is an attracting fixed point since $f(0) = 0$ and $|f'(0)| = 0$. If z is any point such that $|z| < 1$, then the sequence $\{f^{\circ n}(z)\}_{n=0}^{\infty}$ converges to 0 as $n \to \infty$. On the other hand, if $|z| > 1$, then the sequence $\{f^{\circ n}(z)\}_{n=0}^{\infty}$ goes to infinity as $n \to \infty$. To see what happens to values of z having modulus equal to 1, let's write $z = e^{i\theta}$. Then $f(z) = e^{2i\theta}$, which also has modulus 1. Thus all iterates of points on the unit circle $|z| = 1$ stay on the unit circle. The point $z^* = 1$ is a repelling fixed point since $f(1) = 1$ and $|f'(1)| = 2$. The period-2 points are found by solving $f(f(z)) = z^4 = z$. We can rewrite this equation as

$$z(z^3 - 1) = 0$$

One solution to this equation is $z^* = 0$, corresponding to the attracting fixed point, and another solution is $z^* = 1$, corresponding to the repelling fixed point. Notice that the fixed points of $f(z)$ remain fixed points of $f(f(z))$, or equivalently, are also period-2 points of $f(z)$. To find the other two solutions, we write $z = e^{i\theta}$ to get the equation

☞ Any period-n point is also a periodic point of all periods which are positive integer multiples of n.

$$e^{3i\theta} = 1$$

which we need to solve for θ. Since we are working in polar coordinates, we note that $1 = e^{i2n\pi}$ where n is an integer. This implies that $3\theta = 2n\pi$ and from this we find a second pair of period-2 points at $z = e^{2\pi i/3}$ and $z = e^{4\pi i/3}$. Both of these are repelling.

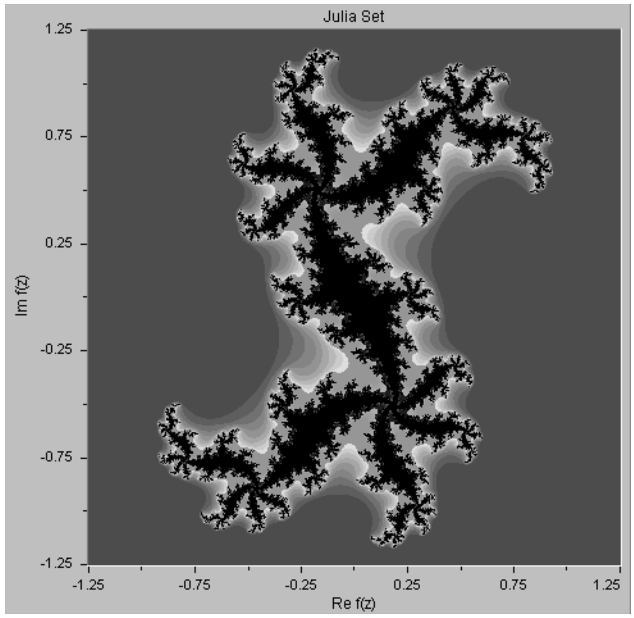


Figure 13.5: The filled Julia set for $f(z) = z^2 + c$, where $c = 0.4012 - 0.3245i$

✓ Show that $e^{2\pi i/3}$ and $e^{4\pi i/3}$ are repelling period-2 points of $f = z^2$. Show that $f^{\circ n}(z) \to 0$ as $n \to \infty$ if $|z| < 1$, and that $|f^{\circ n}(z)| \to \infty$ if $|z| > 1$. What is the "basin of attraction" of the fixed point $z = 0$?

◆ Julia Sets, the Mandelbrot Set, and Cantor Dust

The set of repelling periodic points of the function $f = z^2$ is dense on the unit circle, although we don't show that here. This leads us to the definition of the Julia set.

☞ The *closure* of a set A consists of the points of A together with all points that are limits of sequences of the points of A.

DEFINITION The *filled Julia set* K of a complex-valued function f is the set of all points whose iterates remain bounded. The *Julia set* J of f is the closure of the set of repelling periodic points.

