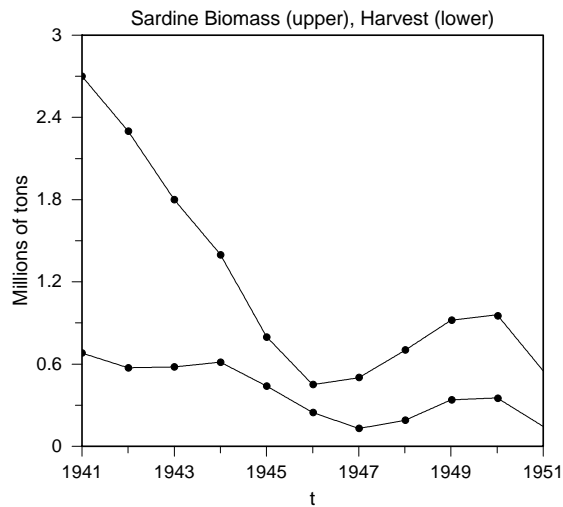


1

CHAPTER

Modeling with the ODE Architect



Pacific sardine population and harvest.


Overview In the two decades from 1932 to 1951, the Pacific sardine fishery completely collapsed. In this chapter you will learn to use the ODE Architect to construct a mathematical model which describes this event rather well. This will have two purposes: it will familiarize you with the menus and features of the ODE Architect, and it will acquaint you with the principles of mathematical modeling.

First we'll construct a model for the Pacific sardine population during the years 1930–1950 as if it were unharvested. Then we will focus on the harvesting that actually took place and see how it contributed greatly to the destruction of the sardine population.

Key words Modeling; Pacific sardine; population model; initial conditions; exponential growth; carrying capacity; logistic equation; harvesting

See also Chapter 9 for more on population models.

◆ Building a Model of the Pacific Sardine Population

 In this chapter we will build a model for the sardine fishery in California and also introduce features of the ODE Architect Tool for solving differential equations. Consult the *User's Guide* for a full description of all features of the tool.

Step 1: State the problem and its context

The Pacific sardine (*Sardinops sagax caerulea*) has historically experienced long-range cycles of abundance and depletion off the West Coast of California. It was during one of the abundant periods, 1920 through 1951, that a huge sardine fishing and canning industry developed. The total catch for the California coastline reached a peak of 726,124 tons during the 1936–37 season (June through the following May). The Pacific sardine population then began a serious decline during the 1940s until, as one estimate has it, by 1959 the sardine biomass was 5% (0.2 million tons) of the 1934 level (4 million tons). (The *biomass* is the amount of a particular organism in its habitat.) There is general agreement that heavy harvesting played a role in the decimation of the Pacific sardine during that period. The fishing industry had a serious decline after the 1950–51 season: increasing numbers of fishermen went bankrupt or moved to other fisheries. Undoubtedly the canneries were also affected.

After 50 years of fishing for the Pacific sardine, a moratorium was imposed by the California legislature in 1967. The Pacific sardine seems to be making a comeback as of the mid-1980s, though the numbers are not yet near the abundant levels of the 1930s.


Here are the goals of your model:

1. Determine the extent to which the precipitous decline of the Pacific sardine population was due to over-harvesting from 1941 to 1951.
2. Ascertain an optimal harvest rate that would stabilize and sustain the sardine population during that time period.

Step 2: Identify and assign variables

Assigning the variables in a mathematical model is a skill that requires some practice. Doing some background reading and studying the context of the problem and the problem statement helps to clarify which are the most important features of the system you wish to model.

It turns out that there tend to be long-range cycles of Pacific sardine abundance and scarcity. These cycles are not yet completely understood, but it is certain that factors such as ocean temperature, nutrient upwelling from deep waters, currents that aid fish migration, predator populations of larger fish and sea lions, and, of course, fishing, play a vital role in the cycles. In this model we will focus on a period when the harvesting of the sardine was very heavy. Due to the large magnitude of the harvesting, its effect is dominant for the period of time we will model, 1941–1951, so we will neglect the other factors. That the other factors still operate on the population is evidenced by the difficulty in getting the model to match the data perfectly. Nevertheless, you'll see how modeling, while not always explaining every aspect, provides insight into the dynamics of an otherwise very complex biological relationship.

 Problem statement.

Given the information we have at this point we need the following variables and parameters in our model:

☞ Sardine biomass is a state variable, the other quantities are parameters.

1. Sardine biomass (in units of million tons)
2. Growth rate for the Pacific sardine (in units of million tons/year)
3. Maximum biomass, or *carrying capacity* (in units of million tons)
4. Sardine harvesting (in units of million tons/year)

Note that we opted to define sardine biomass in million tons, rather than numbers of fish, to be consistent with the data and estimates used.

It's good practice to introduce as few parameters as necessary into a model at first. Additional parameters can be added if they are needed to improve the accuracy of the model. The model may be refined until the desired level of accuracy is achieved.

With the variables and parameters identified, the next step is to construct an equation for the rate of change of the state variable in terms of the state variable itself, the model's parameters, and possibly also time. This equation is known as a differential equation (abbreviated ODE). When an ODE is entered into ODE Architect along with an initial value of the state variable, the Architect Tool displays a graphical representation of the solution.

☞ Graphical representation of the problem.

Figure 1.1 represents the estimated Pacific sardine biomass and harvest during the period 1941–1951. Note again the use of sardine biomass in million tons, rather than numbers of fish.

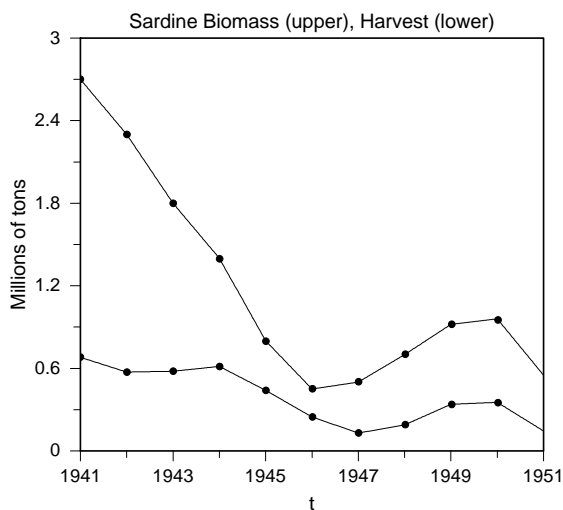


Figure 1.1: Sardine biomass and harvest.

