(5.2) The Definite Integral

Home work 17 - 20, 35, 39, 41, 42,
47 - 50, 53, 59

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students

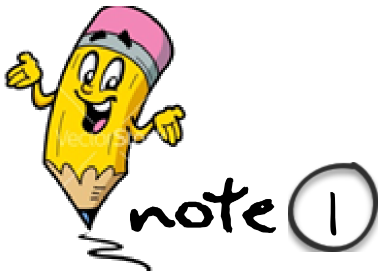
Definition of a Definite Integral

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a, x_1, x_2, \dots, x_n = b$ be the endpoints of these subintervals and we let x_1^*, x_2^*, x_3^*,…, x_n^* be any **sample points** in these subintervals, so x_i^* lies in the i^{th}subinterval $[x_{i-1}, x_i]$.

Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

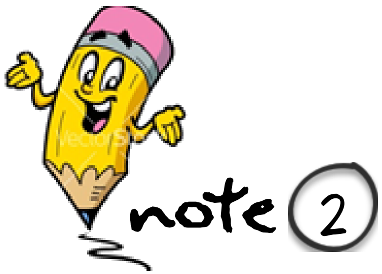


- The symbol $\int$ was introduced by Leibniz and is called an **integral sign**.

- $\int_a^b f(x)dx$ *is* called the **integrand** and a and b are called the **limits of integration**;

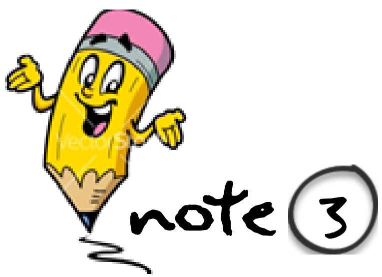
- $\int_a^b f(x)dx$ is all one symbol. The dx simply indicates that the independent variable is x.

- The procedure of calculating an integral is called **integration**.



- The definite integral $\int_a^b f(x)dx$ **is a number**; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(t)dt \; , \qquad \int_a^b f(z)dz \; ,$$



- The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ that occurs in Definition of a definite integral is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826 – 1866).

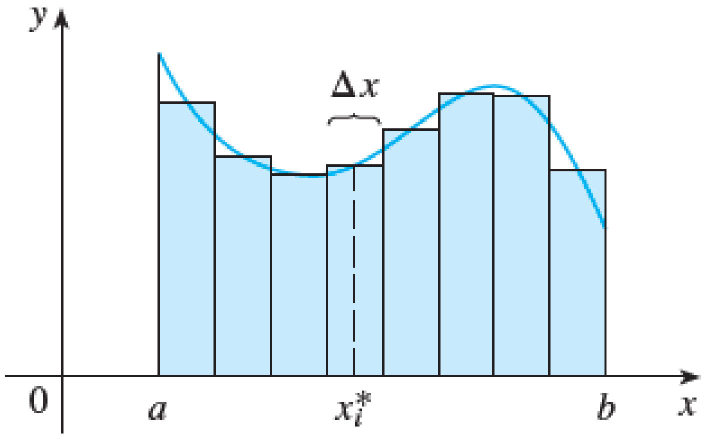


Figure (1)

If $f(x) \geq 0$,

$the\ Riemann\ sum \sum f(x_i^*)\Delta x$

is the sum of areas of rectangles.

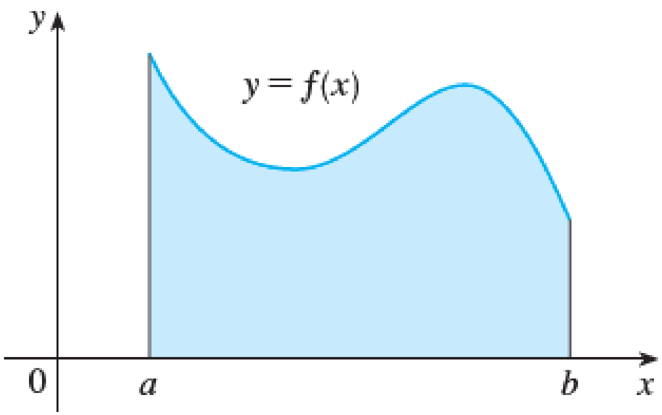


Figure (2)

If $f(x) \geq 0$, the integral $\int_a^b f(x)dx$

is the area under the curve $y = f(x)$ from a to b.