For $f = z^2$, the filled Julia set K of f is the set of all complex numbers z with $|z| \leq 1$, while the Julia set J of f is the unit circle $|z| = 1$. This is a very simple example of a Julia set. In general, Julia sets are highly complicated objects having a very intricate fractal structure. For example, see Figure 13.5 and Screens 3.3 and 3.4 of Module 13.

In the above example, the Julia set J divides those points that iterate to infinity (points outside the unit circle) and those that converge to the attracting fixed point (points inside the unit circle). This division of the domain by the Julia set is often the case in complex dynamics and provides a way of numerically computing the filled Julia set of a given function f. Assign a complex number to each screen pixel. Then use each pixel (i.e., complex number) as an initial condition and iterate to determine whether the orbit of that point exceeds some predetermined bound (for example $|z| = 50$). If it does, we say the orbit diverges and we color the point black. If not, we color the point red to indicate it is in the filled Julia set.

☞ In the Discrete Tool of ODE Architect, the coloring is reversed. Points in the Julia set are colored black and points whose orbits diverge past the predetermined bound are colored with various colors according to their divergence rates (e.g., red is the fastest, dark blue the slowest).

Earlier in this chapter we saw the importance of attracting periodic orbits in building a bifurcation diagram for a real map f. Although we didn't mention it then, we can home in on an attracting periodic orbit of f (if there is one) by starting at $x_0 = \tilde{x}$ if $f'(x)$ is zero at $\tilde{x}$ and nowhere else. Complex functions $f(z)$ for which $f'(\tilde{z}) = 0$ at exactly one point $\tilde{z}$ have the same property as the following theorem shows.

☞ A complex function f is *analytic* if its derivatives of every order exist. A point $\tilde{z}$ is a *critical point* of f if $f'(\tilde{z}) = 0$.

THEOREM 13.3 Let f be an analytic complex-valued function with a unique critical point $\tilde{z}$. If f has an attracting periodic orbit then, the forward orbit of $\tilde{z}$ converges to this orbit.

Let's look at some of the implications of this theorem with the family of functions $f_c(z) = z^2 + c$ where $c = a + ib$ is a complex parameter. For each value of c the only critical point is $\tilde{z} = 0$. To find an attracting periodic orbit for a given value of c we need to compute the orbit

$$\{0, c, c^2 + c, \dots\}$$

and see if the orbit converges or not. If it does, we found the attracting periodic orbit; if not, there doesn't exist one. Let's see what happens when we set $c = 1$ to give the function $f_1(z) = z^2 + 1$. The orbit of the critical point is $\{0, 1, 2, 5, 26, \dots\}$, which goes to infinity. Thus, f_1 has *no* attracting periodic orbit and the Julia set does not divide points that converge to a periodic orbit from points that iterate to infinity. In fact, it can be shown that this Julia set is totally disconnected; it is sometimes referred to as *Cantor dust*. Click on outlying points on the edge of the Mandelbrot (defined below) set in Screen 3.5 of Module 13 and you will generate Cantor dust in the upper graphics screen.

This leads to another question. If some functions in the family f_c have connected Julia sets (such as $f_0 = z^2$) and other functions in the family have totally disconnected Julia sets (such as f_1), what set of points in the c plane separates these distinctive features? This set is the *boundary* of the Mandelbrot set. The *Mandelbrot set M* of the function $f_c(z) = z^2 + c$ is defined as the set of all complex numbers c such that the orbit $\{f_c^{\circ n}(0)_{n=1}^{\infty}\}$ remains bounded, that is, $|f^{\circ n}(0)| \leq K$ for some positive number K and n.

This definition leads us to an algorithm for computing the Mandelbrot set M. Assign to each pixel a complex number c. Choose a maximum number of iterations N and determine whether $|f_c^{\circ n}(0)| < 2$ for all $n \leq N$ (it can be proven that if $|f_c^{\circ n}(0)| > 2$ for some n, then the orbit goes to infinity). If so, then color this point green to indicate that it is in the Mandelbrot set. Otherwise, color this point black. It is this computation that gives the wonderfully intricate Mandelbrot set; see Figure 13.6 and Screens 3.4 and 3.5 of Module 13.