Our first task is to use ODE Architect to build a mathematical model to simulate the growth and decline of the Pacific sardine biomass without harvesting. Then we can explore the impact of harvesting on that biomass.

Click on the spinning orb to go directly to the Architect Tool.

Place the ODE Architect CD-ROM in your computer and start the ODE Architect Tool. Four quadrants will be displayed on the screen (see Figure 1.2). The upper left quadrant is the equation quadrant; it should be empty now. The two right-hand quadrants should be empty. These are plot quadrants that will display 2- or 3-dimensional plots when you solve differential equations. The lower left quadrant currently shows the initial conditions (IC) display. Notice that it is selectable using the four tabs (IC, Sweep, Solver, Equilibrium) on the lower edge of the quadrant. For now leave IC selected.

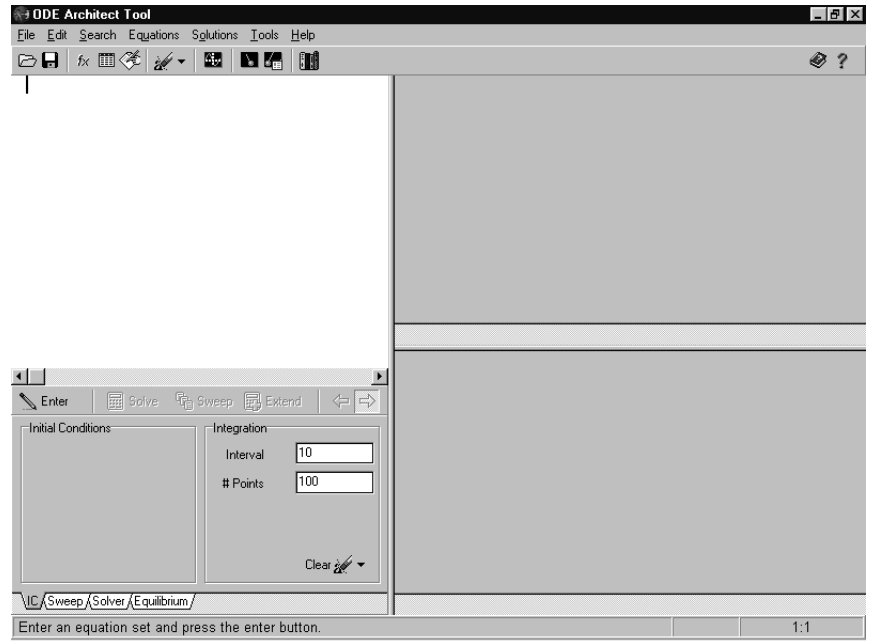


Figure 1.2: ODE Architect tool screen

Step 3: State the relationships that govern the variables

We begin by **simulating** the unchecked growth (**no harvesting**) of the Pacific sardine population, which we will designate as the state variable *sardine*. Basic biology suggests that **it is reasonable to assume that the rate of biomass growth (i.e., the derivative $d(\text{sardine})/dt$) at a given time t is proportional to the quantity of sardines (the size of the biomass) present at that time t .**

Step 4: Translate the laws into equations

Since $\text{sardine}'$ is a common notation for the derivative (rate of change) of *sardine*, we can write

$$\text{sardine}' = r * \text{sardine}$$

where r is a proportionality factor that we will refer to as the **growth rate factor**.

Fishery and biomass data collected over the period 1932–1958 indicate that the Pacific sardine population has had a volatile history. The Pacific sardine biomass, if not manipulated or constrained, can grow at a rate of between 10% and 40% per year. We will assume a moderate position and set the growth rate factor at $r = 0.20$. A modeler often has to make assumptions and guess parameter values to get a model started; you can refine the assumptions later.

Step 5: Solve the resulting differential equations

➞ Entering differential equations.

Point and click the cursor in the equation quadrant and type in

$$\text{sardine}' = r * \text{sardine} \quad (1)$$

using an apostrophe for the prime, and an asterisk for multiplication; hit **Return** (or **Enter**) and assign the value 0.20 to the parameter r by typing in

$$r = 0.20$$

Now click the cursor on the box marked **Enter** just below the equation quadrant. Notice that this causes scales to appear in the two plot quadrants.

➞ Setting initial conditions.

Now go to the lower left quadrant to set the initial conditions. Double click in the appropriate box to select a variable; then type in the new value. Set t (time) to start at 1930 and set **sardine** to be 1 (unit of million tons). We'll go back later and put in a more realistic estimate for **sardine**. In the **Integration** panel, set the solve time to 20 by inserting the number 20 in the **Interval** box. Leave the default value of 100 in the **# Points** box.

➞ Setting the time interval.

Click the **Solve** icon and notice that the right arrow is automatically selected. Your screen will look something like this (Figure 1.3):

➞ Starting the solution.

