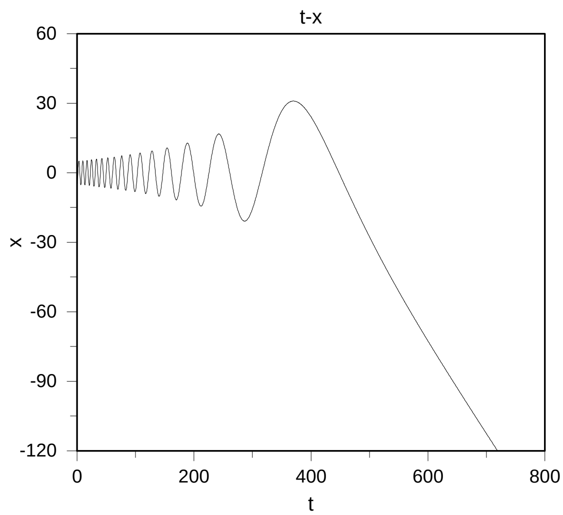
11 CHAPTER Applications of Series Solutions



An aging spring stretches.

Overview Many phenomena, especially those explained by Newton's Second Law, can be modeled by second-order linear ODEs with variable coefficients, for example:

1. Robot arms, which are modeled by a spring-mass equation with a time-varying damping coefficient; and
2. Aging springs, which are modeled by a spring-mass equation with a time-varying spring constant.

These two applications illustrate very different ways in which series solutions can be used to solve linear ODEs with nonconstant coefficients.

Key words Infinite series; recurrence formula; ordinary point; singular point; regular singular point; Bessel's equations; Bessel functions; aging spring; lengthening pendulum

See also Chapter 4 for second-order linear ODEs with constant coefficients (i.e., without the time-dependence).

◆ Infinite Series

Certain second-order linear ODEs with nonconstant coefficients have been studied extensively, so their properties are well-known. We will look at some of these ODEs in the chapter.

If the general linear homogeneous (undriven) second-order ODE

$$x'' + p(t)x' + q(t)x = 0 \tag{1}$$

has coefficients p and q that are not both constants, the methods of Chapter 4 don't work. However, sometimes we can write a solution $x(t)$ as a power series:

$$x(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n \tag{2}$$

where we use ODE (1) to determine the coefficients a_n. Much useful information can be deduced about an ODE when its solutions can be expressed as power series.

☞ Look in your calculus book for Taylor series. The term "analytic" is frequently used for functions with convergent Taylor Series.

If a function $x(t)$ has a convergent Taylor series $x(t) = \sum a_n(t - t_0)^n$ in some interval about $t = t_0$, then $x(t)$ is said to be *analytic* at t_0. Since all derivatives of analytic functions exist, the derivatives x' and x'' of x can be obtained by differentiating that series term by term, producing series with the same radius of convergence as the series for x. If we substitute these series into ODE (1), we can determine the coefficients a_n. To begin with, a_0 and a_1 are equal to the initial values $x(t_0)$ and $x'(t_0)$, respectively.

✓ "Check" your understanding by evaluating the series (2) at $t = t_0$ to show that $a_0 = x(t_0)$. Now differentiate series (2) term by term to obtain a series for $x'(t)$; evaluate this series at t_0 to find that $a_1 = x'(t_0)$. Does a_2 equal $x''(t_0)$?

◆ Recurrence Formulas

A *recurrence formula* for the coefficients a_n is a formula that defines each a_n in terms of the coefficients $a_0, a_1, \ldots, a_{n-1}$. To find such a formula, we have to express each of the terms in ODE (1) [i.e., x'', $p(t)x'$, and $q(t)x$] as power series about $t = t_0$, which is the point at which the initial conditions are given. Then we combine these series to obtain a single power series which, according to ODE (1), must sum to zero for all t near t_0. This implies that the coefficient of each power of $t - t_0$ must be equal to zero, which yields an equation for each a_n in terms of the preceding coefficients $a_0, a_1, \ldots, a_{n-1}$.

Example: Finding a recurrence formula

☞ We chose a first-order ODE for simplicity.