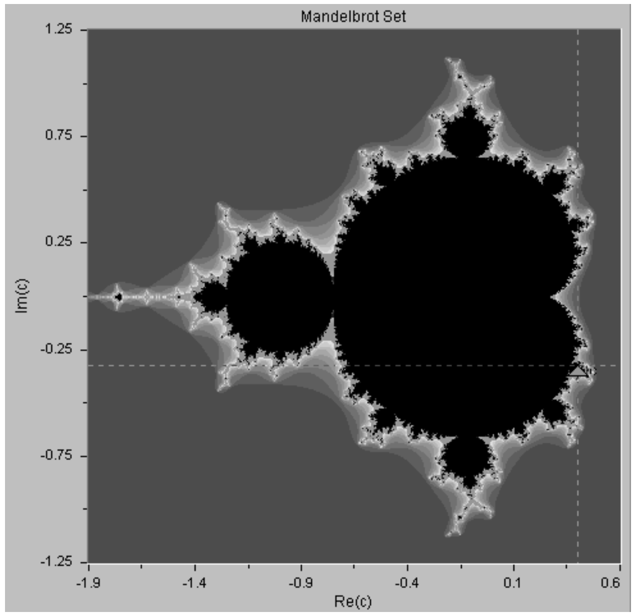


Figure 13.6: The Mandelbrot set; the cross-hairs are set on the point $0.4012 - 0.3245i$, which gives the Julia set shown in Figure 13.5.

The Mandelbrot set actually contains much more information than is described here. It is, in fact, the bifurcation diagram for the family of functions $f_c(z) = z^2 + c$. Each "blob" of the set corresponds to an attracting periodic orbit of a particular period. Values of c in the big cardioid shown on Screen 3.4 of Module 13 give attracting period-1 orbits for f_c. Values of c in the circle immediately to the left of this cardioid give attracting period-2 orbits for f_c. Other "blobs" give other attracting periodic behaviors.

Although we have only defined the Mandelbrot set for the specific family $f_c = z^2 + c$, it can be defined in an anlagous way for other families of complex functions (see the Discrete Tool). One final note on Julia sets and the Mandelbrot set. You've probably seen intricately colored versions of these objects on posters or elsewhere. The coloring is usually determined by how "fast" orbits tend to infinity. The color scheme is, of course, up to the programmer.

Module 13 introduces and lets you play with three important discrete dynamical systems—linear, logistic, and a third that uses complex numbers. Explorations 13.1–13.4 extend these ideas and introduce other maps with curious behavior under iteration.

References

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Doebeli, M. and Ruxton, G. "Controlling Spatial Chaos in Metapopulations with Long-Range Dispersal" in *Bulletin of Mathematical Biology*, Vol. 59 (1997) pp. 497–515. Emphasizes the frequent use of the linear map and the logistic map in population dynamics.

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Saha, P. and Strogatz, S.H. "The Birth of Period 3" in *Mathematics Magazine* Vol. 68 (1995) pp. 42–47

Strogatz, S., *Nonlinear Dynamics and Chaos*, (1994: Addison-Wesley). A very readable text, mostly on ODEs, but Chapter 10 on One-Dimensional Maps is particularly relevant to this chapter/module of ODE Architect.

Yorke, J. and Li, T.Y., "Period 3 Implies Chaos" in the *American Mathematical Monthly* 82 (1975) pp. 985–992

Here are three sources on chaos that are less technical and more accessible, but still give an accurate description of what chaos is all about.

Peak, D. and Frame, M., *Chaos under Control*, (1994: W.H. Freeman)

Stewart, I., *Does God Play Dice?*, (1990, Blackwell)

Stoppard, T., the play "Arcadia", (1993: Faber and Faber). Yes! A play about chaos.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 13.1. One-Dimensional Maps and the Discrete Tool

1. Go to the Discrete Tool and enter the proportional growth model $x_{n+1} = cx_n$, where c is the parameter. For the range $0 \leq n \leq 30$ and the initial condition $x_0 = 0.5$, explore and describe what happens to the iteration map, time series, and bifurcation diagram as the parameter is increased from -2 to 2. For what values of c is there a sudden change in the behavior of the iterates (the bifurcation values of c)? For what values of c are there 1, 2, or infinitely many fixed or periodic points? Which of there points are attractors? Repellers?