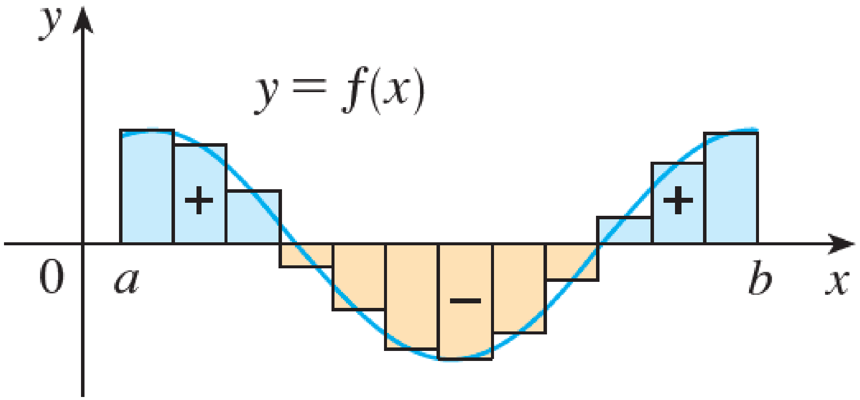


Figure (3)

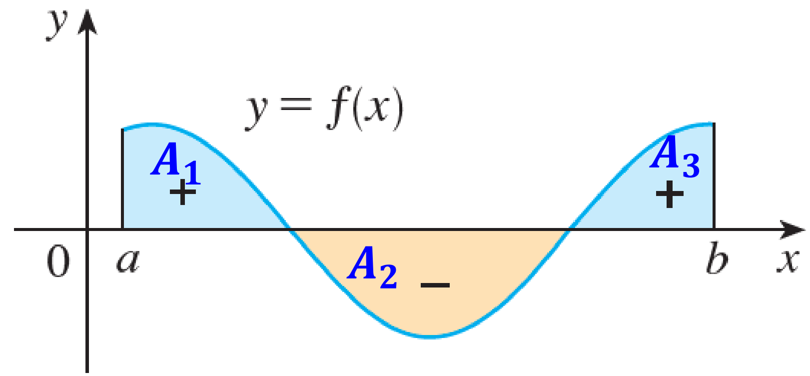
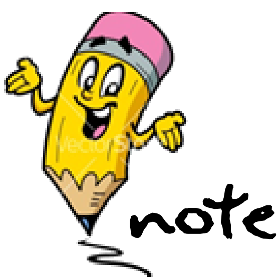


Figure (4)

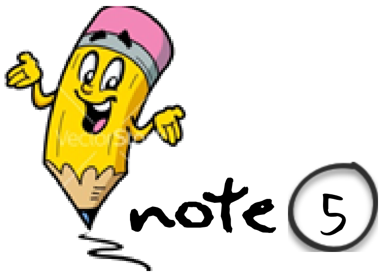
$\sum f(x_i^*)\Delta x$ is an **approximation of the net area.**

$\int_a^b f(x)dx$ **= the net area.**

$$\int_a^b f(x)dx \quad = \quad \boxed{}$$



- The area that lies above the $x - axis$ is $+ve$.
- The area that lies below the $x - axis$ is $-ve$.



We have defined the definite integral for an integrable function, but not all functions are integrable (see Exercises 71–72). The following theorem shows that the most commonly occurring functions are in fact integrable.

