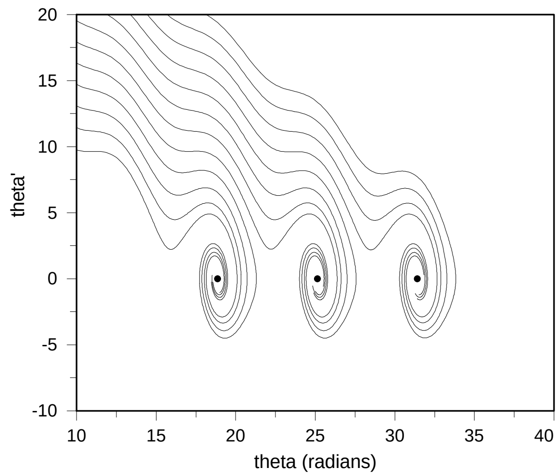
CHAPTER

10 The Pendulum and Its Friends



High energy trajectories of a damped pendulum ODE swing over the top and then settle into decaying oscillations about rest points.

Overview The whole range of fixed-length pendulum models—linear, nonlinear, damped, and forced—are presented in this chapter, and their behaviors are compared using insights provided by integrals. After discussing fixed-length pendulum ODEs, the effects of damping, and separatrices, we turn to a variable-length model. A child pumping a swing alters the length of its associated pendulum as the swing moves. We present a nontraditional autonomous model and show that phase-plane analysis leads to a successful description of the effects of the pumping action. Finally, the problem of finding geodesics (the paths of minimum length between points) on a torus leads to an ODE with a striking resemblance to the pendulum ODE.

Key words Linear pendulum; nonlinear pendulum; damping; energy; pumping (a swing); conservation laws; torus; geodesic; limit cycle; bifurcation

See also Chapter 4 for a spring-mass system which has the same ODE as the linear pendulum; Chapter 11 for a study of damping effects in the Robot and Egg submodule, and a lengthening pendulum in Exploration 11.4; and Chapter 12 for elaboration on the forced, damped pendulum resulting in chaos (and control).

◆ Modeling Pendulum Motion

☞ The volumes by Halliday and Resnick (refs.) are good general references for physical models (including the pendulum).

For a pendulum bob of mass m at the end of a rod of negligible weight and fixed length L at an angle θ to the vertical, Newton's second law gives

$$\text{mass} \cdot \text{acceleration} = \text{sum of forces acting on the bob}$$

☞ Since the tensile force in the rod and the radial component of the gravitational force are equal and opposite, the radial acceleration is zero and the pendulum moves along a circular arc.

The bob moves along an arc of a circle of radius L. The tangential component of the bob's velocity and acceleration at time t are given by $L\theta'(t)$ and $L\theta''(t)$, respectively. The tangential component, $-mg\sin\theta$, of the gravitational force acts to restore the pendulum to its downward equilibrium. The viscous damping force, $-bL\theta'$, is proportional to the velocity and acts in a direction tangential to the motion, but oppositely directed. Other forces such as repeated pushes on the bob may also have components $F(t)$ in the tangential direction

Equating the product of the mass and the tangential acceleration to the sum of the tangential forces, we obtain the pendulum ODE.

$$mL\theta'' = -mg\sin\theta - bL\theta' - F(t) \tag{1}$$

The equivalent pendulum system is

$$\begin{aligned} \theta' &= y \\ y' &= -\frac{g}{L}\sin\theta - \frac{b}{m}y + \frac{1}{mL}F(t) \end{aligned} \tag{2}$$

The angle θ is positive if measured counterclockwise from the downward vertical, and is negative otherwise; θ is measured in radians (1 radian is $360/2\pi$ or about 57°). We allow θ to increase or decrease without bound because we want to keep track of the number of times that the pendulum swings over the pivot, and in which direction. For example, if $\theta = -5$ radians then the pendulum was swung clockwise (the minus sign) once over the top from $\theta = 0$ because the angle -5 is between $-\pi$ (at the top clockwise from 0) and -3π (reaching the top a second time going clockwise).

We will work with the *undriven* pendulum ODE ($F = 0$) in this chapter. Since $\sin\theta \approx \theta$ if $|\theta|$ is small, we will on occasion replace $\sin\theta$ by θ to obtain a linear ODE. We treat both undamped ($b = 0$) and damped ($b > 0$) pendulum ODEs:

☞ The first two ODEs in this list have the form of the mass-spring ODEs of Chapter 4.

$$\theta'' + \frac{g}{L}\theta = 0 \quad \text{(undamped, linear)} \tag{3a}$$

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{L}\theta = 0 \quad \text{(damped, linear)} \tag{3b}$$

$$\theta'' + \frac{g}{L}\sin\theta = 0 \quad \text{(undamped, nonlinear)} \tag{3c}$$

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{L}\sin\theta = 0 \quad \text{(damped, nonlinear)} \tag{3d}$$

$$\theta'' + \frac{b}{m}\theta' + \frac{g}{L}\sin\theta = \frac{1}{mL}F(t) \quad \text{(damped, nonlinear, forced)} \tag{3e}$$

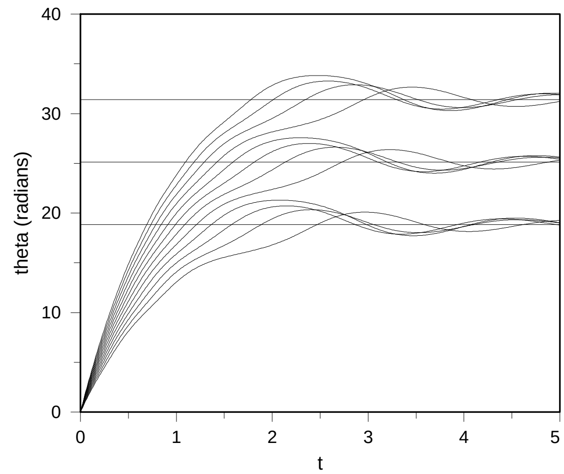


Figure 10.1: Solution curves of a damped pendulum system. What is the meaning of the horizontal solution curves?

Figure 10.1 and the chapter cover figure, respectively, show some solution curves and trajectories of the damped, nonlinear pendulum ODE, $\theta'' + \theta' + 10\sin\theta = 0$. Although the two linear ODEs are only good models of actual pendulum motions when $|\theta|$ is small, these ODEs have the advantages that their solutions have explicit formulas (see Chapter 4). The nonlinear ODEs model pendulum motions for all values of θ, but there are no explicit solution formulas.

Now fire up your computer, go to Screen 1.2 of Module 10, and visually explore the behavior of solution curves and trajectories of linear, nonlinear, damped, and undamped pendulum ODEs. Pay particular attention to the behavior of the animated pendulum at the upper left, and relate its motions to the trajectories and to the solution curves, and to what you think a real pendulum would do. Explore all the options in order to understand the differences.

