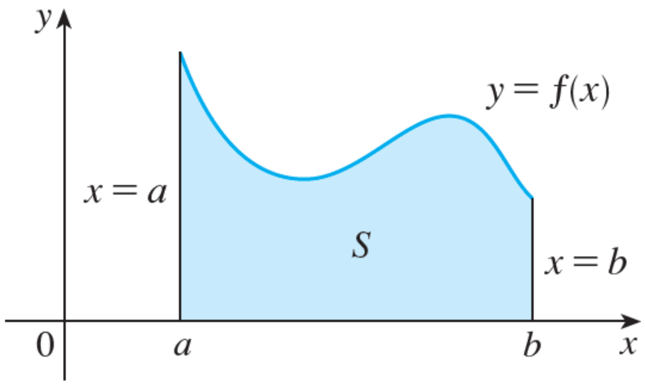
(5.1) Areas And Distances

Dr. Rola Asaad Hijazi

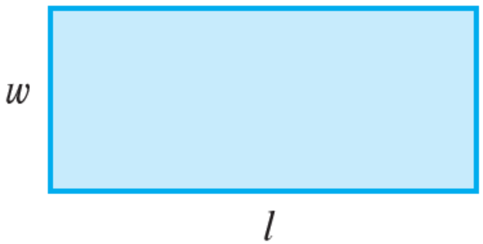
The Area Problem

Find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, is bounded by the graph of a continuous function f [where $f(x) > 0$], the vertical lines $x = a$ and $x = b$, and the $x - axis$.

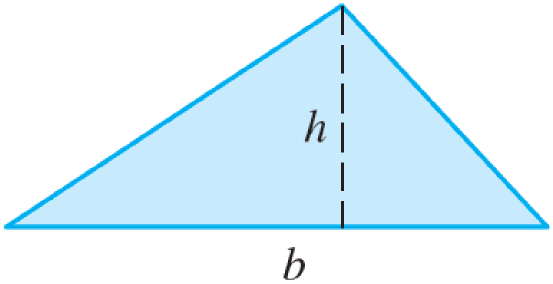


In trying to solve the area problem we have to ask ourselves: What is the meaning of the word *area*?

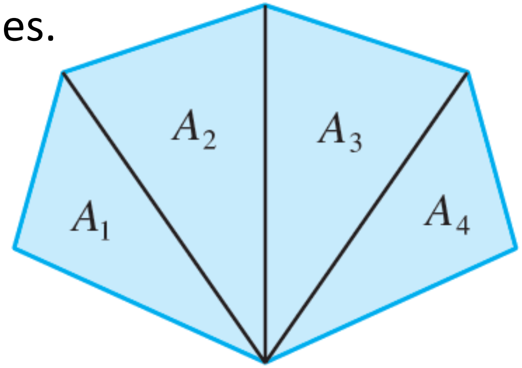
This question is easy to answer for regions with straight sides.
For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles and adding the areas of the triangles.



$A = lw$



$A = \frac{1}{2} bh$

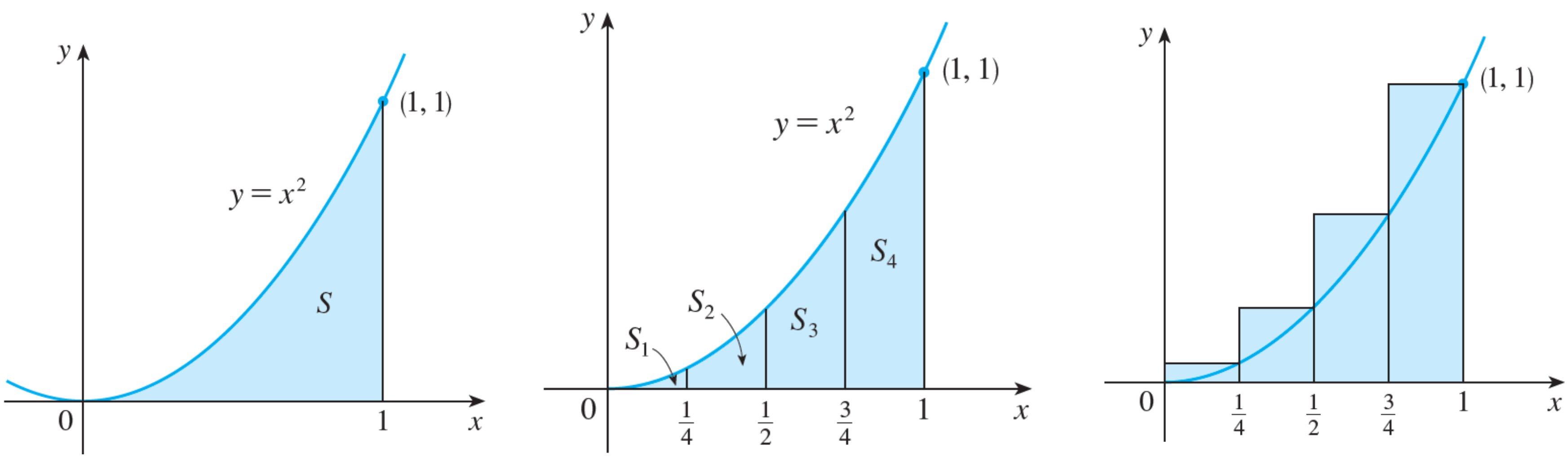


$A = A_1 + A_2 + A_3 + A_4$

However, it isn't so easy to find the area of a region with curved sides.