Theorem (3)

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

Theorem (4)

If f is integrable on $[a, b]$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

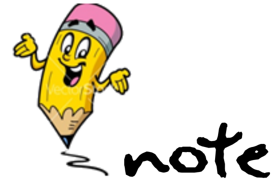
where $\Delta x = \dfrac{b-a}{n}$

and $x_i = a + i\Delta x$

Example 1

Express $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\,\Delta x$

as an integral on the interval $[0,\pi]$.



When we write

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)dx$$

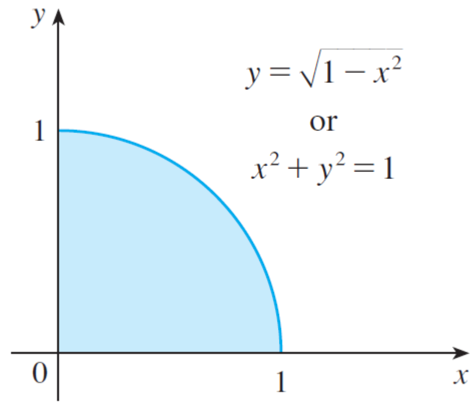
We replace $\lim \sum$ by [ ]

and Δx by [ ]

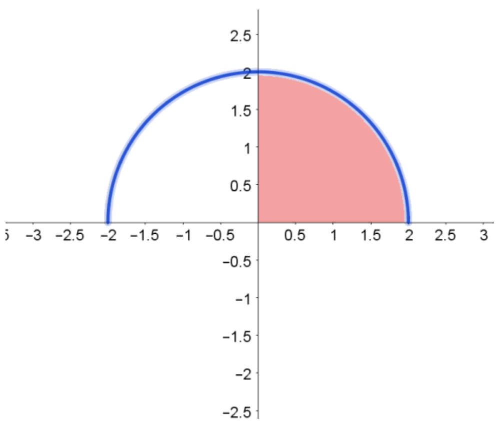
Example 4

Evaluate the following integrals by interpreting each in terms of areas.

(a) $\int_0^1 \sqrt{1-x^2}\ dx$

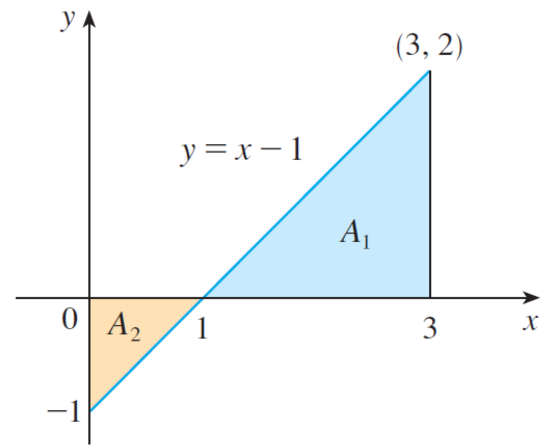


(a) $\int_0^2 \sqrt{4-x^2}dx$



(a) $\int_0^4 \sqrt{16-x^2}dx$

(b) $\int_0^3 (x-1)dx$



Properties of the Definite Integral

Assume that f and g are continuous functions.

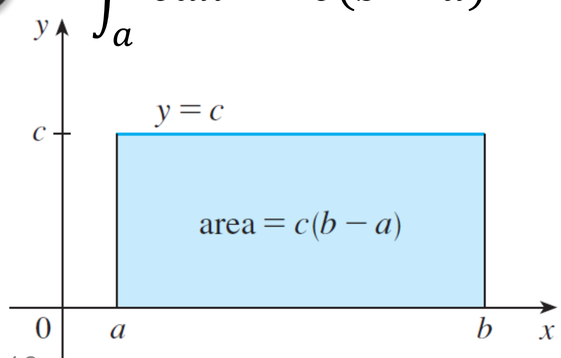
These properties are true for $a < b$, $a = b$, $a > b$.