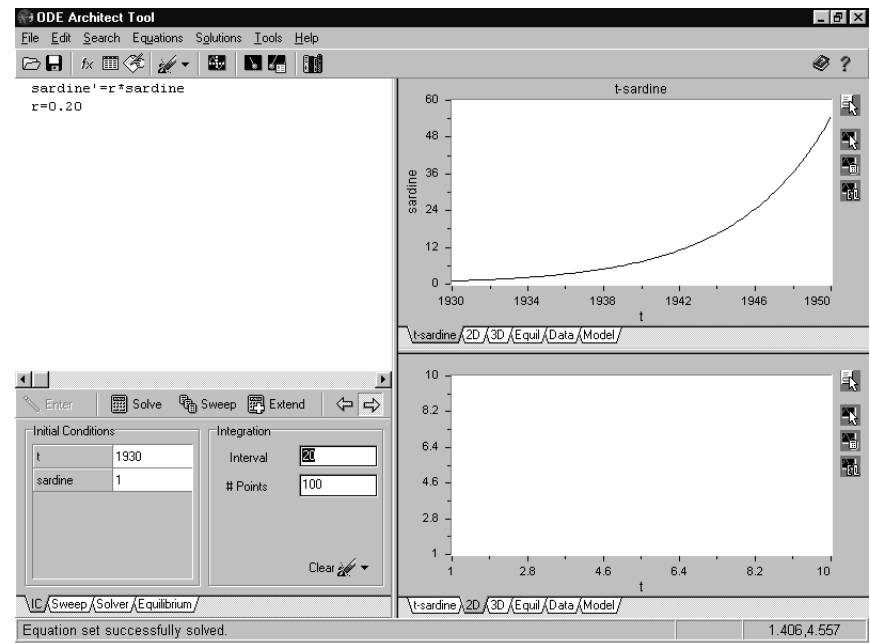


Figure 1.3: Exponentially growing sardine population.



Unbounded growth.

Step 6: Interpret and test the solutions in context

There is now a classical exponential growth curve in the upper right quadrant. This implies that the sardine biomass grows without bound, which can't be true as there is not enough room on the planet! The exponential growth must be limited by factors like available food supplies, disease, predators, and so on; therefore we have to modify our model to reflect this fact. We learned earlier that the sardine biomass has been as large as 4 million tons, but we don't truly know the maximum sustainable biomass (*carrying capacity*), so to start let's assume a carrying capacity of 6 million tons. We can refine this guess later if we have trouble fitting the model to actual data. As we said before, it's not uncommon to have to make informed guesses for values that are not known or available. Then the values can perhaps be deduced by "fine tuning" (refining) the model in subsequent iterations to conform to reality.



Introduce a carrying capacity.

Step 7: Refine the model to predict the empirical data

The following differential equation is sometimes used to exhibit maximum carrying capacity behavior in a population:

$$sardine' = r(capacity - sardine)$$

This equation says that the growth rate at any time is proportional to the "room to grow" factor $capacity - sardine$. Now click the cursor in the equations region. Using our assumed growth rate constant of $r = 0.20$ per year and a carrying capacity of 6 million tons, we modify the sardine growth to be

$$sardine' = r * (6 - sardine) \quad (2)$$

Before entering the new sardine ODE, clear the graphics screens by clicking on **Clear** at the lower left and choosing **Clear All Runs**. A "confirmation" window will pop up; click on **Yes**. Now click in the equation region, make the corrections to your equation, and then click on the box marked **Enter**. Click on the **Solve** icon and notice in the plot window that the graph of the sardine biomass climbs and levels out at the assumed carrying capacity of 6 million tons.

✓ "Check" your understanding by comparing this curve with the earlier one and notice some significant differences: (1.) The first curve was concave up; this one is concave down. Why is that significant? (2.) The first curve grew without bound and had no asymptotes; the second curve has a horizontal asymptote. Explain why.

Examine the two graphs carefully at early values of t , say the first five years. Recall that the slope of a line tangent to the solution curve is the growth rate of the biomass at that time. How do the two curves differ in this regard? When is the rate of change of the biomass the greatest? Is it realistic for a biomass to exhibit its greatest rate of increase when the population is smallest? The answer to these questions is not as simple as you might think. For

many biological populations, rate of change is proportional to the size of the population. The solution of ODE (1) exhibits this proportionality but it is unconstrained and so it's not useful over its whole domain. The solution curve of ODE (2) doesn't exhibit the proportional growth property. Which of the two is most appropriate for the Pacific sardine? We'll come back to that question after a little exploration with ODE (2).

➡ Exceed carrying capacity.

Now let's see if ODE (2) will allow us to exceed the carrying capacity for any length of time. Change the initial biomass to 12 million tons of sardines in the IC window. Click on the **Solve** icon. Notice that the vertical scale in the graph changes to accommodate the revised values and that both the old (lower) and the new (upper) curves are displayed on the graph. Observe what happened to the "overstocked" sardine population. How does this compare to what happened when the initial sardine biomass was 1 million tons? If you examine the two plots closely, you'll see that both plots stabilize at a level of about 6 million tons (see Figure 1.4).

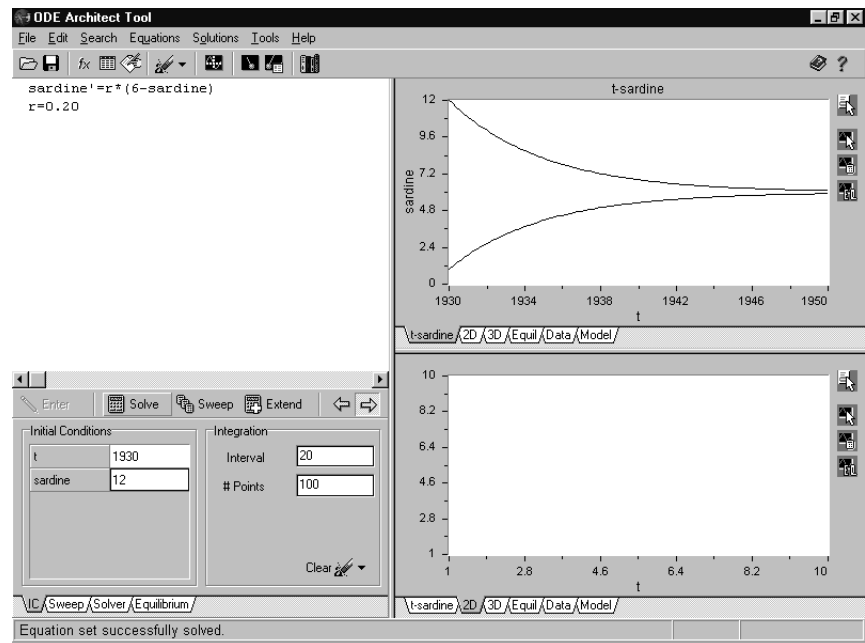


Figure 1.4: Sardine populations approach carrying capacity.