Let's solve the first-order IVP $x' + tx = 0,\ \ x(0) = 1$. First we write $x(t)$ in the form

$$x(t) = \sum_{n=0}^{\infty} a_n t^n$$

where we have chosen $t_0 = 0$. The derivative of $x(t)$ is then

$$x'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting this into the given ODE, we get

$$x' + tx = \sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^{n+1} = 0$$

To make the power of t the same in both sums, replace n by $n-2$ in the second sum to obtain

☞ Notice in the second summation that n starts at 2, rather than 0. Do you see why?

$$\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=2}^{\infty} a_{n-2} t^{n-1} = a_1 + \sum_{n=2}^{\infty} [n a_n + a_{n-2}] t^{n-1} = 0$$

The last equality is true if and only if $a_1 = 0$ and, if for every $n \geq 2$, we have that $n a_n + a_{n-2} = 0$. Therefore, the desired recurrence formula is

$$a_n = \frac{-a_{n-2}}{n}, \quad n = 2, 3, \ldots \tag{3}$$

Since $a_1 = 0$, formula (3) shows that the coefficients $a_3, a_5, \ldots, a_{2k+1}, \ldots$ must all be zero; and $a_2 = -a_0/2,\ \ a_4 = -a_2/4 = a_0/(2 \cdot 4), \ldots$. With a little algebra you can show that the series for $x(t)$ is

$$x(t) = a_0 - \frac{a_0}{2} t^2 + \frac{a_0}{2 \cdot 4} t^4 - \frac{a_0}{2 \cdot 4 \cdot 6} t^6 + \cdots$$

which can be simplified to

$$x(t) = a_0 \left(1 - \frac{t^2}{2} + \frac{1}{2!} \left(\frac{t^2}{2} \right)^2 - \frac{1}{3!} \left(\frac{t^2}{2} \right)^3 + \cdots \right)$$

If the initial condition $a_0 = x(0) = 1$ is used, this becomes the Taylor Series for $e^{-t^2/2}$ about $t_0 = 0$. Although the series solution to the IVP, $x' + x = 0$, $x(0) = 1$, can be written in the form of a familiar function, for most IVPs that is rarely possible and usually the only form we can obtain is the series form of the solution.

✓ Check that $x(t) = e^{-t^2/2}$ is a solution of the IVP $x' + tx = 0,\ \ x(0) = 1$.

◆ Ordinary Points

If $p(t)$ and $q(t)$ are both analytic at t_0, then t_0 is called an *ordinary point* for the differential equation $x''(t) + p(t)x'(t) + q(t)x(t) = 0$. At an ordinary point, the method illustrated in the preceding example always produces solutions written in series form. The following theorem states this more precisely.

Ordinary Points Theorem. If t_0 is an ordinary point of the second-order differential equation

$$x'' + p(t)x' + q(t)x = 0 \tag{4}$$

that is, if $p(t)$ and $q(t)$ are both analytic at t_0, then the general solution of ODE (4) is given by the series

$$x(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n = a_0 x_1(t) + a_1 x_2(t) \tag{5}$$

where a_0 and a_1 are arbitrary and, for each $n \geq 2$, a_n can be written in terms of a_0 and a_1. When this is done, we get the right-hand term in formula (5), where $x_1(t)$ and $x_2(t)$ are linearly independent solutions of ODE (4) that are analytic at t_0. Further, the radius of convergence for each of the series solutions $x_1(t)$ and $x_2(t)$ is at least as large as the smaller of the two radii of convergence for the series for $p(t)$ and $q(t)$.

One goal of Module 11 is to give you a feeling for the interplay between infinite series and the functions they represent. In the first submodule, the position $x(t)$ of a robot arm is modeled by the second-order linear ODE

☞ Note that $p(t) = Ct$ and $q(t) = k$ are analytic for all t.

$$x'' + Ctx' + kx = 0 \tag{6}$$

where C and k are positive constants. Using the methods of the earlier example, we can derive a series solution (with $t_0 = 0$)

$$x(t) = 1 - \frac{kt^2}{2!} + \frac{k(2C+k)t^4}{4!} - \frac{k(2C+k)(4C+k)t^6}{6!} + \cdots \tag{7}$$

that satisfies $x(0) = 1,\ x'(0) = 0$. We then have to to determine how quickly the arm can be driven from the position $x = 1$ to $x = 0.005$ without letting x go below zero. The value of k is fixed at 9, so that only C is free to vary. When $C = k$, it turns out that series (7) is the Taylor series for $e^{-kt^2/2}$ about $t = 0$. It can then be demonstrated numerically, using ODE Architect, that $C = 9$ produces a solution that stays positive and is an optimal solution in the sense of requiring the least time for the value of x to drop from 1 to 0.005.