✓ "Check" your understanding by matching solution curves of Figure 10.1 with the corresponding trajectories in the chapter cover figure. Describe the long-term behavior of the pendulum represented by each curve.

☞ This is also what Problem 1 of Exploration 10.1 is about.

✓ Go to Screen 1.2 of Module 10 and explore what happens to solutions of the undamped, linearized ODE, $\theta'' + \theta = 0$, if θ_0 is 0 and θ'_0 is large. The motion of the animated pendulum is crazy, even though it accurately portrays the behavior of the solutions $\theta(t) = \theta'_0 \sin t$. Explain what is going on. Is the linearized ODE a good model here? Repeat with the undamped, nonlinear ODE, $\theta'' + \sin\theta = 0$, and the same initial data as above. Is this a better model?

There is another way to look at pendulum motion, an approach based on integrals of motion. This approach goes beyond pendulum motion and applies to any physical system which can be modeled by a second-order ODE of a particular type.

◆ Conservative Systems: Integrals of Motion

In this section we will study solutions of the differential equation

☞ V is often called a potential function.

$$q'' = -\frac{dV}{dq} \tag{4}$$

for a generic variable q where $V(q)$ is a given function.

Example 1: The undamped, nonlinear pendulum ODE is the special case where $q = \theta$:

$$\theta'' = -\frac{g}{L}\sin\theta, \qquad V(\theta) = -\frac{g}{L}\cos\theta$$

Example 2: You will see later in this chapter that geodesics on a surface of revolution lead to the differential equation

$$u'' = M^2\frac{f'}{f^3}, \qquad V(u) = M^2\frac{1}{2f^2}$$

where the generic variable q is u in this case, M is a constant, and f is a function of u.

ODE (4) is autonomous and equivalent to the system

$$\begin{aligned} q' &= y \\ y' &= -\frac{dV}{dq} \end{aligned} \tag{5}$$

A solution to system (5) is a pair of functions, $q = q(t)$, $\;y = y(t)$. One way to analyze the behavior of these solutions is by a conservation law. A function $K(q, y)$ that remains constant on each solution [i.e., $K(q(t), y(t))$ is a constant for all t], but varies from one solution to another, is said to be a *conserved quantity*, or an *integral of motion* and the system is said to be *conservative*. For system (5) one conserved quantity is

$$K(q, y) = \frac{1}{2}y^2 + V(q) \tag{6}$$

Here's how to prove that $K(q(t), y(t))$ stays constant on a solution—use the chain rule and system (5) to show that dK/dt is zero:

$$\frac{dK}{dt} = y\frac{dy}{dt} + \frac{dV}{dq}\frac{dq}{dt} = y\left(-\frac{dV}{dq}\right) + \frac{dV}{dq}y = 0$$

Incidentally, if K is any conserved quantity, so also is $\alpha K + \beta$ where α and β are constants and $\alpha \neq 0$.

Example 3: Here's an example where we use $\alpha K + \beta$, rather than K, as the integral to show how integrals sometimes correspond to physical quantities. Look back at the function $V(\theta) = -(g/L)\cos\theta$ for the undamped, nonlinear pendulum of Example 1. Using formula (6), we see that $E(\theta, y)$ is an integral, where

$$\begin{aligned} E(\theta, y) &= mL^2 K(\theta, y) + mgL \\ &= mL^2\left(\frac{1}{2}y^2 - \frac{g}{L}\cos\theta\right) + mgL \\ &= \frac{1}{2}m(Ly)^2 + mgL(1-\cos\theta) \\ &= \text{kinetic energy} + \text{potential energy} \end{aligned}$$

This integral is called the *total mechanical energy* of the pendulum. The constant mgL is inserted so that the potential energy is zero when the pendulum bob is at its lowest point.

✓ Find the conserved quantity E for the undamped, linear pendulum ODE $\theta'' + \theta = 0$. Draw level curves $E(\theta, y) = E_0$, where $y = \theta'$, in the θy-plane, and identify the curves (e.g., ellipses, parabolas, hyperbolas).

☞ So we can draw trajectories of system (5) by drawing level sets of an integral.

Drawing the level curves of a conserved quantity K in the qy-plane for system (5) gives phase plane trajectories of the system and so serves to describe the motions. This may be much easier than finding solution formulas, but even so, we can take some steps toward obtaining formulas. To see this, we have from equation (6) that if K has the value K_0 on a trjajectory of system (5)

$$\frac{1}{2}y^2 + V(q) = K_0, \quad \text{i.e.,} \quad y = q' = \pm\sqrt{2K_0 - 2V(q)}$$

This is a separable first-order differential equation (as discussed in Chapter 2) that can be solved by separating the variables and integrating:

$$\int \frac{dq}{\sqrt{K_0 - V(q)}} = \sqrt{2}t + C$$

◆ The Effect of Damping

Mechanical systems are usually damped by friction, and it is important to understand the effect of friction on the motions. Friction is not well described by the fundamental laws of physics, and any formula we write for it will be more or less *ad-hoc*. The system will now be modeled by a differential equation of the form

$$q'' + f(q, q') + \frac{dV}{dq} = 0$$

or, rewritten as a system of first-order ODEs,

$$\begin{aligned} q' &= y \\ y' &= -f(q,y) - dV/dq \end{aligned} \tag{7}$$

where $-f(q,y)$ represents the frictional force; the function $f(q,y)$ always has the sign of y.

At low velocities, $f(q,y) = by$ is a reasonably good approximation of the friction due to air, but higher powers of y are necessary at higher velocities. This latter fact is why reducing the speed limit actually helps reduce gasoline usage—there is less drag at lower speeds. If friction were only a linear function of velocity, the effects of a higher speed would be cancelled by the distance being covered in a shorter time, and the system would expend the same amount of energy in either case. But if friction depends on the cube of velocity, for instance, you gain a lot by going more slowly. We will examine more elaborate friction laws when we study the pumping of a swing, but for now we will use viscous damping with $f = by$.

Example 4: Let's model the motion of a linearized pendulum with and without damping:

$$\begin{aligned} \theta' &= y \\ y' &= -10\theta - by \end{aligned} \tag{8}$$

where $b = 0$ (no damping), or $b = 1$ (viscous damping). If there is no damping, then one conserved quantity is

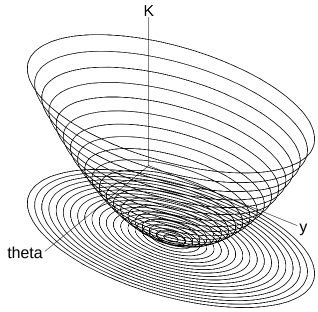
$$K = \frac{1}{2}y^2 + 5\theta^2 \tag{9}$$

The left graph in Figure 10.2 displays the integral surface defined by formula (9). The surface is a bowl whose cross-sections $K = K_0$ are ellipses. Projecting the ellipses downward onto the θy-plane gives the trajectories of system (8) with $b = 0$.