2. Go to the Discrete Tool and explore and describe what happens to the iteration map, the time series, and the bifurcation diagram as the parameter c for the logistic map $g_c(x) = cx(1 - x)$ is incremented from 1 to 4. Use the range $50 \leq n \leq 150$ to avoid an initial wandering, and the initial condition $x_0 = 0.5$. Describe what all three graphs on the tool screen look like at values of c where there is a periodic orbit? What is the period? Go as far forward as you can with the period doubling sequence of values of c: 3, 3.434, ... What are the corresponding periods? [*Suggestion:* Zoom on the bifurcation diagram.] Repeat with the sequence 3.83, 3.842, ...

3. In the Discrete Tool enter the tent map T_c on the interval $0 \leq x \leq 1$:

$$T_c(x) = c(1 - 2\,\mathrm{abs}(x - 0.5)) = \begin{cases} 2cx, & 0 \leq x \leq 0.5 \\ 2c(1 - x), & 0.5 \leq x \leq 1 \end{cases}$$

where the parameter c is allowed to range from 0 to 1. Describe and explain what you see as c is incremented from 0 to 1. [*Suggestion:* use the Edit option in the Menu box for the Bifurcation Diagram and set $200 \leq n \leq 300$ in order to suppress the initial transients.] Any orbits of period 2? Period 3?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 13.2. Circle Maps

Another common type of discrete dynamical system is a *circle map*, which maps the perimeter of the unit circle onto itself. These functions arise when modeling coupled oscillators, such as pendulums or neurons. The simplest types of circle maps are rotations that take the form

$$R_{\omega}(\theta) = (\theta + \omega) \bmod 2\pi$$

where $0 \leq \theta \leq 2\pi$ and ω is a constant.

1. Show that if $\omega = (p/q)\pi$ with p and q positive integers and p/q in lowest terms, then every point has period q.

2. Show that if $\omega = a\pi$ with a an irrational number, then no point on the circle is periodic.

3. What is the long-term behavior of the orbit of a point on the circle if $\omega = a\pi$, where a is an irrational number?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 13.3. Two-Dimensional Maps and the Discrete Tool

A two-dimensional discrete dynamical system looks like thes:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n, c) \\ y_{n+1} &= g(x_n, y_n, c) \end{aligned} \tag{3}$$

where f and g are given functions and c is a "place holder" for parameters. For given values of c, x_0, and y_0, system (3) defines an orbit of points

$$(x_0, y_0), \quad (x_1, y_1), \quad (x_2, y_2), \ldots$$

in the xy-plane. The two dimensional tab in the Discrete Tool allows you to explore discrete systems of the form of (3)

1. Open the Discrete Tool and explore the default system (a version of what is known as the *Hènon Map*):

$$\begin{aligned} x_{n+1} &= 1 + y_n - ax_n^2 \\ y_{n+1} &= bx_n \end{aligned} \tag{4}$$

where a and b are parameters. For fixed values of the parameters a and b find the fixed points. Are they sinks, sources, or neither? How sensitive is the long-term behavior of an orbit to small changes in the initial point (x_0, y_0)? What happens if you increment a through a range of values? If you increment b? Any period doubling sequences? In your judgment, is there any long-term chaotic wandering? [*Suggestion:* Keep the values of a and b within small ranges of their default values to avoid instabilities.]

2. Repeat Problem 1 with the following version of the Hènon map:

$$x_{n+1} = a - x_n^2 + by_n$$
$$y_{n+1} = x_n$$

Start with $a = 1.28$, $b = -0.3$, $x_0 = 0$, $y_0 = 0$.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 13.4. Julia and Mandelbrot Sets and the Discrete Tool

Note that the color schemes for the Julia and Mandelbrot sets in Module 13 differ from those in the discrete tool.

1. Use the Discrete Tool to explore the Mandelbrot set and Julia sets for the complex family $f_c = z^2 + c$. What happens to the filled Julia sets as you move c from inside the Mandelbrot set up toward the boundary, then across the boundary and out beyond the Mandelbrot set? Describe how the Julia sets change as you "walk" along the edge of the Mandelbrot set.

2. Repeat Problem 1 for the complex family $g_c = c \sin z$.

3. Repeat Problem 1 for the family $h_c = ce^z$.