☞ Historically, new functions in engineering, science, and mathematics have often been introduced in the form of series solutions of ODEs.

In the majority of cases, however, it is *not* possible to recognize the series solution as one of the standard functions of calculus. Then the only way to approximate $x(t)$ at a given value of t is by summing a large number of terms

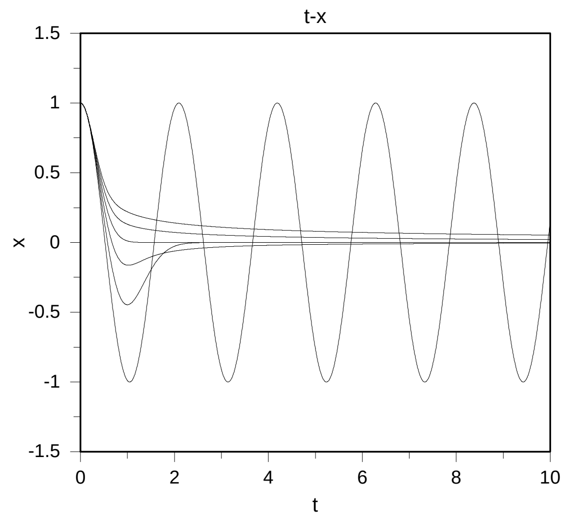


Figure 11.1: Solutions of ODE (6) for $k = 9$, $x(0) = 1$, $x'(0) = 0$, and $C = 0, 3, 6, 9, 12, 15$. Which is the $C = 15$ curve?.

in the series, or by using a numerical solver to solve the corresponding IVP. ODE Architect was used to graph solutions of ODE (6) for several values of C (Figure 11.1).

What if t_0 is *not* an ordinary point for the ODE, $x'' + p(t)x' + q(t)x = 0$, that is, what if $p(t)$ or $q(t)$ is not analytic at t_0? For example, in the ODE $x'' + x/(t-1) = 0$, $q(t)$ is not analytic at $t_0 = 1$. Such a point is said to be a *singular point* of the ODE. For example, $t_0 = 1$ is a singular point for the ODE $x'' + x/(t-1) = 0$. Next we show how to deal with ODEs with certain kinds of singular points.

✓ Is $t = 0$ an ordinary point or a singular point of $x'' + t^2 x = 0$? What about $x'' + (\sin t)x = 0$ and $x'' + x/t = 0$?

◆ Regular Singular Points

A singular point of the ODE $x''(t) + p(t)x'(t) + q(t)x(t) = 0$ is a *regular singular point* if both $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are analytic at t_0. In this case we'll have to modify the method to find a series solution to the ODE.

✓ Is $t = 0$ a regular singular point of $x'' + x'/t + x = 0$? What about $x'' + x' + x/t^2 = 0$ and $x'' + x' + x/t^3 = 0$?

Since $(t-t_0)p(t)$ and $(t-t_0)^2q(t)$ are analytic at t_0, they have power series expansions centered at t_0:

$$(t-t_0)p(t) = P_0 + P_1(t-t_0) + P_2(t-t_0)^2 + \cdots$$
$$(t-t_0)^2q(t) = Q_0 + Q_1(t-t_0) + Q_2(t-t_0)^2 + \cdots$$

As we shall soon see, the constant coefficients, P_0 and Q_0, in these two series are particularly important. The roots of the quadratic equation (called the *indicial equation*)

$$r(r-1) + P_0r + Q_0 = 0 \tag{8}$$

are used in solution formula (9) below.

☞ Assume that the roots of the indicial equation are real numbers.

A theorem due to Frobenius tells us how to modify our original method of constructing power series solutions so that we can obtain series solutions near regular singular points.

Frobenius' Theorem. If t_0 is a regular singular point of the second-order differential equation $x''(t) + p(t)x'(t) + q(t)x(t) = 0$, then there is at least one series solution at t_0 of the form

$$x_1(t) = (t-t_0)^{r_1}\sum_{n=0}^{\infty} a_n(t-t_0)^n = \sum_{n=0}^{\infty} a_n(t-t_0)^{n+r_1} \tag{9}$$

where r_1 is the larger of the two roots r_1 and r_2 of the indicial equation.