Once damping is turned on, the integral K in formula (9) no longer is constant on a trajectory. But the integral concept still gives a good geometric picture of the behavior of a system under damping, because the value of K decreases along trajectories. This fact follows from the following computation (using system (7)):

$$\begin{aligned} \frac{d}{dt}\left(\frac{y^2}{2} + V(q)\right) = y\frac{dy}{dt} + \frac{dV}{dq}\frac{dq}{dt} &= y\left(-f(q,y) - \frac{dV}{dq}\right) + \frac{dV}{dq}y \\ &= -yf(q,y) \le 0 \end{aligned}$$

where the final inequality follows from the fact that $f(q,y)$ has the sign of y. In particular, the value of K along a solution of system (8) decreases, and will either tend to a finite limit, which can only happen if the solution tends to an equilibrium of the system, or the value of K will tend to $-\infty$. If V is bounded from below (as happens for all our examples), the latter does not happen.



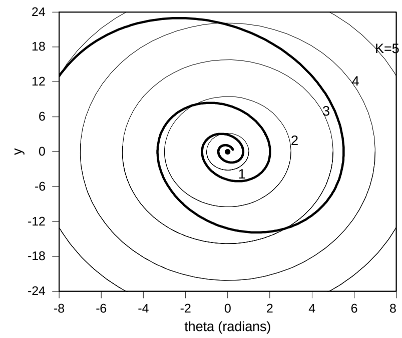


Figure 10.2: The left graph shows the integral surface $K = y^2/2 + 5\theta^2$ for the undamped, linearized system, $\theta' = y$, $y' = -10\theta$, and the projections of the level curves $K = K_0$. The right graph shows a trajectory of the damped, linearized system, $\theta' = y$, $y' = -10\theta - y$, as it cuts across the level curves of K, with K decreasing as it goes.

Example 5: Let's turn on viscous damping (take $b = 1$ in system (8)) and see what happens. The right side of Figure 10.2 shows a trajectory of the damped, linear pendulum system as it cuts across the level curves of the integral function $K = y^2/2 + 5\theta^2$. K decreases as the trajectory approaches the spiral sink at $\theta = 0$, $y = 0$. [The level curves of K are drawn by ODE Architect as trajectories of the undamped system (8) with $b = 0$.]

Now let's turn to the more realistic nonlinear pendulum and see how damping affects its motions.

Example 6: The nonlinear system is

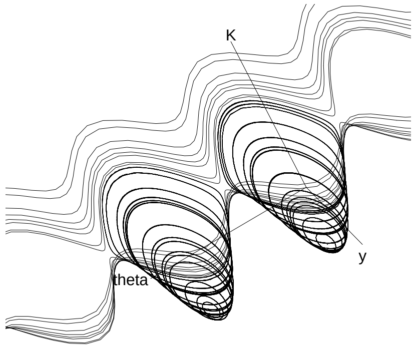
$$\begin{aligned} \theta' &= y \\ y' &= -10\sin\theta - by \end{aligned} \tag{10}$$

where $b = 0$ corresponds to no damping, and $b = 1$ gives viscous damping. In the no-damping case we can take the conserved quantity K to be

$$K = \frac{1}{2}y^2 - 10(\cos\theta - 1) \tag{11}$$

The left side of Figure 10.3 shows part of the surface defined by equation (11).

Example 7: With damping turned on (set $b = 1$ in system (10)) a trajectory with a high initial K-value may "swing over the pivot" several times before settling into a tightening spiral terminating at a sink, $\theta = 2n\pi$, $y = 0$, for some value of n. The right side of Figure 10.3 shows one of these trajectories as it swings over the pivot once, and then heads toward the point, $\theta = 2\pi$, $y = 0$, where $K = 0$.



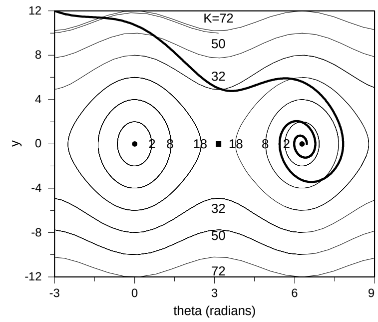


Figure 10.3: The left graph shows the surface $K = y^2/2 - 10(\cos\theta - 1)$ with two of its bowl-like projections that touch the θy-plane at equilibrium points of minimal K-value. The nonlinear pendulum system is $\theta' = y,\ y' = -10\sin\theta - by$ with $b = 0$. Turn on damping ($b = 1$) and watch a trajectory cut across the level sets $K = K_0$, with ever smaller values of K (right graph).

✓ Would you increase or decrease b to cause the trajectory starting at $\theta = -3,\ y = 12$ to approach $\theta = 0,\ y = 0$? How about $\theta = 10\pi,\ y = 0$? What would you need to do to get the trajectory to approach $\theta = -2\pi,\ y = 0$, or is this even possible?

◆ Separatrices

A trajectory is a *separatrix* if the long-term behavior of trajectories on one side is quite different from the behavior on the other side. As we saw in Chapters 6 and 7 each saddle comes equipped with four separatrices: two approach the saddle with increasing time (the *stable* separatrices for that saddle point) and two approach as time decreases (the *unstable* separatrices). These separatrices are of the utmost importance in understanding how solutions behave in the long term.

Example 8: The undamped system

$$\begin{aligned} \theta' &= y \\ y' &= -10\sin\theta \end{aligned} \tag{12}$$

has equilibrium points at $\theta = n\pi,\ y = 0$. According to the equilibrium calculations in ODE Architect, these points are centers if n is even, and saddles if n is odd. Each separatrix at a saddle enters (or leaves) the saddle tangent to an eigenvector of the Jacobian matrix evaluated at the point. ODE Architect gives us these eigenvectors after it has located the saddle.

Example 9: (*Plotting a Separatrix:*) To find a point approximately on a saddle separatrix, just take a point close to a saddle and on an eigenvector. Then solve

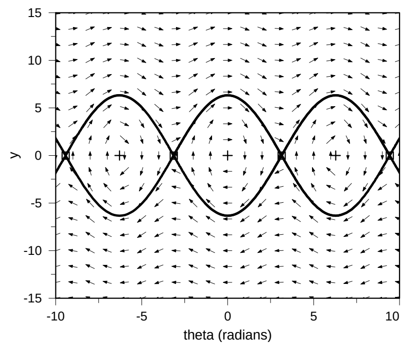


Figure 10.4: Saddle separatrices for the undamped, nonlinear pendulum system enclose centers.