➡ To set scaling of axes.

Sometimes it's advantageous to change the scales of the axes to make graphs easier to read and interpolate, so we'd like to show you how to reset the vertical and horizontal scales. (The default setting for scales for ODE Architect is **Auto Scale**.) Select the upper right graph by placing the cursor arrow on the graph and clicking the *right-most* mouse button (or click on the icon at the upper right corner of the graph). You will see various plot window options presented. Select **Scales** from the resulting dropdown menu using the *left-most* mouse button. Click on **Auto Scale** to toggle it off. (The check in the

box will disappear.) To change other values, double click in the box to select the value and just type to make a change. On the **X-Scale** menu set **Minimum** = 1930; **Maximum** = 1950; **Number of Ticks** = 10; and **Label every** = 2. (Adjust the number of ticks by clicking on the down arrow and selecting, or by double clicking the box and typing in the new value.) Make sure **Linear** is selected (not **Log**). Now select the **Y-Scale** menu (at the top): click the **Auto Scale** to toggle it off; set **Minimum** = 0; **Maximum** = 12; **Number of Ticks** = 10; and **Label every** = 2, and check that the **Linear** button is selected. Your screen should have a window that looks like Figure 1.5:

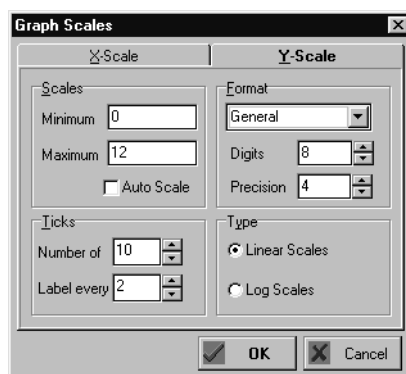


Figure 1.5: Plot scales window.

☞ You must click the **OK** button to enter changes.

☞ Sweeping a variable.

☞ If the **# Points** is set to n , you'll get n overlaid graphs.

Click on **OK** to cause the graph to be rescaled. (In this particular case, it turns out that the scale did not change from the automatically selected value.)

Now click on the **Clear** box in the **Integration** panel and choose **Clear All Runs**. What has changed? Next click on the **Solve** icon again. Notice that you got only the most recent curve (carrying capacity exceeded); you cleared the previous solution.

We can extend the ease of making comparisons by sweeping through several possible initial values for *sardine* and displaying them all on one graph. Click on the **Clear** box and choose **Clear All Runs**. Now notice the **Sweep** tab beneath the **Initial Conditions** panel; click on **Sweep**. Click on **Single** for type of sweep. We will choose **Sweep 1** to be a sweep (multi-plot) of various initial values of *sardine*. In the **Sweep 1** box, click on the down arrow and select *sardine*. Set **Start** = 1; **Stop** = 7; and **# Points** = 3. Now click on the **Sweep** box next to the **Solve** icon (not the **Sweep** tab). (Figure 1.6).

Notice that ODE Architect makes several runs. Notice also that the initial value for *sardine* located in the IC window was ignored and the values we entered in the sweep conditions were used instead.

To better see these results, let's rescale the vertical axis (**Y-Scale**) to **Minimum** = 1 and **Maximum** = 7. Look back at page 7 if you do not recall how to do this. Figure 1.7 shows that multiple runs are easily comparable in this format. Which initial value for *sardine* created the most stable or flattest curve? Does the population always stabilize around the same biomass?

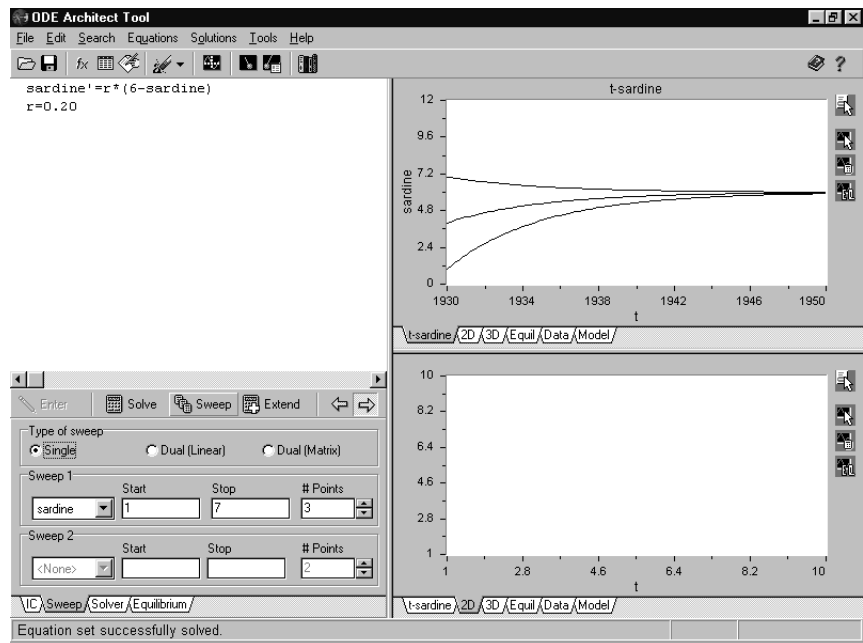
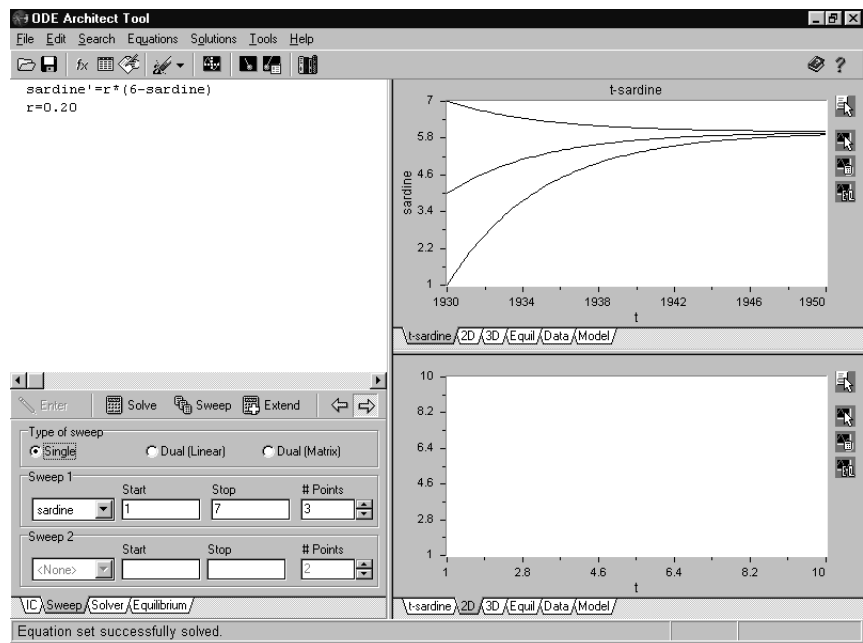
Figure 1.6: Setting up the sweep for *sardine*.