☞ The second summation is called the *Frobenius series*.

☞ Consult the references for detailed instructions on how to find the coefficients a_n.

The coefficients a_n can be determined in the same way as in the earlier example: differentiate twice, substitute the series for qx_1, px_1' and x_1'' into the given differential equation, and then find a recurrence formula.

Here are a few things to keep in mind when finding a Frobenius series.

1. The roots of the indicial equation may not be integers, in which case the series representation of the solution would not be a power series, but is still a valid series.
2. If $r_1 - r_2$ is not an integer, then the smaller root r_2 of the indicial equation generates a second solution of the form

$$x_2(t) = (t-t_0)^{r_2}\sum_{n=0}^{\infty} b_n(t-t_0)^n$$

   which is linearly independent of the first solution $x_1(t)$.
3. When $r_1 - r_2$ is an integer, a second solution of the form

$$x_2(t) = Cx_1(t)\ln(t-t_0) + \sum_{n=0}^{\infty} b_n(t-t_0)^{n+r_2}$$

   exists, where the values of the coefficents b_n are determined by finding a recurrence formula, and C is a constant. The solution $x_2(t)$ is linearly independent of $x_1(t)$.

◆ Bessel Functions

For any nonnegative constant p, the differential equation

$$t^2x''(t) + tx'(t) + (t^2 - p^2)x(t) = 0$$

is known as *Bessel's equation of order* p, and its solutions are the *Bessel functions of order* p. In normalized form, Bessel's equation becomes

☞ If t is very large, Bessel's equation looks like the harmonic oscillator equation, $x'' + x = 0$.

$$x''(t) + \frac{1}{t}x'(t) + \left(\frac{t^2 - p^2}{t^2}\right)x(t) = 0$$

From this we can see that $tp(t) = 1$ and $t^2q(t) = t^2 - p^2$, so that $tp(t)$ and $t^2q(t)$ are analytic at $t_0 = 0$. Therefore zero is a regular singular point and, using equation (8), we find that the indicial equation is

☞ The roots of the indicial equation are are p and $-p$.

$$r(r-1) + r - p^2 = r^2 - p^2 = 0$$

Application of Frobenius' Theorem yields a solution J_p given by the formula

☞ Consult the references for the derivation of the formula for $J_p(t)$.

$$J_p(t) = t^p \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(p+1)(p+2)\cdots(p+n)} t^{2n}$$

The function $J_p(t)$ is called the *Bessel function of order* p *of the first kind*. The series converges and is bounded for all t. If p is not an integer, it can be shown that a second solution of Bessel's equation is $J_{-p}(t)$ and that the general solution of Bessel's equation is a linear combination of $J_p(t)$ and $J_{-p}(t)$.

For the special case $p = 0$, we get the function $J_0(t)$ used in the aging spring model in the second submodule of Module 11:

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{t}{2}\right)^{2n} = 1 - \frac{t^2}{4} + \frac{t^4}{64} - \frac{t^6}{2304} + \cdots$$

Note that even though $t = 0$ is a singular point of the Bessel equation of order zero, the value of $J_0(0)$ is finite [$J_0(0) = 1$]. See Figure 11.2.

✓ Check that $J_0(t)$ is a solution of Bessel's equation of order 0.

When p is an integer we have to work much harder to get a second solution that is linearly independent of $J_p(t)$. The result is a function $Y_p(t)$ called the *Bessel function of order* p *of the second kind*. The general formula for $Y_p(t)$ is extremely complicated. We show only the special case $Y_0(t)$, used in the aging spring model:

☞ Actually γ is an unending decimal (or so most mathematicians believe), and 0.5772 gives the first four digits.

$$Y_0(t) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{t}{2}\right)J_0(t) + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}H_n}{(n!)^2}\left(\frac{t}{2}\right)^{2n}\right]$$

where $H_n = 1 + (1/2) + (1/3) + \cdots + (1/n)$ and γ is *Euler's constant*: $\gamma = \lim_{n\to\infty}(H_n - \ln n) \approx 0.5772$.

