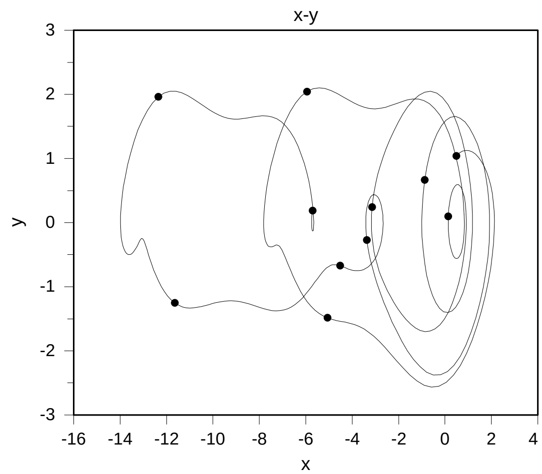
12 Chaos and Control

CHAPTER



Poincaré map of a forced damped pendulum superimposed on a trajectory.

Overview In this Chapter we'll look at solutions of a forced damped pendulum ODE. In the linear approximation of small oscillations, this ODE becomes the standard constant-coefficient ODE $x'' + cx' + kx = F(t)$, which can be solved explicitly in all cases. Without the linear approximation, the pendulum ODE contains the term $k \sin x$ instead of kx. Now the study becomes much more complicated. We'll focus on the special case of the nonlinear pendulum ODE

$$x'' + cx' + \sin x = A \cos t \qquad (1)$$

but our results leave a world of further things to be discovered. We'll show that appropriate initial conditions will send the pendulum on any desired sequence of gyrations, and hint at how to control the chaos by finding such an initial condition.

Key words Forced damped pendulum; sensitivity to initial conditions; chaos; control; Poincaré sections; discrete dynamical systems; Lakes of Wada; control

See also Chapter 10 for background on the pendulum. Chapter 13 for more on discrete dynamical systems and other instances of chaos and sensitivity to initial conditions.

◆ Introduction

How might chaos and control possibly be related? These concepts appear at first to be opposites, but in fact they are two faces of the same coin!

A good way to start discussing this apparent paradox is to think about learning to ski. The beginning skier tries to be as stable as possible, with feet firmly planted far enough apart to give confidence that she or he will not topple over. If you try to ski in such a position, you cannot turn, and the only way to stop, short of running into a tree, is to fall down. Learning to ski is largely a matter of giving up on "stability," bringing your feet together so as to acquire controllability! You need to allow chaos in order to gain control.

Another example of the relation between chaos and control is the early aircraft available at the beginning of World War I, carefully designed for greatest stability. The result was that their course was highly predictable, an easy target for antiaircraft fire. Very soon the airplane manufacturers started to build in enough instability to allow maneuverability!

◆ Solutions as Functions of Time

The methods of analysis we will give can be used for many other differential equations, such as Duffing's equation

$$x'' + cx' + x - x^3 = A\cos\omega t, \tag{2}$$

or the differential equation

$$x'' + cx' + x - x^2 = A\cos\omega t, \tag{3}$$

which arises when studying the stability of ships. The explorations at the end suggest some strategies for these problems.

Let's begin to study ODE (1) with $c = 0.1$:

$$x'' + .1x' + \sin x = A\cos t \tag{4}$$

Let's compute some solutions, starting at $t = 0$ with $A = 1$ and various values of $x(0)$ and $x'(0)$, and observe the motion out to $t = 100$, or perhaps longer (see Figure 12.1). We see that most solutions eventually settle down to an oscillation with period 2π (the same period as the driving force). This xt-plot actually shows oscillations which differ by multiples of 2π.

This settling down of behaviors at various levels is definitely a feature of the parameter values chosen: for the amplitude $A = 2.5$ in ODE (4), for instance, there does not appear to be any steady-state oscillation at all.

Looking at such pictures is quite frustrating: it is very hard to see the pattern for which initial conditions settle down to which stable oscillations, and which will not settle down at all.