We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates the procedure.

Example (1): Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1



(a) (b) (c)

- We divide S into four strips $S_1, S_2, S_3, and\ S_4$ by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure (b).

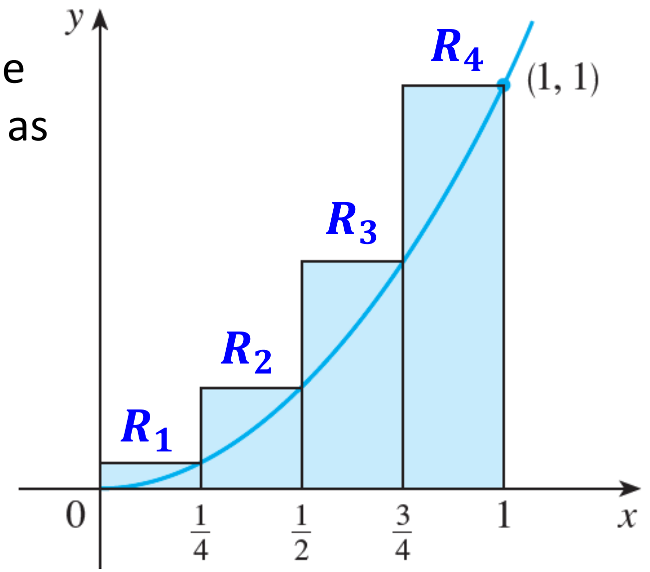
- We approximate each strip by a rectangle that has the same base as the strip and whose height is the same as **the right edge** of the strip [see Figure (c)].

- Each rectangle has width $\frac{1}{4}$ and the heights are $(\frac{1}{4})^2$, $(\frac{1}{2})^2$, $(\frac{3}{4})^2$, $(1)^2$

- Let $R_4 = R_1 + R_2 + R_3 + R_4$

$$= (\tfrac{1}{4})(\tfrac{1}{4})^2 + (\tfrac{1}{4})(\tfrac{1}{2})^2 + (\tfrac{1}{4})(\tfrac{3}{4})^2 + (\tfrac{1}{4})(1)^2$$

$$= \frac{15}{32} = 0.46875$$



(c)

From Figure (c) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

- We divide S into four strips $S_1, S_2, S_3, and\ S_4$ by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure (b).

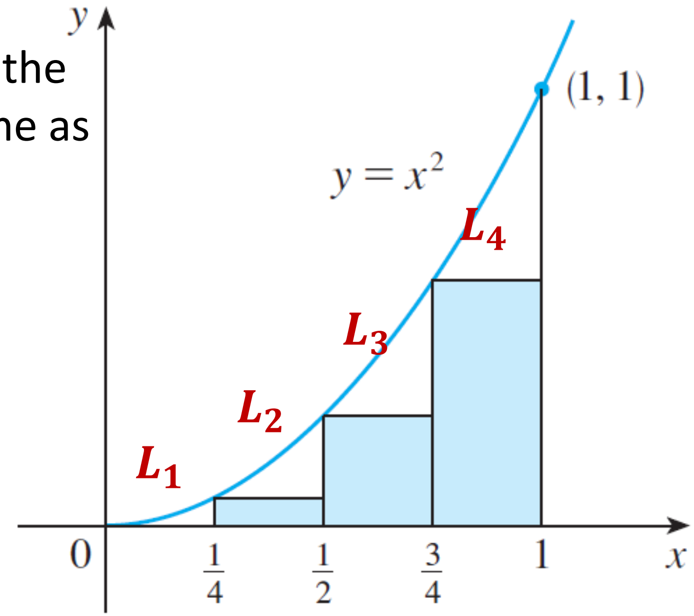
- We approximate each strip by a rectangle that has the same base as the strip and whose height is the same as **the left endpoints** of the strip [see Figure (d)].