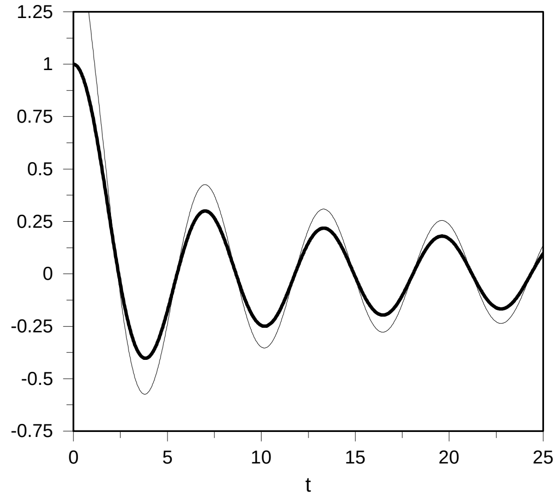


Figure 11.2: The graph of $J_0(t)$ [dark] looks like the graph of the decaying sinusoid $\sqrt{2/\pi t}\cos(t-\pi/4)$ [light].

The general solution of Bessel's equation of integer order p is

$$x(t) = c_1 J_p(t) + c_2 Y_p(t) \tag{10}$$

for arbitrary constants c_1 and c_2. An important thing to note here is that the value of $Y_p(t)$ at $t = 0$ *does* reflect the singularity at $t = 0$; in fact, $Y_p(t) \to -\infty$ as $t \to 0^+$, so that a solution having the form given in (10) is bounded only if $c_2 = 0$.

Bessel functions appear frequently in applications involving cylindrical geometry and have been extensively studied. In fact, except for the functions you studied in calculus, Bessel functions are the most widely used functions in science and engineering.

◆ Transforming Bessel's Equation to the Aging Spring Equation

☞ See "Aging Springs" in Module 11.

Bessel's equation of order zero can be transformed into the aging spring equation $x'' + e^{-at}x = 0$. To do this, we take

$$t = (2/a)\ln(2/as) \tag{11}$$

where the new independent variable s is assumed to be positive. Then we can use the chain rule to find the first two derivatives of the displacement x of the aging spring with respect to s:

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{dx}{dt}\left(-\frac{2}{as}\right)$$

$$\begin{aligned}
\frac{d^2x}{ds^2} &= \frac{d}{ds}\left[\frac{dx}{dt}\left(-\frac{2}{as}\right)\right] + \frac{dx}{dt}\frac{d}{ds}\left(-\frac{2}{as}\right) \\
&= \frac{d^2x}{dt^2}\frac{dt}{ds}\left(-\frac{2}{as}\right) + \frac{dx}{dt}\frac{2}{as^2} \\
&= \frac{d^2x}{dt^2}\left(-\frac{2}{as}\right)\left(-\frac{2}{as}\right) + \frac{dx}{dt}\frac{2}{as^2} \\
&= \frac{d^2x}{dt^2}\frac{4}{(as)^2} + \frac{dx}{dt}\frac{2}{as^2}
\end{aligned}$$

☞ We use w in place of x in the aging spring section of Module 11.

Bessel's equation of order $p = 0$ is given by:

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + s^2x = 0$$

and when we substitute in the derivatives we just found, we obtain

$$s^2\left(\frac{d^2x}{dt^2}\frac{4}{(as)^2} + \frac{dx}{dt}\frac{2}{as^2}\right) + s\frac{dx}{dt}\left(-\frac{2}{as}\right) + s^2x = 0$$

Using the fact that

$$s = (2/a)e^{-at/2} \tag{12}$$

(found by solving equation (11) for s) in the last term, when we simplify this monster equation it collapses down to a nice simple one:

$$\frac{d^2x}{dt^2}\frac{4}{a^2} + \frac{4}{a^2}e^{-at}x = 0$$

Finally, if we divide through by $4/a^2$, we get the aging spring equation, $x'' + e^{-at}x = 0$.

The other way around works as well, that is, a change of variables will convert the aging spring equation to Bessel's equation of order zero. That means that solutions of the aging spring equation can be expressed in terms of Bessel functions. This can be accomplished by using $x = c_1J_0(s) + c_2Y_0(s)$ as the general solution of Bessel's equation of order 0, and then using formula (12) to replace s. Take another look as Experiments 3 and 4 on Screens 2.5 and 2.6 of Module 11. That will give you a graphical sense about the connection between aging springs and a Bessel's equation.