Figure 1.7: Sweeping and solving gives plots with initial sardine tonnages of 1, 4, 7 (in millions).

Do the resulting curves accurately represent the growth you'd expect over the whole range of values for t ? Growth is usually proportional to population size when well below the carrying capacity. However, when you look at your graph notice that for small populations of sardines, the growth rate is rather steep. As the sardine population approaches the carrying capacity the biomass should level off, which the preceding curves do reflect.

Now we'll examine the properties of the model we created in ODE (2). The growth rate is proportional to $(\text{capacity} - \text{sardine})$ and so for small sardine biomass, the biomass grows at a nearly constant rate. Near the carrying capacity, the factor $(\text{capacity} - \text{sardine})$ causes a leveling off (see Figure 1.7): the factor forces growth to be proportional to the distance from capacity.


◆ The Logistic Equation

Combining the elements of the proportional growth model given by ODE (1) and the restricted growth model given by ODE (2) leads to what is called the *logistic equation for growth* (or the *Verhulst equation*, after the nineteenth century Belgian mathematician and biologist P. F. Verhulst):

$$\text{sardine}' = r \text{sardine} \frac{6 - \text{sardine}}{6} \quad (3)$$

Notice that for values of *sardine* very near zero, the factor $r * \text{sardine}$ dominates the computation, causing the behavior to approximate exponential growth. This is because the factor $(6 - \text{sardine})/6$ has a value very near 1. For values of *sardine* near 6 (the carrying capacity), the factor $(6 - \text{sardine})/6$ is near zero, and so growth slows to approach zero. Therefore we can expect exponential growth for small biomass with growth tapering off as the biomass approaches carrying capacity. Let's see if this refinement improves the model.


Click on the **IC** tab to clear the graph and enter a new equation. After clicking on the **Clear** box, and choosing **Clear All Runs**, click in the equations quadrant and modify the growth ODE to read:

 Changing the equation.

$$\text{sardine}' = r * \text{sardine} * (6 - \text{sardine})/6 \quad (4)$$

Don't forget to click the box labeled **Enter**. Reset the initial sardine biomass to **1**. Finally, click on the **Solve** icon. Your screen should look something like Figure 1.8.

Notice that the graph now displays a mathematical representation more like what we expect of the sardine biomass over the long term. It is an elongated S-shaped curve with slow growth for small biomass, maximum growth near the midrange, and slow growth near the carrying capacity.

 Try various initial values. As before, use a **Single** sweep.

Use the sweep feature now to see how the logistic growth curve responds for various initial conditions for the variable *sardine*. **Sweep 1 sardine; Start = 1; Stop = 7; # points = 4**. Click on **Sweep**. Figure 1.9 shows the four solution curves.

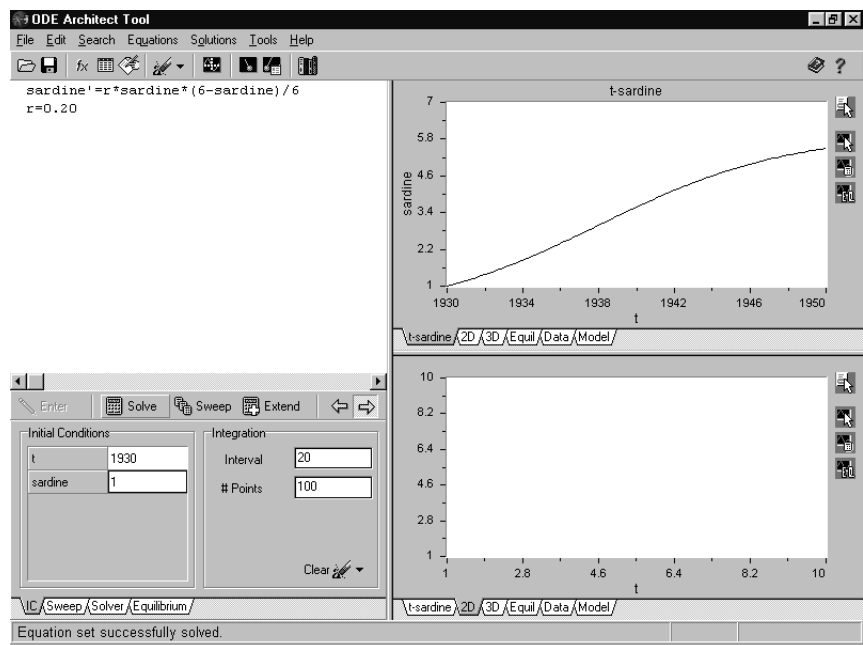


Figure 1.8: A logistic growth curve.