1. $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2. $$\int_a^a f(x)dx = 0$$

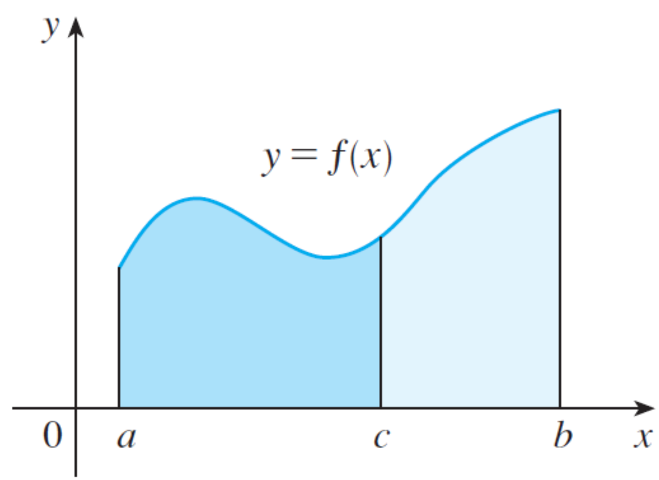
3. $$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

4. $$\int_a^b c\,dx = c(b-a)$$



5. $$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

6. $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$$



Example 6

Use the properties of integrals to evaluate

$$\int_0^1 (4+3x^2)dx$$

$$\int_0^1 (4+3(x-1))dx$$

Example 7

If $\int_0^{10} f(x)dx = 17$ and $\int_0^{8} f(x)dx = 12$,

Find $\int_8^{10} f(x)dx$.

Comparison Properties Integral

These properties are true only if $a \le b$.

(7) If $f(x) \ge 0$ for $a \le x \le b$

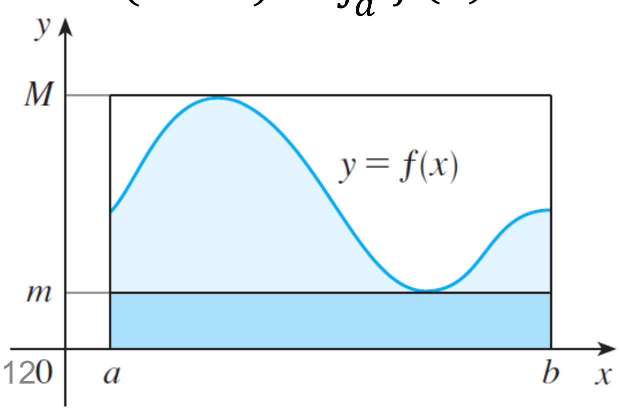
then $\int_a^b f(x)dx \ge 0$

(8) If $f(x) \ge g(x)$ for $a \le x \le b$

then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$

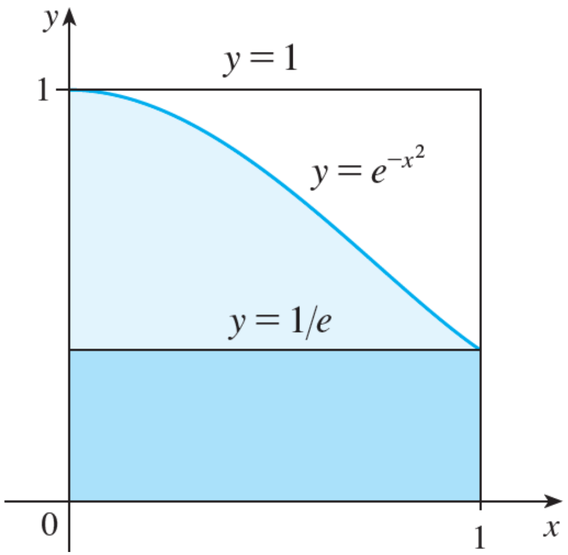
(9) If $m \le f(x) \le M \; for \;\; a \le x \le b$