- Each rectangle has width $\frac{1}{4}$ and the heights are $(0)^2, (\frac{1}{4})^2, (\frac{1}{2})^2, (\frac{3}{4})^2$

- Let $L_4 = L_1 + L_2 + L_3 + L_4$

$$0 + \frac{1}{4}(\frac{1}{4})^2 + (\frac{1}{4})\ (\frac{1}{2})^2 + (\frac{1}{4})\ (\frac{3}{4})^2$$

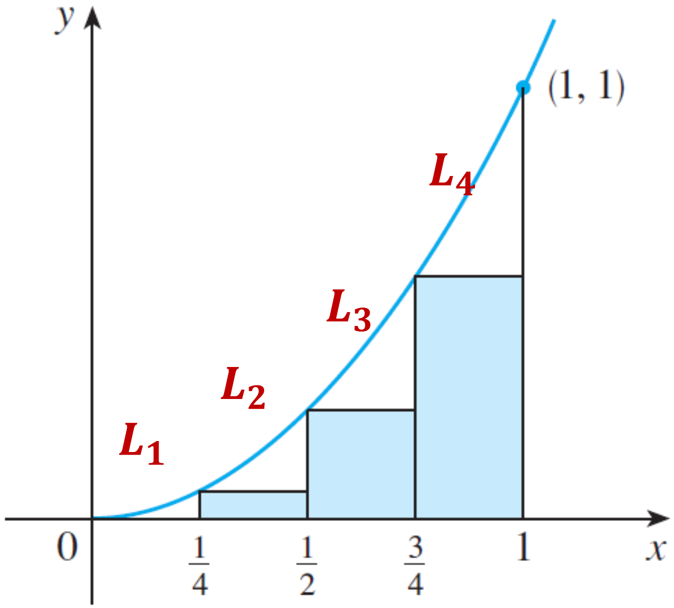
$$= \frac{7}{32} = 0.21875$$



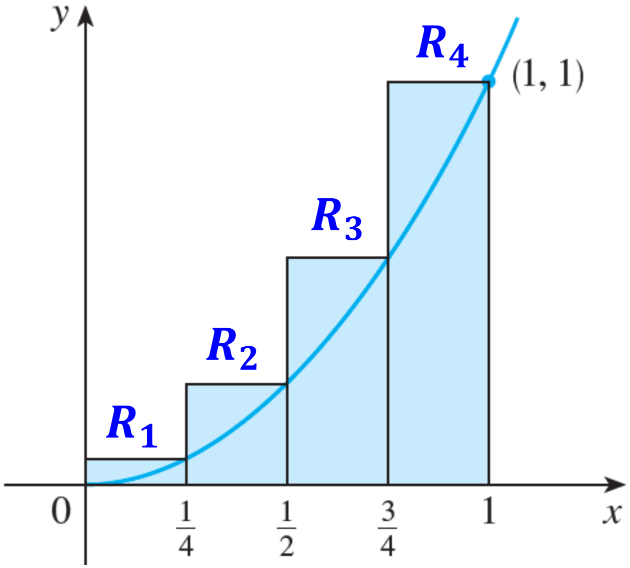
(d)

From Figure (d) we see that the area A of S is larger than L_4, so

$$A > 0.21875$$



(d)