References

Borrelli, R. L., and Coleman, C. S., *Differential Equations: A Modeling Perspective*, (1998: John Wiley & Sons, Inc.)

Boyce, W. E., and DiPrima, R. C., *Elementary Differential Equations and Boundary Value Problems*, 6th ed., (1997: John Wiley & Sons, Inc.)

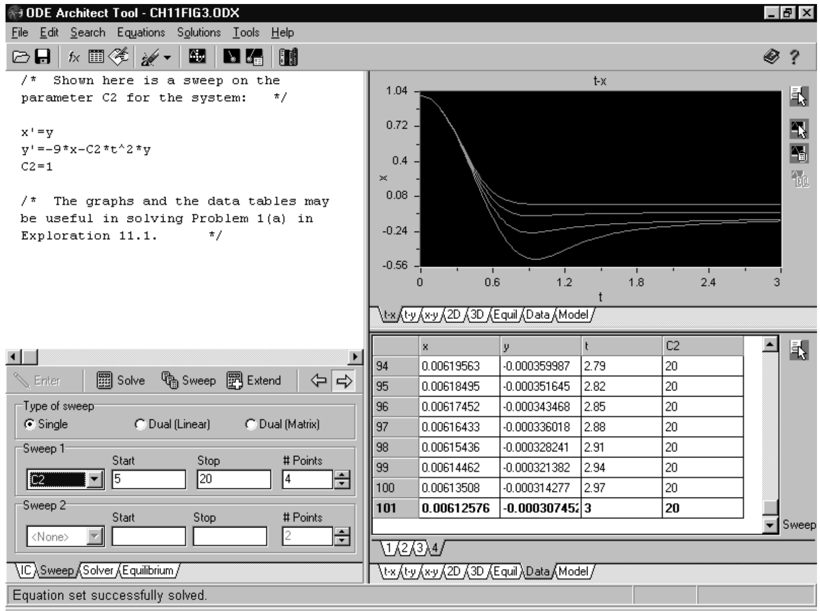


Figure 11.3: Here are some typical graphs for the solution of $x'' + C_2t^2x' + 9x = 0$ for various values of C_2. The graphs and the data tables are useful in Problem 1 of Exploration 11.1.

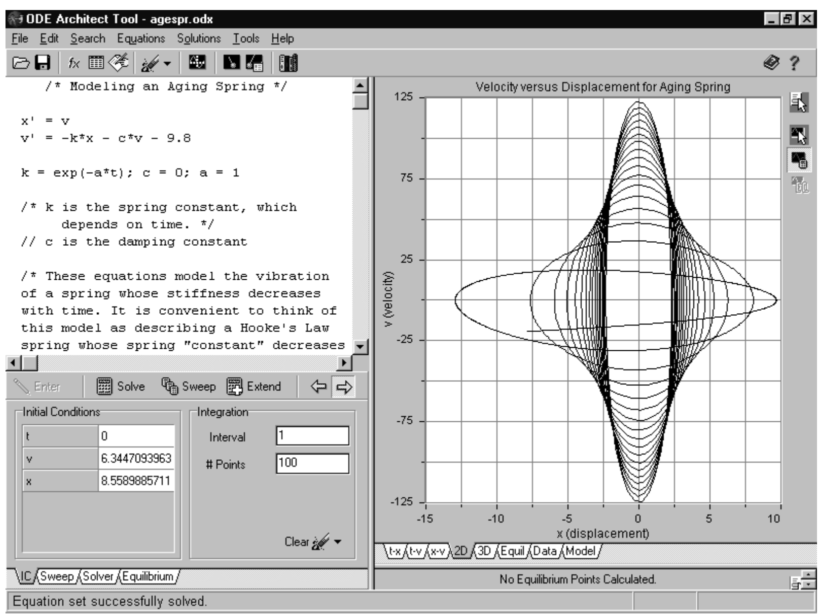


Figure 11.4: Here is a phase-plane portrait for an aging spring ODE, $x'' + e^{-t}x = -9.8$. See "Modeling an Aging Spring" in the library folder "Physical Models" and also Problem 1 in Exploration 11.3.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date ______________________________

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Exploration 11.1. Damping a Robot Arm

In each of the following problems it is assumed that the displacement x of a robot arm satisfies an IVP of the form

$$x'' + b(t)x' + 9x = 0, \quad x(0) = 1, \quad x'(0) = 0$$

An optimal damping function $b(t)$ is one for which the solution $x(t)$ reaches 0.005 in minimal time t^* without ever going below zero.