Then $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$



Example 8

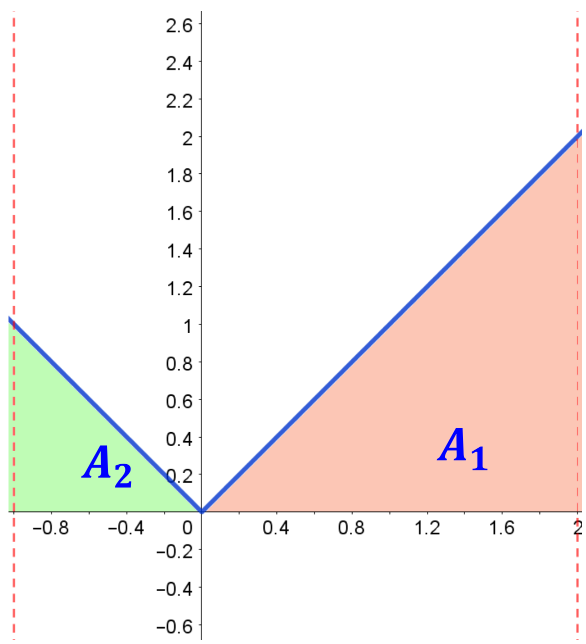
Use property 9 to estimate $\int_0^1 e^{-x^2} dx$



Exercise 39

Evaluate the integral by interpreting it in terms of areas.

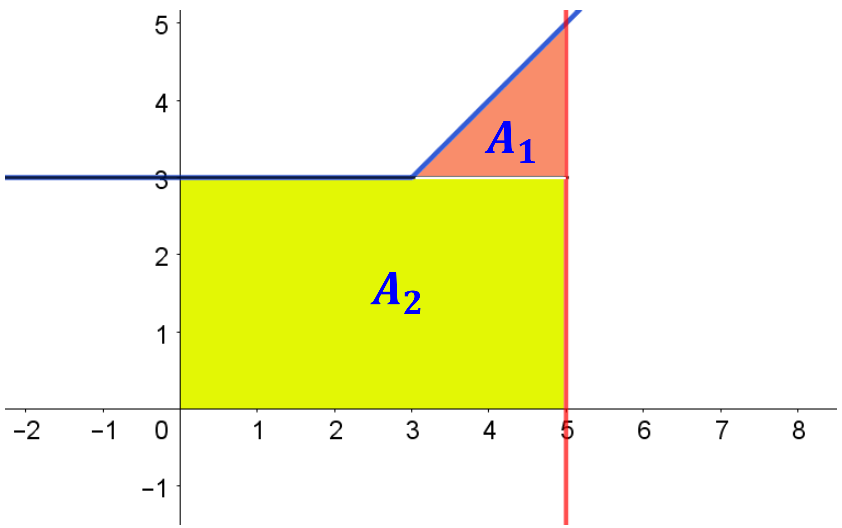
$$\int_{-1}^{2} |x|dx$$



Exercise 50

Find $\int_0^5 f(x)dx$ if

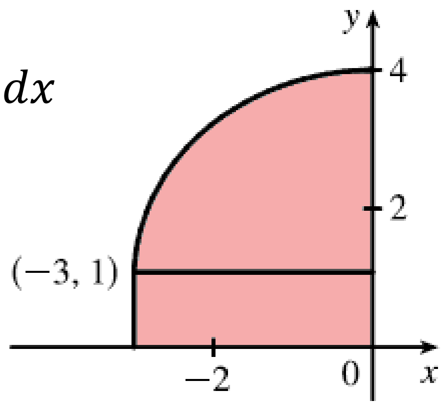
$$f(x) = \begin{cases} 3 & for \quad x < 3 \\ x & for \quad x \geq 3 \end{cases}$$



Evaluate the integral by interpreting it in terms of areas.

Exercise 37

$$\int_{-3}^{0} 1+\sqrt{9-x^2}\,dx$$



Exercise 35

$$\int_{2}^{4} (1-x)\,dx$$