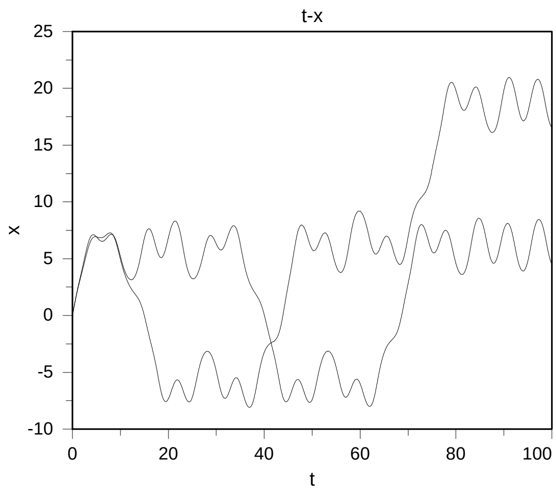


Figure 12.1: Solution curves of ODE (4) with $x(0) = 0,\ \ x'(0) = 2, 2.1$.

◆ Poincaré Sections

Poincaré found a way to understand and visualize the behavior of our differential equation: he sampled solutions of ODE (4) at multiples of the period 2π of the driving function:

$$0, 2\pi, 4\pi, \dots, 2k\pi, \dots$$

This is much like taking pictures with a strobe light.

An equivalent way of saying this is to say that we will iterate[1] the mapping $P : \mathbb{R}^2 \to \mathbb{R}^2$ which takes a point (a, b) in $\mathbb{R}^2$, computes the solution $x(t)$ with $x(0) = a,\ \ x'(0) = b$, and sets

☞ Note that the clock starts at $t_0 = 0$ when generating Poincaré plots.

$$P(a, b) = (x(2\pi), x'(2\pi)) \tag{5}$$

This mapping P is called a *Poincaré mapping*. If you apply the operator P to (a, b) k times in succession, the result is $P^k(a, b)$ and we see that

$$P^k(a, b) = (x(2k\pi), x'(2k\pi))$$

☞ When the xx'-plane is used to chart the evolution of the points $P^k(a, b)$, $k = 1, 2, \dots$, it is called the Poincaré plane.

In sense, the Poincaré section is simply a crutch: every statement about Poincaré sections corresponds to a statement about the original ODE, and vice versa. But this crutch is invaluable the orbits of a nonautonomous ODE such as (4) intersect each other and themselves in a hopelessly tangled way.

[1]Chapter 13 discusses iterating maps $f : \mathbb{R} \to \mathbb{R}$; there you will find that already the map $f(x) = \lambda x(1 - x)$ is filled with surprises. Before trying to understand the iteration of P, which is quite complicated indeed, the reader should experiment with several easier examples, like linear maps $\mathbb{R}^2 \to \mathbb{R}^2$. The notion of *basin* will also be much clarified by considering the iteration of Newton's method in one complex variable, perhaps for cubic polynomials.

◆ Periodic Points

A good way to start investigating the Poincaré mapping P (or for that matter, the iteration of any map) is to ask: what periodic points does it have? Setting $x' = y$, a *periodic point* is a point (x, y) in $\mathbb{R}^2$ such that for some integer k we have $P^k(x, y) = (x, y)$. *Fixed* points are periodic points with $k = 1$, and are particularly important.

Periodic points of period k for P are associated with periodic solutions of ODE (4) of period $2k\pi$. In particular, if $x(t)$ is a solution which is periodic of period 2π, then

$$(x(0), x'(0)) = (x(2\pi), x'(2\pi))$$

is a fixed point of P. If you observe this solution with a strobe which flashes every 2π, you will always see the solution in the same place.

◆ The Unforced Pendulum

If there is no forcing term in ODE (4), then we have an autonomous ODE like those treated in Chapter 10.