1. Consider damping functions of the form $b(t) = C_k t^k$. For a positive integer k, let C_k^* be the value of C_k that gives the optimal solution, and denote the corresponding minimal time by t_k^*. In Module 11, Screen 1.4 and TTA 3 on Screen 1.7 you found that the optimal solution for $k = 1$ is $x(t) = e^{-9t^2/2}$, with $C_1^* = 9$ and $t_1^* \approx 1.0897$.

(a) Use ODE Architect to find an approximate optimal solution and values of C_k^* and t_k^* when $k = 2$. [*Suggestion:* Look at Figure 11.3.]

(b) Repeat with $k = 3$.

(c) Compare the optimal damping functions for $k = 1, 2, 3$, in the context of the given physical process.

2. For *quadratic damping*, $b(t) = C_2 t^2$, derive a power series solution $x(t) = \sum_{n=0}^{\infty} a_n t^n$. Show that the recurrence formula for the coefficients is

$$a_{n+2} = \frac{-[9a_n + C_2(n-1)a_{n-1}]}{(n+1)(n+2)}, \quad n \geq 1$$

and $a_2 = -9a_0/2$. Recall that $a_0 = x(0)$ and $a_1 = x'(0)$.

3. Let $P_6(t)$ be the Taylor polynomial $\sum_{n=0}^{6} a_n t^n$, where the a_n are given by the recurrence formula in Problem 2.

(a) Write out $P_6(t)$ with C_2 as a parameter; briefly describe how the graph of $P_6(t)$ changes as C_2 increases.

☞ You will need results from Problem 1(a) here.

(b) Graph the apparently optimal solution from Problem 1(a) over the interval $0 \le t \le t_2^*$ and compare it to the graph of $P_6(t)$ with $C_2 = C_2^*$.

4. If the robot arm is totally undamped, its position at time t is $x(t) = \cos 3t$; therefore the arm cannot reach $x = 0$ for all t, $0 \le t \le \pi/6$. In this situation the undamped arm can't remain above $x = 0$. The optimal damping functions $C_k^* t^k$ found in Problem 1 look more like step functions as the degree k increases. Try to improve the time t^* by using a step function for damping.

Assume the robot arm is allowed to fall without damping until just before it reaches $x = 0$, at which time a constant damping force is applied. This situation can be modeled by defining

$$b(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{\pi}{6} - \varepsilon \\ B_\varepsilon & \text{for } t \ge \frac{\pi}{6} - \varepsilon \end{cases}$$

for $\varepsilon = 0.2$, 0.1, and 0.05. Use ODE Architect to find values of B_ε that give an approximate optimal solution. Include a graph showing your best solution for each ε and give your best value of t^* in each case. What happens to the "optimal" B_ε as $\varepsilon \to 0$?

5. Find a formula for the solution for the situation in Problem 4. The value of ε should be treated as a parameter. Assume that $x(t) = \cos 3t$ for $t < (\pi/6) - \varepsilon$. Then the IVP to be solved is

$$\begin{aligned} &x'' + B_\varepsilon x' + 9x = 0 \\ &x(\pi/6 - \varepsilon) = \cos[3(\tfrac{\pi}{6} - \varepsilon)] = \sin 3\varepsilon \\ &x'(\pi/6 - \varepsilon) = -3\sin[3(\tfrac{\pi}{6} - \varepsilon)] = -3\cos 3\varepsilon \end{aligned}$$

The solution will be of the form $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $r_1 < r_2 < 0$, but the optimal solution requires that $c_2 = 0$. Why? For a fixed ε, find the value of B_ε so that $x(t)$ remains positive and reaches 0.005 in minimum time.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

Name/Date ______________________________

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Exploration 11.2. Bessel Functions

1. Bessel functions resemble decaying sinusoids. Let's compare the graph of $J_0(t)$ with that of one of these sinusoids.