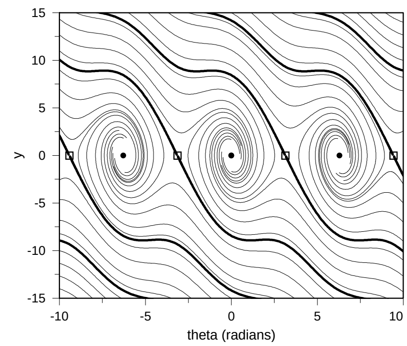


Figure 10.5: Basins of attraction of spiral sinks are bounded by stable saddle separatrices.

forward and backward to obtain a reasonable approximation to a separatrix. For example, at $\theta = \pi,\ \ y = 0$, ODE Architect tells us that $(0.3015, 0.9535)$ is an eigenvector corresponding to the eigenvalue 3.162, and so the saddle separatrix is unstable. To graph the corresponding separatrix we choose as the initial point $\theta_0 = \pi + 0.003015,\ \ y = 0.009535$ which is in the direction of the eigenvector and very close to the saddle point. Figure 10.4 shows several separatrices of system (12). The squares indicate saddle points, and the plus signs inside the regions bounded by separatrices indicate centers.

✓ Describe the motions that correspond to trajectories inside the regions bounded by separatrices. Repeat with the region above the separatrices. Can a separatrix be both stable and unstable? [*Hint:* Each separatrix in Figure 10.4 begins and ends at different points.]

Example 10: Add in some viscous damping and the picture completely changes: Figure 10.5 shows the stable separatrices at the saddle points for the system

$$\begin{aligned} \theta' &= y \\ y' &= -10\sin\theta - y \end{aligned} \tag{13}$$

The equilibrium points at $\theta = 2n\pi,\ \ y = 0$ are no longer centers, but attracting spiral points (the solid dots). The *basin of attraction* of each sink (i.e., the points on the trajectories attracted to the sink) is bounded by the four stable saddle separatrices.

✓ With a fine-tipped pen, draw the *unstable* separatrices at each saddle in Figure 10.5.

That's all we have to say about the motions of a constant-length pendulum for now. More (much more) is discussed in Chapter 12, where we add a driving term $F(t)$ to the pendulum equations.

◆ Pumping a Swing

Recall that in an *autonomous* differential equation, the time variable t does not appear explicitly. The central thing to realize is that *the ODE that models pumping a swing must be autonomous:* a child pumping the swing does not consult a watch when deciding how to lean back or sit up; the movements depend only on the position of the swing and its velocity. The swinger may change pumping strategies, deciding to go higher or slow down, but the modeling differential equation for any particular strategy should be autonomous, depending on various parameters which describe the strategy.

☞ If you use a different pumping strategy, make up a differential equation of your own!

If you observe a child pumping a swing, or do it yourself, you will find that one strategy is to lean back on the first half of the forward swing and to sit up the rest of the time. If you stand on the seat, the strategy is the same: you crouch during the forward down-swing, and stand up straight the rest of the time. The work is done when you bring yourself back upright during the forward up-swing, either by pulling on the ropes (if sitting), or simply by standing.

The pumping action effectively changes the length of the swing, which complicates the ODE considerably, for two reasons. Newton's second law must be stated differently, as will be shown below, and we must find an appropriate equation to model the changing length.

The question of friction is more subtle. Of course, the air creates a drag, but that is not the most important component of friction. We believe that things are quite different for a swing attached to the axle by something flexible, than if it were attached by rigid rods. Circus acrobats often drive swings right over the top; they always have rigid swings. We believe that a swing attached flexibly to the axle cannot be pumped to go over the top. Suppose the swing were to go beyond the horizontal—then at the end of the forward motion, the swinger would go into free-fall instead of swinging back; the jolt (heard as "ka-chunk") when the rope becomes tight again will drastically slow down the motion. If you get on a swing, you will find that this effect is felt before the amplitude of the swing reaches $\pi/2$; the ropes become loose near the top of the forward swing, and you slow down abruptly when they draw tight again.

We will now turn this description into a differential equation.

◆ Writing the Equations of Motion for Pumping a Swing

Modeling the pendulum with changing length requires a more careful look at Newton's second law of motion. The equation $F = ma = mq''$ is not correct when the mass is changing (as when you use a leaky bucket as the bob of a pendulum), or when the distance variable is changing with respect to position and velocity (as for the child on the swing). In cases such as this, force is the rate of change of the *momentum* mq':

$$\text{Force} = (mq')' \tag{14}$$

When the mass and pendulum length are constant, equation (14) indeed reduces to the more familiar $F = ma$.

The analog in rotational mechanics about a pivot, where $q = L\theta$, is that the *torque* equals the rate of change of angular momentum:

$$\text{Torque} = (I\theta')'$$

where I is the moment of inertia (the rotational analog of the mass). If a force $\mathbf{F}$ is applied at a point p, then the torque about the pivot is the vector product $\mathbf{r} \times \mathbf{F}$, where $\mathbf{r}$ is the position vector from the pivot to p. For the undamped and nonlinear pendulum, the gravitational torque can be treated as the scalar $-mgL\sin\theta$, and the moment of inertia is $I = mL^2$. Then Newton's second law becomes

$$(mL^2\theta')' = -mgL\sin\theta \tag{15}$$

When L and m are constant, equation (15) is precisely the ODE of the undamped, nonlinear pendulum. In the case of the child pumping a swing, the mass m remains constant (and can be divided out of the equation), but L is *not* constant, so we must differentiate $L^2\theta'$ in (15) using the chain rule to get

$$2L\left(\frac{\partial L}{\partial \theta}\theta' + \frac{\partial L}{\partial \theta'}\theta''\right)\theta' + L^2\theta'' = -gL\sin\theta$$

or, in system form

$$\begin{aligned} \theta' &= y \\ y' &= -\frac{2y^2\partial L/\partial\theta + g\sin\theta}{2y\partial L/\partial y + L} \end{aligned} \tag{16}$$

The person pumping the swing is changing L as a function of θ and y. For the reasons given in Screen 2.3 of Module 10 we will use the following formula for L:

$$L = L_0 + \frac{\Delta L}{\pi^2}\left(\frac{\pi}{2} - \arctan 10\theta\right)\left(\frac{\pi}{2} + \arctan 10y\right) \tag{17}$$

where L_0 is the distance from the axle to the center of gravity of the swinger when upright, and ΔL is the amount by which leaning back (or crouching) increases this distance. Note that

$$\frac{1}{\pi}\left(\frac{\pi}{2} - \arctan 10\theta\right)$$

is a smoothed-out step function: roughly 1 when $\theta < 0$ and 0 when $\theta > 0$. The jump from one value to the other is fairly rapid because of the factor 10; other values would be appropriate if you were to sit (or stand) up more or less suddenly. A similar analysis applies to the second arctan factor in formula (17).