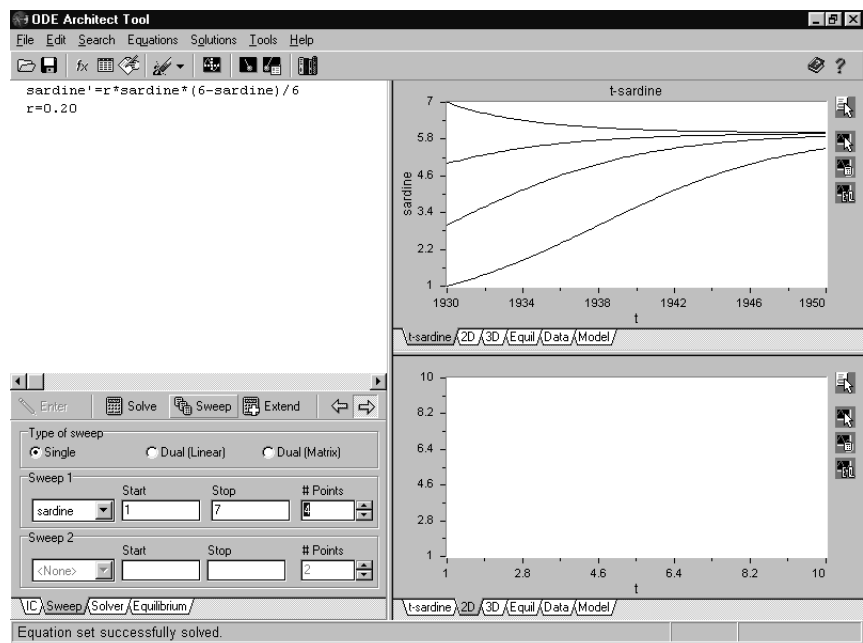


Figure 1.9: Four logistic solution curves.

✓ Does the model respond to your initial conditions in a reasonable manner? Do you think that this is a good population model to use for modeling the biomass of the Pacific sardine?

Let's now use the model given in equation (3) to explore the harvesting that took place in the years 1941–1951.


◆ Introducing Harvesting via Landing Data

In Figure 1.1 you saw a graph of the Pacific sardine harvest and the resulting biomass decline during the years 1941–51. We have not yet taken into account this harvest (or *landing*) data in our model; so our model does not yet reflect the collapse of the sardine fishery that occurred. We'll now incorporate the landing data into our model in the form of a lookup table.

The tutorial in Module 1 provides you with landing data for the Pacific sardine over the time period 1941–1951 in the form of a table with 11 rows and 2 columns (tutorial steps 13 and 14). This data can be entered as a *lookup table* named HTABLE by following these directions (which also appear in the tutorial):

- Start by clicking on the **Equations** entry on the menu bar and choosing **Lookup Tables** to display the lookup table manager window.
- Double click on **<Create New Table>** to display the new table window. Enter the name HTABLE, and specify 11 rows and 2 columns in the appropriate boxes. Then click the **OK** button. An array of empty cells will appear with 11 rows and 2 columns.
- Enter the data (provided in the tutorial window) in the array by clicking on cell [1, 1] to start. When all of the data is entered, click the **OK** button.
- Close the lookup table manager window.


Now you have a lookup table called HTABLE.

 Defining the harvest.

Go to the equation quadrant and, on a new line, add the following

$$harvest = \text{lookupval}(\text{HTABLE}, 1, t, 2)$$

Be sure to click the **Enter** box. The value returned by `lookupval` is the data in column 2 of HTABLE corresponding to the t -value of the data in column 1. (This value is computed by linear interpolation.)

 Using the 2D custom tab.

Let's now look at this harvest data. Since *harvest* is not an ODE state variable, the Architect does not automatically generate a plot tab; we will have to make it by hand. Click on the **2D** tab at the lower right to select what we want to plot on each of the two axes. Place the cursor on the lower right plot quadrant, and after clicking the *right-most* mouse button, select **Edit** with the *left-most* mouse button. Leave the **X-Axis** variable set to t . For the **Y-Axis**

click on the down arrow after **1. <None>** and select **harvest**. Now click on **Titles** at the top of the edit window and type in the **Graph Title** box: Harvest. In the **X-axis Title** box type: Year; and in the **Y-axis Title** box: Harvest. See Figure 1.10. Click **OK**. Using the *right-most* mouse button again on the lower graph, select **Scales**. Set the **X-Scale** as follows: deselect **Auto Scale**; set **Minimum** = 1940; **Maximum** = 1955; **Number of Ticks** = 3; **Label every** = 2. Select the **Y-Scale**, deselect **Auto Scale**, and set: **Minimum** = 0; **Maximum** = 1; **Number of Ticks** = 5; and **Label every** = 1. Click **OK**.

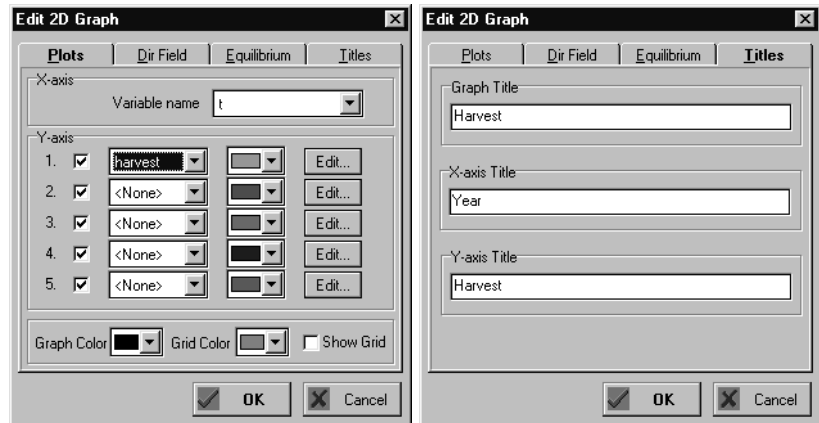


Figure 1.10: **Plots** and **Titles** panels for 2D tab.