(a) On the same set of axes, graph the Bessel function $J_0(t)$ and the function

$$\sqrt{\frac{2}{\pi t}}\cos\left(t - \frac{\pi}{4}\right)$$

over the interval $0 \le t \le 10$.

(b) Now graph these same two functions over the interval $0 \le t \le 50$.

(c) Describe what you see.

[*Suggestion:* You can use ODE Architect to plot a good approximation of $J_0(t)$ by solving an IVP involving Bessel's equation in system form:

$$x' = y, \quad y' = -x - y/t, \qquad x(t_0) = 1, \quad x'(t_0) = 0$$

with $t_0 = 0.0001$. Actually, $J_0(0) = 1$ and $J_0'(0) = 0$, but $t_0 = 0$ is a singular point of the system so we must move slightly away from zero. You can plot the decaying sinusoid on the same axes as $J_0(t)$ by entering $a = \sqrt{\frac{2}{\pi t}}\cos(t - \frac{\pi}{4})$ in the same equation window as the IVP, selecting a custom 2D plot, and plotting both a and x vs. t.]

2. Repeat Problem 1 for the functions $Y_0(t)$ and $\sqrt{\frac{2}{\pi t}}\sin\left(t - \frac{\pi}{4}\right)$. To graph a good approximation of $Y_0(t)$, solve the system equivalent of Bessel's equation of order zero (from Problem 1) with initial data $t_0 = 0.89357$, $x(t_0) = 0$, $x'(t_0) = 0.87942$. As in Problem 1, we have to avoid the singularity at $t_0 = 0$, especially here because $Y_0(0) = -\infty$. The given initial data are taken from published values of Bessel functions and their derivatives.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 11.3. Aging Spring Models

1. Check out the Library file "Modeling an Aging Spring" in the "Physical Models" folder (see Figure 11.4). The ODE in the file models the motion of a vertically suspended damped and aging spring that is subject to gravity. Carry out the suggested explorations.

2. Show that

$$x(t) = \sqrt{\frac{t+1}{3}}\sin\left(\frac{\sqrt{3}}{2}\ln(t+1)\right) - \sqrt{t+1}\cos\left(\frac{\sqrt{3}}{2}\ln(t+1)\right)$$

is an analytic solution of the initial value problem

$$x''(t) + \frac{x(t)}{(t+1)^2} = 0, \quad x(0) = -1, \quad x'(0) = 0$$

Explain why this IVP provides another model for the motion of an aging spring that is sliding back and forth (without damping) on a support table.

3. Graph the solution $x(t)$ from Problem 2 over the interval $0 \leq x \leq 10$ and compare the graph to the one obtained in Module 11 using ODE Architect.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 11.4. The Incredible Lengthening Pendulum

☞ The ODE for a pendulum of varying length is derived in Chapter 10.

Suppose that we have an undamped pendulum whose length $L = a + bt$ increases linearly over time. Then the ODE that models the motion of this pendulum is

$$(a + bt)\theta''(t) + 2b\theta'(t) + g\theta(t) = 0 \tag{13}$$

where θ is small enough that $\sin\theta \approx \theta$, the mass of the pendulum bob is 1, and the value of the acceleration due to gravity is $g = 32$.[1]

1. With $a = b = 1$ and initial conditions $\theta(0) = 1$ and $\theta'(0) = 0$, use ODE Architect to solve ODE (13) numerically. What happens to $\theta(t)$ as $t \to +\infty$?

2. Under the same conditions, what happens to the oscillation time of the pendulum as $t \to +\infty$? (The oscillation time is the time between successive maxima of $\theta(t)$.)

[1]See the article "Poe's Pendulum" by Borrelli, Coleman, and Hobson in *Mathematics Magazine*, Vol. 58 (1985) No. 2, pp. 78–83. See also "Child on a Swing" in Module 10.

3. Show that the change of variables

$$s = (2/b)\sqrt{(a+bt)g}, \quad x = \theta\sqrt{a+bt}$$

transforms Bessel's equation of order 1

$$s^2\frac{d^2x}{ds^2} + s\frac{dx}{ds} + (s^2-1)x = 0$$

into ODE (13) for the lengthening pendulum. [*Suggestion:* Take a look at the section "Transforming Bessel's Equation to the Aging Spring Equation" in this chapter to help you get started. Use the change of variables given above to express the solution of the IVP in Problem 1 using Bessel functions.]