As for friction with the swing, we will use

$$f(\theta, y) = \varepsilon y + \left(\frac{\theta}{1.4}\right)^6 y$$

The first term corresponds to some small viscous air resistance. Admittedly, the second term is quite ad-hoc, but it serves to describe some sort of insurmountable “brick wall,” which somewhat suddenly takes effect when $\theta > 3/2 \sim \pi/2$. So it does seem to reflect our qualitative description.

Writing the differential equation as an autonomous system is now routine—an unpleasant routine since we need to differentiate L, which leads to pretty horrific formulas. But with this summary, we have tried to make the structure clear. Now let’s get real and insert friction into modeling system (16):

$$\begin{aligned} \theta' &= y \\ y' &= -\frac{2y^2 \partial L/\partial\theta + g\sin\theta + \text{friction term}}{2y\partial L/\partial y + L} \end{aligned} \tag{18}$$

where L is given by formula (17) and

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -10\Delta L \frac{\frac{\pi}{2} + \arctan(10y)}{\pi^2(1+100\theta^2)} \\ \frac{\partial L}{\partial y} &= 10\Delta L \frac{\frac{\pi}{2} - \arctan(10\theta)}{\pi^2(1+100y^2)} \\ \text{friction term} &= \varepsilon y + \left(\frac{\theta}{1.4}\right)^6 y \end{aligned} \tag{19}$$

Example 11: Now set $g = 32$, $L_0 = 4$, $\Delta L = 1$, and $\varepsilon = 0$ (no viscous damping), and use ODE Architect to solve system (18). Figure 10.6 shows that you can pump up a swing from rest at an initial angle of 0.25 radian (about 14°) within a reasonable time, but not from the tiny angle of 0.01 radian. Do you see the approach to a stable, periodic, high-amplitude oscillation? This corresponds to an attracting limit cycle in the θy-plane.

What happens if we put viscous damping back in? See for yourself by going to Screen 2.5 of Module 10 and clicking on several initial points in the θy-screen. You should see *two* limit cycles now:

- a large attracting limit cycle representing an oscillation of amplitude close to $\pi/2$, due to the “brick wall” friction term, and (for $\varepsilon > 0$)
- a small repelling limit cycle near the downward equilibrium, due to friction and viscous air resistance.

In order to get going, the child must move the swing outside the small limit cycle, either by cajoling someone into pushing her, or backing up with her feet on the ground. Once outside the small limit cycle, the pumping will push the trajectory to the attracting limit cycle, where it will stay until the child decides to slow down.

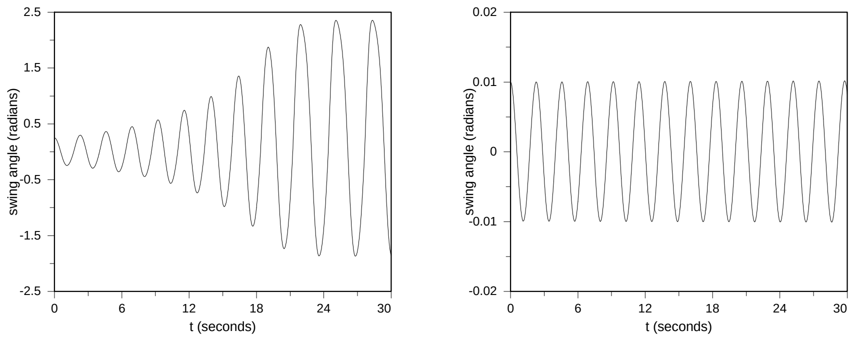


Figure 10.6: Successful pumping (left graph) starts at a moderately high angle ($\theta = 0.25$ radian). If θ_0 is small (e.g., $\theta_0 = 0.01$ rad), then pumping doesn't help much (right graph).

Please note that this structure of the phase plane, with two limit cycles, is necessary in order to account for the observed behavior: the origin must be a sink because of air resistance, and you cannot have an attracting limit cycle surrounding a sink without another limit cycle in between.

✓ Does the system without viscous damping have a small repelling limit cycle?

◆ Geodesics

Geodesics on a surface are curves that minimize length between sufficiently close points on the surface; they may, but need not, minimize length between distant points.

Example 12: Straight lines are geodesics on planes, and they minimize the distance between arbitrary points. Great circles are geodesics on the unit sphere, but they only minimize length between pairs of points if you travel in the right direction. If you travel along the equator your path will be shortest until you get half-way around the world; but further along, you would have done better to go the other way.

To look for geodesics, we use the fact that parametrization of a curve γ by its arc length s results in traversing a curve at constant speed 1, that is, $|d\gamma/ds|$ is always 1.

On a surface in three-dimensional space, a geodesic γ is a curve for which the vector $d^2\gamma/ds^2$ is perpendicular to the surface at the point $\gamma(s)$. For now, let's assume that all curves are parameterized by arc length, so γ' means $d\gamma/ds$.

If any curve γ (not necessarily a geodesic) on the surface is parametrized at constant speed, we are guaranteed that γ'' is perpendicular to γ', but not necessarily to the surface. To see this, observe that $\gamma' \cdot \gamma' = 1$, where γ' is the velocity vector for the curve γ and the "dot" indicates the *dot* (or *scalar* product) of two vectors. Differentiaing the dot product equation we have $\gamma'' \cdot \gamma' + \gamma' \cdot \gamma'' = 0$, so γ'' is perpendicular to γ' (or else is the zero vector).

The statement that γ'' is perpendicular to the *surface* says that γ is going as "straight" as it can in the surface, and that the surface is exerting no force which would make the curve bend away from its path. Such a curve is a geodesic. See the book by Do Carmo for a full explanation of why geodesics defined as above minimize the distance between nearby points.

Example 13: On a sphere, the parallels of latitude yield acceleration vectors in the plane of the parallel and perpendicular to the parallel (but not in general perpendicular to the surface), whereas any great circle yields acceleration vectors pointing toward the center of the sphere and hence perpendicular to both the great circle and to the surface. The great circles are geodesics, but the parallels (except for the equator) are not.