Click on the **Solve** icon. Notice that a graphical representation of the Pacific sardine landings appears in the lower graph but the upper graph has not been affected. That's because we have not included harvest (landings) in the sardine model yet. (Note: the two graphs have different vertical axis scales.)

✎ Including *harvest* in the sardine model.

After clicking on **Clear** and choosing **Clear All Runs**, go to the equation quadrant and modify the *sardine* ODE as follows:

$$sardine' = r * sardine * (6 - sardine) / 6 - harvest \quad (5)$$

You may have to scroll the equations quadrant (on some computers) in order to see the whole equation. (You can also move the dividing line between the right and left quadrants, at the slight expense of the graphing resolution.) Click the **Enter** box. Before we run the model, we must change the initial conditions to reflect the reality of the Pacific sardine population at that time. In the literature, the most reliable data for the Pacific sardine biomass starts in 1941. Thus set the **IC** for *t* to 1941 and the **IC** for *sardine* to 2.71. Reset **Interval** to 10. Now click the **Solve** icon and note the results. For best viewing of the top right graph window choose the **X-Scale**; deselect **Auto Scale**; set

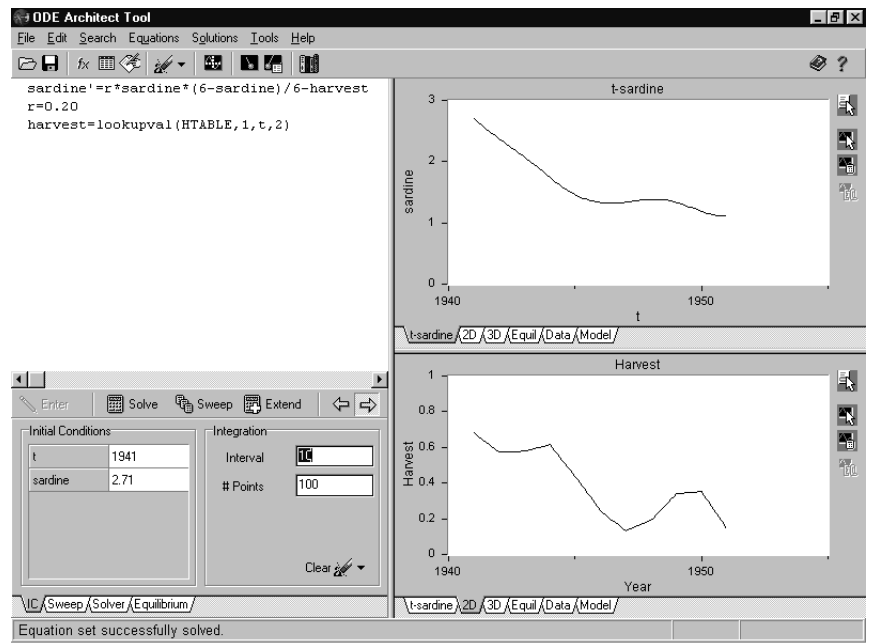


Figure 1.11: Model sardine biomass (upper), actual sardine biomass (lower).

Minimum = 1940; Maximum = 1955; Number of Ticks = 3; Label every = 2. Rescale the Y-Scale axis to: Minimum = 0; Maximum = 3; Number of Ticks = 3; Label every = 1. See Figure 1.11 for the graphs.

☞ Fishery collapse.

How do these model results compare with the expected behavior at the beginning of the chapter? While the overall behavior is captured in general terms by the model, it is unusual to have a model match the estimated data exactly.

✓ What are your thoughts about the model as it relates to historical behavior? Explain any discrepancies.

Step 8: Interpret the implications of the model

☞ Analysis.

It is now clear that while over-exploitation of the sardine landings was not the sole factor, it played a very large role in the collapse of the California fishery in the early 1950s. Since we now have a functioning model of that ten-year period in time, you have the amazing power to use your computer to revise history and attempt to save the fishing industry. What limit on the landings would have allowed a sizable sardine harvest¹ but not a collapse of the fishery?

¹Historical note: A limit to the total catch of sardines at between 200,000 and 300,000 tons was recommended as early as 1929, and repeatedly over the next several years.

Now it is your turn to examine some options and try some alternate scenarios in the Explorations that follow.

◆ How to Model in Eight Steps

Modeling a situation mathematically involves many ideas and activities, but modeling is not always straightforward. There are many times when you may be puzzled, confused, and frustrated and you must retrace or rethink the steps involved. We summarize the steps in an order that allows for easy reference, but keep in mind the need to retreat, reassess, and redefine your thinking.

1. State the problem and its context.
2. Identify and assign variables.
3. State the laws that govern the relationships between the variables.
4. Translate the laws into equations.
5. Solve the resulting equations.
6. Interpret and test the solutions in the context of the natural environment.
7. Refine the model until it predicts the empirical data.
8. Interpret the implications of the model.

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Exploration 1.1. Constant Harvesting of a Biomass

1. *No harvesting.*

Let's examine the rate of growth (the derivative) of the sardine biomass using the logistic model of ODE (3). To do this we'll look at the values of $sardine'$ as a function of sardine biomass size. Go to the equations quadrant and type in the ODE

$$sardine' = r * sardine * (6 - sardine)/6$$
$$r = 0.20$$

Click the **Enter** box.

To create a plot of $sardine$ vs. $sardine'$, select the **2D** tab (if necessary), place the cursor over the lower right graph, press the *right-most* mouse button, and select **Edit**. For the **X-Axis** use the down arrow to select $sardine$. For **Y-Axis 1**, select $sardine'$. Click now on the **Titles** tab at the top of the edit window. Type Rate of Growth vs. Biomass as the Graph Title, Sardine as the X-axis label and Sardine' as the Y-axis label. Click **OK**. Place the cursor over the lower right graph again, press the *right-most* mouse button and select **Auto Scales: Both** (if necessary). Next set the **IC** for $sardine$ to 1. Click **Clear** and select **Clear All Runs** (if necessary), then click the **Solve** icon.