Example: The ODE

$$x'' + x' + 10 \sin x = 0$$

models a damped pendulum without forcing. A phase plane portrait is shown in Figure 12.2. Note that the equilibrium points (of the equivalent system) at $x = 2n\pi,\ x' = 0$ are spiral sinks, but the equilibrium points at $x = (2n+1)\pi$, $x' = 0$ are saddles. Note also that the phase plane is divided into slanting regions, each of which has the property that its points are attracted to the equilibrium point inside the region. These regions are called *basins of attraction*. If a forcing term is supplied, these basins become all tangled up (Figure 12.4 on page 227).

There is a Poincaré mapping P for the unforced damped pendulum, which is fairly easy to understand, and which you should become familiar with before tackling the forced pendulum. In this case, two solutions of ODE (4) with $A = 0$ stand out: the equilibria $x(t) = 0$ and $x(t) = \pi$ for all t. Certainly if the pendulum is at one of these equilibria and you illuminate it with a strobe which flashes every T seconds, where T is a positive number, you will always see the pendulum in the same place. Thus these points are fixed points of the corresponding Poincaré mapping P. In the xx'-plane, the same thing happens at the other equilibrium points, that is, at the points $\dots, (-2\pi, 0), (0, 0)(2\pi, 0), \dots$ for the "downward" stable equilibria, and at the points $\dots, (-3\pi, 0), (-\pi, 0), (\pi, 0), \dots$ for the unstable equilibria.

The analysis in Module 10 using an integral of motion should convince you that for the unforced damped pendulum, these are the only periodic points: if the pendulum is not at an equilibrium, the value of the integral decreases with time, and the system cannot return to where it was.

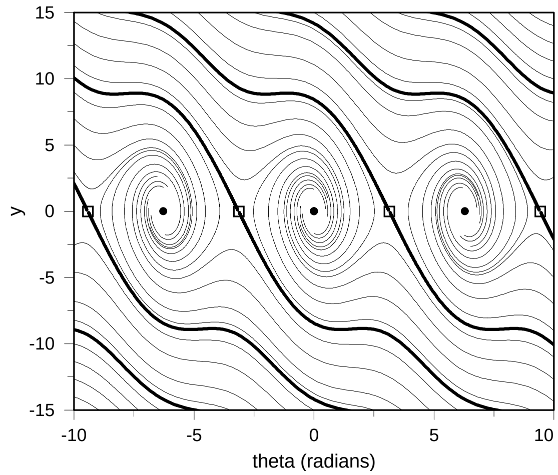


Figure 12.2: Basins of attraction of the downward equilibrium positions of the unforced damped pendulum are bounded by separatrices.

If you start the pendulum with both $x(0)$ and $x'(0)$ small, the damping will simply kill off the motion, and the pendulum will be attracted to the downward equilibrium. The point $(0,0)$ in state space is called a *sink*.

The behavior is more interesting near an unstable equilibrium. Imagine imparting an initial velocity to the bob by kicking it. For a small kick, it will swing back. Now kick it a little harder: it will rise higher, and still swing back. Kick it harder still, and it will make it over the top, and hit you in the back if you aren't careful. Dividing the kicks which don't make it over from those that do is a very special kick, where pendulum rises forever, more and more slowly, tending to the unstable equilibrium. Thus there are initial conditions which generate solutions that tend to the unstable equilibrium; in the Poincaré plane these solutions form two curves which meet end to end at the fixed point corresponding to the unstable equilibrium. Together they form the *stable separatrix* of the fixed point. There are also curves of initial conditions which come from the unstable equilibrium; together they form the *unstable separatrix* of the unstable equilibrium. See Figure 12.3

As stated earlier, a good first thing to do when iterating a map is to search for the periodic points; a good second thing is to find the periodic points which correspond to unstable equilibria (*saddles*, in the case of the pendulum) and find their separatrices.

For the unforced damped pendulum, the equilibria of the differential equation and the fixed points of any Poincaré map coincide; so, too, do the separatrices of the unstable equilibria (in the phase plane) and the separatrices of the corresponding saddle fixed points in the Poincaré plane. These separatrices separate the trajectories which approach a given sink from the trajectories that approach a different sink.