◆ Geodesics on a Surface of Revolution

Suppose that

$$x = f(u), \quad z = g(u)$$

is a parametrization by arc length u of a curve in the xz-plane. One consequence of this parametrization is that $(f'(u))^2 + (g'(u))^2 = 1$. Let's rotate the curve by an angle θ around the z-axis, to find the surface parametrized by

$$P(u,\theta) = \begin{bmatrix} f(u)\cos\theta \\ f(u)\sin\theta \\ g(u) \end{bmatrix}$$

Let's suppose that curves γ on the surface are parametrized by arc length s and, hence, these curves have

$$\gamma(s) = \begin{bmatrix} f(u(s))\cos\theta(s) \\ f(u(s))\sin\theta(s) \\ g(u(s)) \end{bmatrix}$$

and we need to differentiate this twice to find

$$\gamma'(s) = \begin{bmatrix} f'(u(s))u'(s)\cos\theta(s) - f(u(s))\sin\theta(s)\theta'(s) \\ f'(u(s))u'(s)\sin\theta(s) + f(u(s))\cos\theta(s)\theta'(s) \\ g'(u(s))u'(s) \end{bmatrix}$$

$$\gamma''(s) = u''\begin{bmatrix} f'(u)\cos\theta \\ f'(u)\sin\theta \\ g'(u) \end{bmatrix} + (u')^2\begin{bmatrix} f''(u)\cos\theta \\ f''(u)\sin\theta \\ g''(u) \end{bmatrix} + 2u'\theta'\begin{bmatrix} -f'(u)\sin\theta \\ f'(u)\cos\theta \\ 0 \end{bmatrix}$$
$$-(\theta')^2\begin{bmatrix} f(u)\cos\theta \\ f(u)\sin\theta \\ 0 \end{bmatrix} + \theta''\begin{bmatrix} -f(u)\sin\theta \\ f(u)\cos\theta \\ 0 \end{bmatrix} \tag{20}$$

This array is pretty terrifying, but the two equations

$$\gamma'' \cdot \frac{\partial P}{\partial u} = 0, \quad \text{and} \quad \gamma'' \cdot \frac{\partial P}{\partial \theta} = 0 \tag{21}$$

which express the fact that γ'' is perpendicular to the surface, give

$$u'' - (\theta')^2 f(u) f'(u) = 0 \quad \text{and} \quad 2u' f(u) f'(u)\theta' + \theta''(f(u))^2 = 0 \tag{22}$$

That γ'' is perpendicular to the surface if formulas (21) hold follows because the vectors $\partial P/\partial u$ and $\partial P/\partial\theta$ span the tangent plane to the surface at the point (θ, u). Formulas (22) follow from formulas (21), from the fact that $(f'(u))^2 + (g'(u))^2 = 1$, and from the formulas

$$\frac{\partial P}{\partial u} = \begin{bmatrix} f'(u)\cos\theta \\ f'(u)\sin\theta \\ g'(u) \end{bmatrix}, \qquad \frac{\partial P}{\partial \theta} = \begin{bmatrix} -f(u)\sin\theta \\ f(u)\cos\theta \\ 0 \end{bmatrix}$$

The quantity

$$M = (f(u))^2\theta' \tag{23}$$

is conserved along a trajectory of system (22) since

$$\frac{d}{ds}[(f(u))^2\theta'] = (f(u))^2\theta'' + 2u'\theta' f' f = 0$$

The integral M behaves like *angular momentum* (see Exploration 5 for the central force context that first gave rise to this notion).

Substituting $\theta' = M/(f(u))^2$ into the first ODE of equation(22) gives

$$u'' - \frac{M^2 f'(u)}{(f(u))^3} = 0 \tag{24}$$

Using equations (23) and (24), we obtain a system of ODEs for the geodesics on a surface of revolution:

$$\begin{aligned} \theta' &= \frac{M}{(f(u))^2} \\ u' &= w \\ w' &= \frac{M^2 f'(u)}{(f(u))^3} \end{aligned} \tag{25}$$

We recognize that ODE (24) is of the form of ODE (4), so

$$u'' = -\frac{d}{du}\frac{M^2}{(f(u))^2}$$

and we can analyze this ODE by the phase plane and conservation methods used earlier. Let us now specialize to the torus.

◆ Geodesics on a Torus

Rotate a circle of radius r about a line lying in the plane of the circle to obtain a torus. If R is the distance from the line to the center of the circle, then in the above equations we can set

$$\begin{aligned} x &= f(u) = R + r\cos u \\ z &= g(u) = r\sin u \end{aligned}$$

If we set $r = 1$, then we have $(f')^2 + (g')^2 = 1$ as required in the derivation of the geodesic ODEs. The system of geodesic ODEs (25) becomes

$$\begin{aligned} \theta' &= \frac{M}{(M+\cos u)^2} \\ u' &= w \\ w' &= -\frac{M^2 \sin u}{(R+\cos u)^3} \end{aligned} \tag{26}$$

where M is a constant. The variable u measures the angle up from the outer equator of the torus, and θ measures the angle around the outer equator from some fixed point. Figure 10.7 shows seventeen geodesics through the point $\theta_0 = 0,\ u_0 = 0$ with w_0 sweeping from -8 to 8. In Figure 10.7 and subsequent figures we take $R = 3$ and $M = 16$. Figure 10.8 shows the geodesic curves in the θu-plane (left graph) and in the uu'-plane (right). Note the four outlying geodesics that coil around the torus, repeatedly cutting both the outer $[u = 2n\pi]$ and the inner $[u = (2n+1)\pi]$ equators and periodically going through the hole of the donut. Twelve geodesics oscillate about the geodesic along the outer equator.

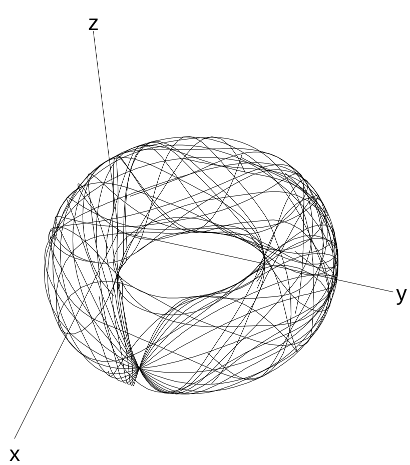


Figure 10.7: Seventeen geodesics through a point on the outer equator.

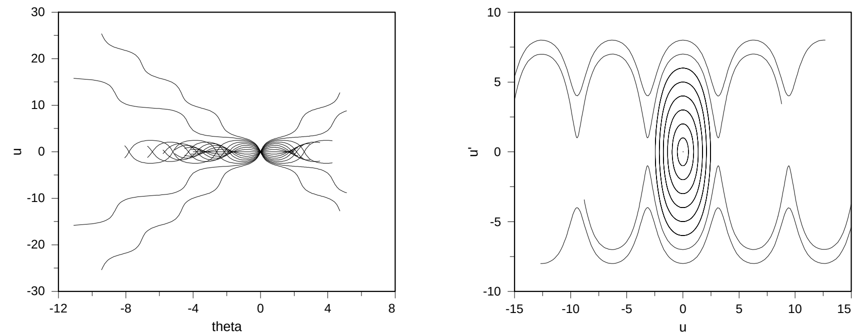


Figure 10.8: The seventeen geodesics of Figure 10.7 drawn in the θu-plane (left) and in the uu'-plane (right).