The top graph shows (by default) $sardine$ vs. time. Notice in the lower graph that the sardine growth rate, $sardine'$, is maximized somewhere near a midsized sardine population of about 3 million tons. Rescale the Y-axis of the top graph (if necessary) to **Minimum** = 0; **Maximum** = 6; **Number of Ticks** = 6. Verify that the sardine biomass grows at the rate of approximately 10% to 40% per year, depending upon the size of the biomass.


2. Constant harvesting.

Let's analyze the effect of constant harvesting on the logistic sardine population of Problem 1. Since the sardine biomass was 2.71 million tons in 1941, reset the *sardine* IC to 2.71 and keep the *sardine* vertical scale set on the range 0 to 6 and re-solve to observe the relative stabilization of the population.

Now insert a constant harvesting term in the model by modifying the ODE in the equation quadrant to read

$$\text{sardine}' = r * \text{sardine} * (6 - \text{sardine}) / 6 - \text{harvest}$$

Try a harvest value that is slightly less than the biomass growth amount for 2.71 million tons by setting a constant harvest in the equation quadrant. For example you could try $\text{harvest} = 0.28$ (280,000 tons per year) and solve the model. (Be sure to click on the **Enter** box first.)

 Clearing a 2D custom graph. The same procedure clears a 3D custom graph.

Now click on the **2D** tab in the lower graph quadrant. Clear the graph in that quadrant by setting all axes to **<None>** in the **Plots** tab of the **Edit** box, then going to the **Titles** tab and deleting all titles. For the upper right graph you can set up and run a sweep of *harvest* over the values 0.1 (100,000 tons/year) to 0.7 (700,000 tons/year) using 7 points in the sweep. Describe the biomass behavior for harvest levels of 0.1; 0.3; 0.5; 0.7. From your exploration, determine what constant harvest amount provides a large harvest yet does not jeopardize the long-term viability of the Pacific sardine population. Did the harvest levels suggested by fishery researchers stand up?

3. How does the IC affect the optimum harvest level?

Is the optimum harvest level that you determined in Problem 2 affected by the initial biomass of the sardine in 1941? Try some different values for the IC and explain what you learn about the relationship between initial biomass and the optimum constant harvest amount.

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 1.2. Constant Effort Harvesting

1. *Using a constant effort harvesting function.*

Another model for harvesting is to land a certain percentage of the existing biomass each year. This is called *constant effort harvesting*. Introduce constant effort harvesting into ODE (5) by setting

$$\text{harvest} = 0.25 * \text{sardine}$$

to harvest 25% of the sardine population each year. Try a run. What happens?

Go back and revise the harvest function to

$$\text{harvest} = k * \text{sardine}$$

and sweep through several values of your choosing for the harvest percentage k . Summarize your results. What is the optimum harvest percentage?

2. *How does the IC affect the optimum harvest percentage?*

Run some experiments to determine if the optimum harvest percentage you select in Problem 1 is sensitive to the initial biomass of the sardine in 1941. Explain your results. How do your results compare to the results of Problem 3 in Exploration 1.1?

Answer questions in the space provided, or on attached sheets with carefully labeled graphs. A notepad report using the Architect is OK, too.

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Exploration 1.3. Investigating a harvesting function

1. *A unifying harvest strategy.*

We can combine the strategies used in Explorations 1.1 (Problem 2) and 1.2 (Problem 1) by using a function that approximates each strategy at the appropriate time: proportional harvest for small sardine biomass and constant harvest for sufficiently large sardine biomass. A function suitable to this purpose is

$$\text{harvest} = \frac{\alpha * \text{sardine}}{\beta + \text{sardine}}$$

Use some algebra to demonstrate that the function does behave as claimed. Approximately what is the proportion? Approximately what is the constant harvest level?

2. *Testing the function.*

Determine values for α and β suitable for the Pacific sardine based on what you learned from Explorations 1.1 (Problem 2) and 1.2 (Problem 1). Is the optimal choice of α and β dependent on the initial biomass of the sardine?

Exploration 1.4. The Ricker Growth Rate Model

☞ Compare the Ricker with the logistic function: $R = rP(1 - P/K)$ for positive constants r and K .

Biologists commonly use the *Ricker function* to model fish population reproduction. The Ricker function is $R = \alpha P e^{(P_r - P)/P_m}$, where R is the reproduction rate, α is a constant, P is the parental or spawning stock population, P_r is the stock size at which $R = P$, and P_m is the stock size that yields maximum reproduction in the absolute sense. Calibrated for the Pacific sardine during the time period 1941 through 1951, this function is: $R = 0.15 P e^{(2.4 - P)/1.7}$.

1. *The Ricker population model.*

Replace the logistic term in ODE (5) with the Ricker function to obtain

$$sardine' = 0.15 * (sardine) * \exp((2.4 - sardine)/1.7)$$

This function exhibits “compensatory behavior” that biologists know many fish populations exhibit. Plot two sardine populations vs. time on the same set of axes for comparison: $sardine1'$ as per the Ricker function above and $sardine2'$ as per the logistics growth model used earlier. You have to select the **2D** tab on the graphics window when defining the graph to get both populations on the same graph. To compare their respective growth patterns, plot the two sardine populations from 1920 to 1960 (# **Points** = 40) with **IC** set to 1 on both plots and with no harvesting. Based upon this comparison, speculate what “compensatory” behavior is as envisioned by the biologists and reflected by the Ricker function.

2. *Repeat the harvest experiments.*

Repeat Exploration 1.1, Problem 2, using the Ricker function in the *sardine* ODE. What harvest level would provide a stable sustainable Pacific sardine population? Test whether the optimal harvest rate depends on the population IC. Are the results significantly different than when you used the logistic function?