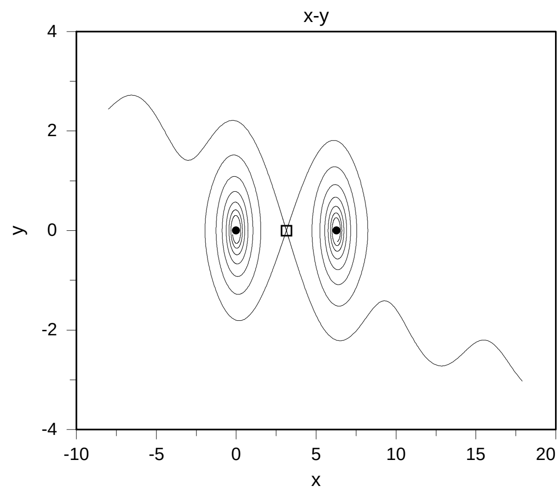


Figure 12.3: Stable and unstable separatrices at a saddle for an unforced damped pendulum. Which are the stable separatrices?

✓ "Check" your understanding by reproducing the plot in Figure 12.3.

◆ The Damped Forced Pendulum

We described above the Poincaré plane for the unforced pendulum. The same description holds for the forced pendulum. A figure showing a Poincaré map for a forced pendulum appears as the chapter cover figure. Thus, in the Poincaré plane, we expect to see a collection of fixed points corresponding to the oscillations to which the pendulum "settles down", and each has a basin: the set of initial conditions which will settle down to it. The basins appear to be extraordinarily tangled and complicated, and they are. The reader should put up the picture of the basins (Screen 2.6 in Module 12), and practice superimposing iterations on the figure, checking that if you start in the blue basin, the entire orbit remains in the blue basin, perhaps taking a complicated path to get near the sink, but making it in the end.

◆ Tangled Basins, the Wada Property

In the tangled basins Screen 3.3 of Module 12, each basin appears to be made of a central piece, and four canals which go off and meander around the plane. The meandering appears to be completely random and chaotic, and the only thing the authors really know about the shapes of the basins of our undamped pendulum is the following fact: The basins have the *Wada property*: every

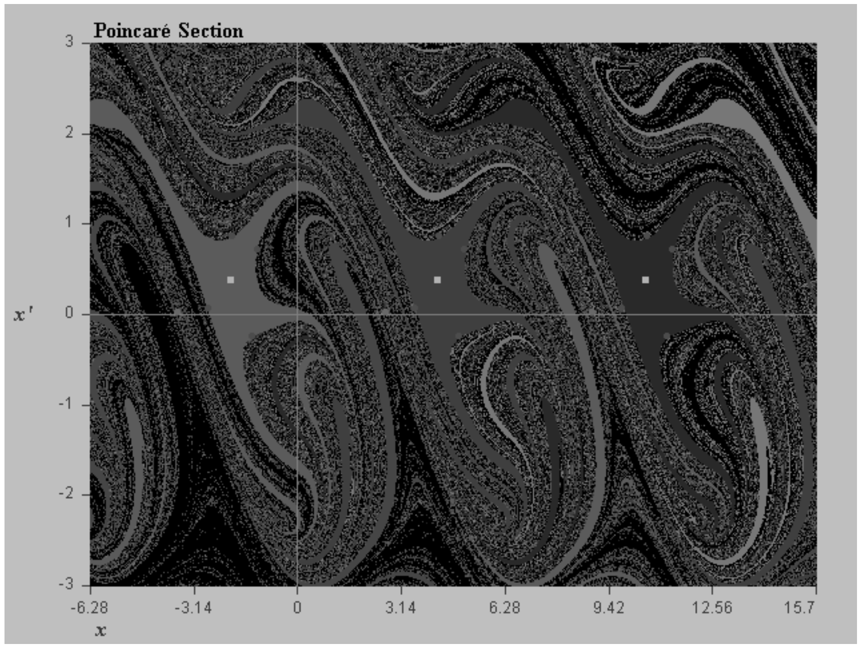


Figure 12.4: Tangled basins for a forced damped pendulum.

point of the boundary[2] of any basin is in the boundary of all the others. Thus if you start at a boundary point of any basin, and perturb the initial condition an arbitrarily small amount, you can land in any of the infinitely many basins.