Figure 10.9 shows the outer and inner equatorial geodesics (the horizontal lines) in the θu-plane, as well as three curving geodesics starting at $\theta_0 = 0$, $u_0 = 0$. One oscillates about the outer equator six times in one revolution (i.e., as θ increases from 0 to 2π). The other two start with values of y_0 that take them up over the torus and near the inner equator. One of these geodesics turns back and slowly oscillates about the outer equator. The other starts with a slightly larger value of y_0, cuts across the inner equator, and slowly coils around the torus. This suggests that the inner equator ($u = \pi$) is a *separatrix* geodesic, dividing the geodesics into those that oscillate about the outer equator from those that coil around the torus. This separatrix is

☞ Try this and see how hard it is to stay on the inner equator.

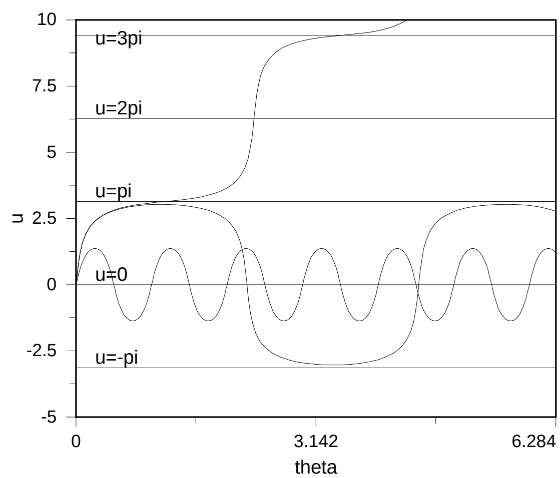


Figure 10.9: Equatorial geodesics (lines), a geodesic that rapidly oscillates around the outer equator, another that oscillates slowly around the outer equator, and a third that slowly coils around the torus.

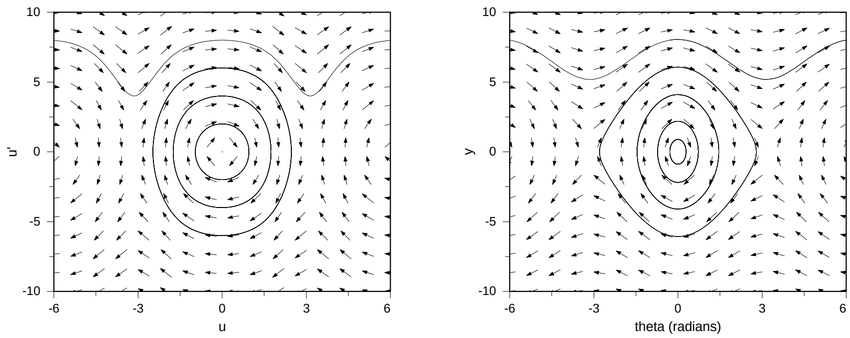


Figure 10.10: The graphs of the toroidal geodesics in the uu'-plane (left) look like the trajectories of an undamped, nonlinear pendulum (right).

unstable in the sense that if you start a geodesic near the inner equator (say at $\theta_0 = 3.14,\ y_0 = 0$) and solve the system (26), then the geodesic moves away from the separatrix.

Why do we call this geodesic model a "friend of the pendulum"? Take a look at the u' and w' ODEs in system (26). Note that if we delete the term "$\cos u$" from the denominator of the w' equation, then we obtain the system

$$\begin{aligned} u' &= w \\ w' &= -\frac{M^2}{R^3}\sin u \end{aligned} \tag{27}$$

which is precisely the system for an undamped, nonlinear pendulum with $g/L = M^2/R^3$. This fact suggests that geodesics of (26) plotted in the uu'-plane will look like trajectories of the pendulum system (27). Figure 10.10 compares the two sets of trajectories and shows how much alike they are. This illustrates a general principle (which, like most principles, has its exceptions): *If two systems of ODEs resemble one another, so will their trajectories.*

References

Arnold, V.I., *Ordinary Differential Equations* (1973: M.I.T.)

Note: Arnold's book is the classical text. Much of the considerable literature on modeling swings has been influenced by his description, which uses a nonautonomous length function, $L(t)$, instead of $L(\theta, \theta')$. In Sections 27.1 and 27.6, Arnold shows that if the child makes pumping motions at the right frequency at the bottom position of the swing, the motion eventually destabilizes, and the swing will start swinging without any push. This is correct (we have seen it done, though it requires pumping at an uncomfortably high frequency), but appears to us to be completely unrelated to how swings are actually pumped. Since the usual swing-pumping is done without reference to a clock, a proper model must certainly give an autonomous equation.

Do Carmo, M.P., *Differential Geometry of Curves and Surfaces* (1976: Prentice-Hall)

Halliday, D., Resnick, R., *Physics*, (v. I and II), 3rd ed. (1977-78: John Wiley & Sons, Inc.)

Hubbard, J.H., and West, B.H., *Differential Equations: A Dynamical Systems Approach, Part II: Higher Dimensional Systems*, (1995: Springer-Verlag) especially Ch. 6.5 and 8.1, which have more on conservation of energy.

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Stewart, I., *Does God Play Dice?* The Mathematics of Chaos (1989: Blackwell Publishers) Ch. 5 The One Way Pendulum.

Werkes, S., Rand, R., Ruina, A., *Modeling the Pumping of a Swing* C·ODE·E Newsletter, Winter/Spring ’97

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 10.1. Explorations of Basic Pendulum Equation

1. If the nonlinear pendulum ODE (3c) is approximated by the linear ODE (3a), how closely do the trajectories and the component curves of the two ODEs match up? Screen 1.2 in Module 10 will be a big help here.

2. What would motions of the system, $x' = y, \quad y' = -V(x)$, look like under different potential functions, such as $V(x) = x^4 - x^2$? What happens if a viscous damping term $-y$ is added to the second ODE of the system? Use graphical images like those in Figures 10.2 and 10.3 to guide your analysis. Use ODE Architect to draw trajectories in the xy-plane for both the undamped and damped case. Identify the equilibrium points in each case as saddles, centers, sinks, or sources. Plot the stable and the unstable saddle separatrices (if there are any) and identify the basin of attraction of each sink. [*Suggestion:* Use the Equilibrium feature of ODE Architect to locate the equilibrium points, calculate Jacobian matrices, find eigenvalues and eigenvectors, and so help to determine the nature of those points.]

3. Find all solutions of the undamped and linearized pendulum ODE,

$$\theta'' + (g/L)\theta = 0$$

Show that all solutions are periodic of period $2\pi\sqrt{L/g}$. Are all solutions of the corresponding nonlinear pendulum ODE, $\theta'' + (g/L)\sin\theta = 0$, periodic? If the latter ODE has periodic solutions, compare the periods with those of solutions of the linearized ODE that have the same initial conditions.