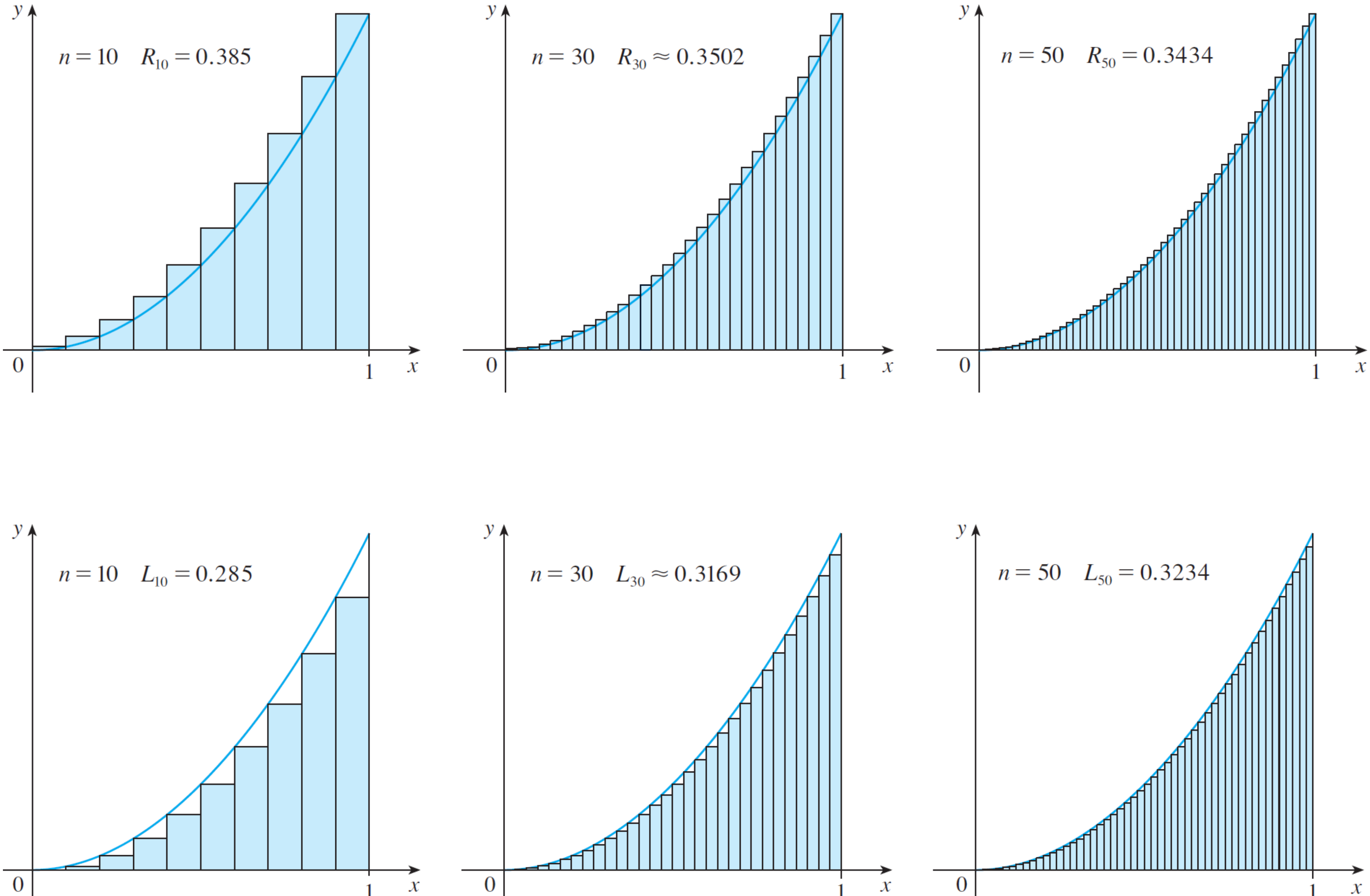


(c)

$A > 0.21875$

$A < 0.46875$

$0.21875 < A < 0.46875$

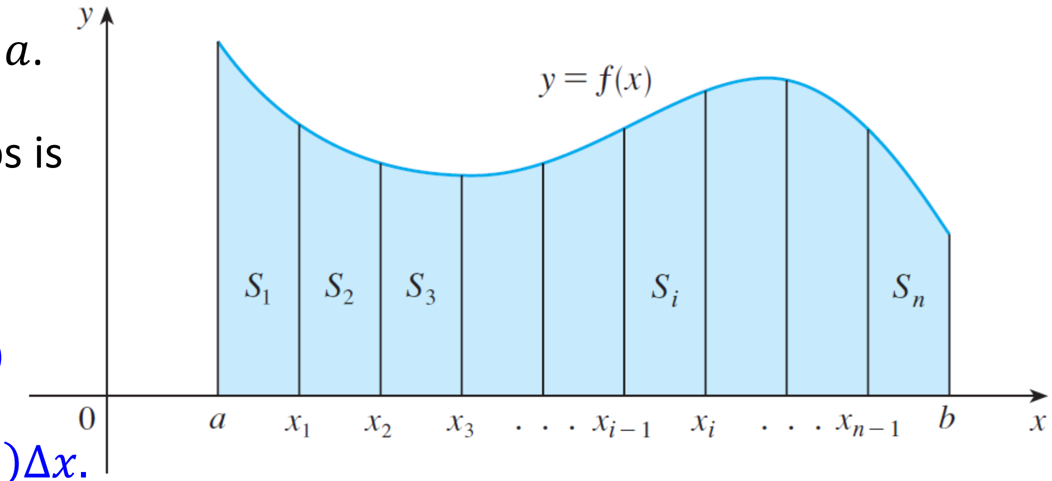


n = 10 R10 = 0.385
n = 30 R30 ≈ 0.3502
n = 50 R50 = 0.3434
n = 10 L10 = 0.285
n = 30 L30 ≈ 0.3169
n = 50 L50 = 0.3234
y
x
0
1

- The width of the interval $[a, b]$ is $b - a$.
- So the width of each of the $n -$ strips is

$$\text{base} \quad \Delta x = \frac{b-a}{n}$$

- The height of the rectangle $= f(x_i)$
- The area of the i^{th} rectangle $= f(x_i)\Delta x$.



- Area of $\boldsymbol{S} = \boldsymbol{R_n} = \boldsymbol{f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x}$
- Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.Therefore we define the area A of the region S in the following way.

Definition The **area A of the region** S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$\boldsymbol{A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}[f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x] = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x}$$