A careful look at Figure 12.4 should convince you that this stands a good chance of being true: Each region of a canal boundary point includes pieces on many curves. It isn't clear, of course, that there are canals of *all* the basins between any two canals.

It is one thing to think that the Wada property is likely true, and quite another to prove it. It isn't clear how you would prove anything whatsoever about the basins: they do not appear to be amenable to precise study.

To get a grip on these basins, the first step is to understand why they appear to be bounded by smooth curves, and to figure out what these smooth curves are. For each sink (solid white squares in Figure 12.4), there are in fact four periodic points (open squares), each of period two, which are saddles, and such that for each saddle one of the two unstable separatrices is entirely contained in the corresponding basin.

[2]The boundary ∂U of an open set $U \subset \mathbb{R}^2$ is a point $\mathbf{x} \in \mathbb{R}^2$ which is not in U, but such that there exists a sequence of points $\mathbf{x}_n \in U$ which converges to $\mathbf{x}$. Later we will encounter the notion of *accessible boundary*: the points $\mathbf{x} \in \partial U$ such that there exists a parametrized curve $\gamma : (0, 1] \to U$ such that $\lim_{t \to 0} \gamma(t) = \mathbf{x}$. For simple open sets, the boundary and the accessible boundary coincide, but not for our basins.

The next step is to show that the *accessible boundary* of the basin is made up of the stable separatrices of these saddles. This uses the technique of *basin cells*, as pioneered by Kennedy, Nusse and Yorke. To see a fleshed out sketch, see the C·ODE·E article referenced at the end of this chapter.

◆ Gaining Control

The statement about the basins having the Wada property is, in some sense, a negative statement, saying that there is maximum possible disorder. Is there some positive statement one can make about the forced pendulum (for these parameter values)? It turns out that there is. The precise statement is as follows.

During one period of the forcing term, say during

$$t \text{ in the interval } I_k = [2k\pi, 2(k+1)\pi]$$

the pendulum will do one of the following four things:

- It will cross the bottom position exactly once moving clockwise (count this possibility as -1);
- It will cross the bottom position exactly once moving counterclockwise (count this possibility as $+1$);
- It will not cross the bottom position at all (count this possibility as 0);
- It will do something else (possibility NA).

Note that most solutions appear to be attracted to sinks, and that the stable oscillation corresponding to a sink crosses the bottom position twice during each I_k, and hence these oscillations (and most oscillations after they have settled down) belong to the NA category.

The essential control statement we can make about the pendulum is the following:

> For any biinfinite sequence $\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots$ of symbols ε_i selected from the set $\{-1, 0, 1\}$, there exists $x(0), x'(0)$ such that the solution with this initial condition will do ε_k during the time interval I_k.

The chaos game in Module 12 suggests why this might be true; the techniques involved in the proof were originally developed by Smale[3].

[3]Stephen Smale is a contemporary mathematician who was awarded a Fields medal (the mathematical equivalent of a Nobel prize) in the early 1960's. See Devaney in the references at the end of this chapter.