4. Use the sweep and the animate features of ODE Architect to make "movies" of the solution curves and the trajectories of the nonlinear pendulum ODE, $\theta'' + b\theta + \sin\theta = 0$, where $\theta_0 = 0$, $\theta_0' = 10$, and b is a nonnegative parameter. Interpret what you see in terms of the motions of a pendulum. In this regard, you may want to use the model-based pendulum animation feature of ODE Architect.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 10.2. Physical Variations for Child on a Swing

1. Module 10 and the text of this chapter describe a swing-pumping strategy where the swinger changes position only on the first half of the forward swing (i.e., where θ is negative but θ' is positive). Is this the strategy you would use to pump a swing? Try pumping a swing and then describe in words your most successful strategy.

2. Rebuild the model for the length function $L(\theta, \theta')$ of the "swing pendulum" to model your own pumping scenario. [*Suggestion:* Change the arguments of the arctan function used in Module 10 and the text of this chapter.] Use the ODE Architect to solve your set of ODEs. From plots of $t\theta$-curves and of $\theta\theta'$-trajectories, what do you conclude about the success of your modeling and your pumping strategy?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 10.3. Bifurcations

In these problems you will study the bifurcations in the swing-pumping model of Module 10 and this chapter as the viscous damping constant ε or the incremental pendulum length ΔL is changed.

1. There is a Hopf bifurcation for the small-amplitude repelling limit cycle at $\varepsilon = 0$. for the swing-pumping system (18) and (19) Plot lots of trajectories near the origin $\theta = 0, \quad y = 0$ for values of ε above and below $\varepsilon = 0$ and describe what you see. What does the ODE Architect equilibrium feature tell you about the nature of the equilibrium point at the origin if $\varepsilon < 0$? If $\varepsilon = 0$? If $\varepsilon > 0$?

2. Now sweep ΔL through a series of values and watch what happens to the large-amplitude attracting limit cycle. At a certain value of ΔL you will see a sudden change (called a *homoclinic, saddle-connection bifurcation*). What is this value of ΔL? Plot lots of trajectories for various values of ΔL and describe what you see.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 10.4. Geodesics on a Torus

The basic initial value problem for a geodesic starting on the outer equator of a torus is

$$\begin{aligned} \theta' &= \frac{M}{(M+\cos u)^2} \\ u'' &= -\frac{M^2 \sin u}{(R+\cos u)^3} \\ u(0) = 0, &\quad u'(0) = \alpha, \quad \theta(0) = 0 \end{aligned} \tag{28}$$

where M is a constant.

1. Make up your own "pretty pictures" of geodesic sprays on the surface of the torus by varying $u'(0)$. Explain what each geodesic is doing on the torus. If two geodesics through $u_0 = 0$, $\theta_0 = 0$ intersect at another point, which provides the shortest path between two points? Is every "meridian", $\theta =$ const., a geodesic? Is every "parallel", $u =$ const., a geodesic?

2. Repeat Problem 1 at other initial points on the torus, including a point on the inner equator.

3. Explore different values for R (between 2 and 5) for the torus—what does it mean for the solutions of the ODEs for the geodesics? To what extent does the ugly denominator in the ODEs mess up the similarity to the nonlinear pendulum equation?

Answer by discussing effects on θu-phase portraits.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 10.5. The Central Force and Kepler's Laws

An object at position $\mathbf{r}(t)$ (relative to a fixed coordinate frame) is moving under a central force if the force points toward or away from the origin, with a magnitude which depends only on the distance r from the origin. This is modeled by the differential equation $\mathbf{r}'' = f(r)\,\mathbf{r}$, where we will take $\mathbf{r}(t)$ to be a vector moving in a fixed plane.

Example 14: (*Newton's law of gravitation*) This, as applied to a planet and the sun, is perhaps the most famous differential equation of all of science. Newton's law describes the position of the planet by the differential equation

$$\mathbf{r}'' = -\frac{AG}{r^3}\mathbf{r},$$

where $\mathbf{r}$ is the vector from the center of gravity of the two bodies (located, for all practical purposes, at the sun) to the planet, G is the universal gravitational constant, and $A = M^3/(m+M)^2$, where m is the mass of the planet (so for all practical purposes, A is the mass of the sun).

1. Newton's law of gravitation is often called the "inverse square law," not the "inverse cube law." Explain.

2. The way to analyze a central force problem is to write it in polar coordinates, where

☞ Another way to write the vector $\mathbf{r}$ is $\mathbf{r} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta$, where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along the positive x- and y-axes, respectively.

$$\mathbf{r} = r[\cos\theta, \sin\theta]$$
$$\mathbf{r}' = r'[\cos\theta, \sin\theta] + r\theta'[-\sin\theta, \cos\theta]$$
$$\mathbf{r}'' = (r'' - r(\theta')^2)[\cos\theta, \sin\theta] + (2r'\theta' + r\theta'')[-\sin\theta, \cos\theta]$$

Show that the central force equation $\mathbf{r}'' = f(r)\mathbf{r}$ yields

$$2r'\theta' + r\theta'' = 0 \tag{29}$$

and

$$r'' - r(\theta')^2 = rf(r) \tag{30}$$

3. Show that the quantity $M = r^2\theta'$ is constant as a function of time during a motion in a central force system, using equation (29).

The quantity M (now called the angular momentum of the motion) was singled out centuries ago as a quantity of interest precisely because of the derivation above. You should see that the constancy of M is equivalent to *Kepler's second law*: the vector **r** sweeps out equal areas in equal times.

4. Substitute $\theta' = M/r^2$ into equation (30) and show that, for each value of the particular central force $f(r)$ and each angular momentum M, the resulting differential equation is of the expected form.

5. Specialize to Newton's inverse square law with $k = AG$ and show that the resulting system becomes

$$r'' = -\frac{k}{r^2} + \frac{M^2}{r^3}$$

or the system

$$\begin{aligned} r' &= y \\ y' &= -\frac{k}{r^2} + \frac{M^2}{r^3} \end{aligned}$$

Make a drawing of the phase plane for this system, and analyze this drawing using the conserved quantity K, where

$$K(r, y) = \frac{y^2}{2} + \frac{k}{r} - \frac{M^2}{2r^2}$$

K is evidently defined only for $r > 0$, and K has a unique minimum, so the level curves of K are simple closed curves for $K << 0$, corresponding to the elliptic orbits of *Kepler's first law*, an unbounded level curve when $K = 0$ corresponding to a parabolic orbit, and other unbounded curves for $K > 0$ which correspond to hyperbolic orbits. (For discussion of these three cases and their relation to conic sections, see Hubbard and West, Part II, Section 6.7 pp. 43–47.)