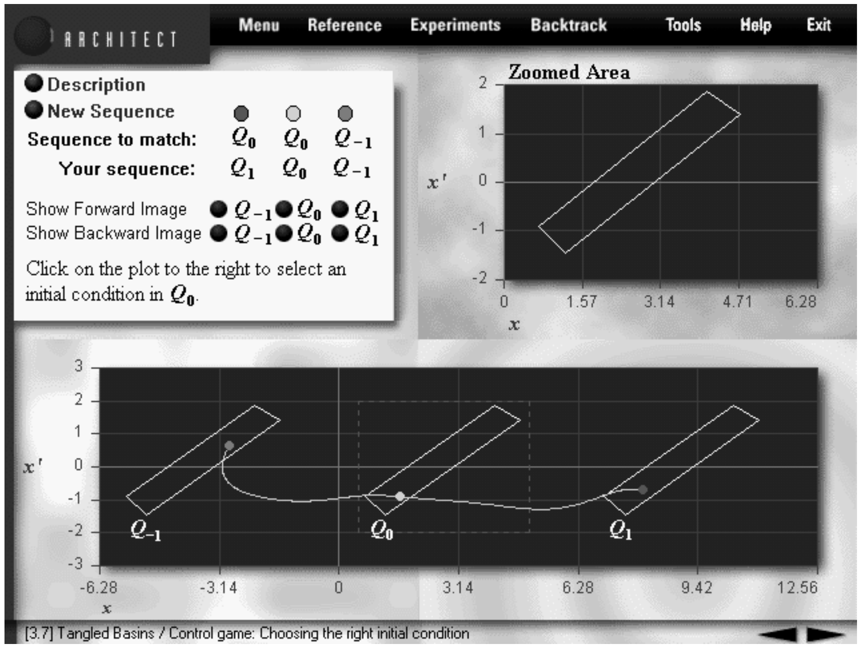


Figure 12.5: Start in quadrilateral Q_0 and reach forward into Q_1 and backward into Q_{-1}.

We start by drawing quadrilaterals Q_k around the kth saddle, long in the unstable direction and short in the stable direction, such that it crosses a good part of the tangle. We can now translate our symbols ε_i, which refer to the differential equation, into the Poincaré mapping language:

If at time $t = 2k\pi$ the pendulum is in Q_k and at time $2(k+1)\pi$ it is in $Q_{k+\varepsilon_k}$, then during I_k the pendulum does ε_k. So it is the same thing to require that a trajectory of the pendulum realize a particular symbol sequence, and to require that an orbit of the Poincaré map visit a particular sequence of quadrangles, just so long as successive quadrangles be neighbors or identical.

Draw the forward image of that quadrilateral, and observe that it grows much longer in the unstable direction and shrinks in the stable direction; we will refer to $P(Q_k)$ as the kth *snake*, S_k. The entire proof comes down to understanding how S_k intersects Q_{k-1}, Q_k and Q_{k+1}.

The thing to be checked is that S_k intersects all three in subquadrangles going from top to bottom, and that the top and bottom of Q_k map to parts of the boundary of S_k which are outside $Q_{k-1} \cup Q_k \cup Q_{k+1}$. See Figure 12.5 for an example of a winning strategy for three adjacent quadrilaterals.

Once you have convinced yourself that this is true, you will see that every symbolic sequence describing a history of the pendulum is realized by an intersection of thinner and thinner nested subquadrangles.

A similar argument shows that a symbol describing a future of the pendulum corresponds to a sequence of thinner and thinner subquadrangles going from left to right. The details are in the C·ODE·E paper in the references.

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Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date ______________________________

Course/Section ______________________________

Exploration 12.1.

1. Choose a value for $c \neq 0.1$, take $A = 1$ in ODE (1), and produce graphs like those in the chapter cover figure and Figure 12.1.

2. Choose a value for $A \neq 1$ and $c = 0.1$ in ODE 1 and produce graphs like those in the chapter cover figure and Figure 12.1.

3. Choose a value for $\omega \neq 1$ in the ODE

$$x'' + 0.1x' + \sin x = \cos \omega t$$

and produce graphs like those in the chapter cover figure and Figure 12.1.

4. Repeat Problems 1 and 2, but for the Duffing ODE,

$$x'' + cx' + x - x^3 = A\cos t$$

5. Repeat Problems 1 and 2, but for the ODE with a quadratic nonlinearity,

$$x'' + cx' + x - x^2 = A\cos t$$