5.2 The Definite Integral

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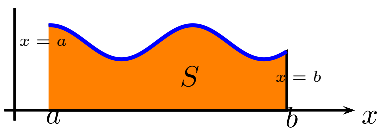
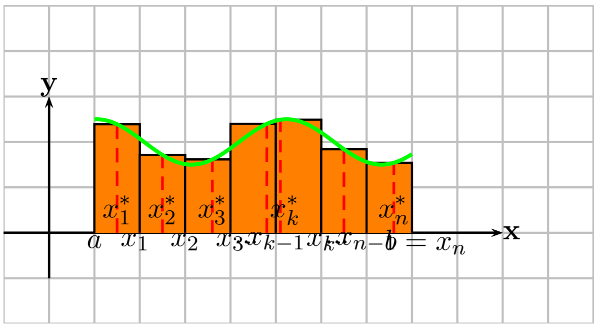
Definition 1: [Definition of A Definite Integral]

If $f(x)$ is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subinterval with length $\triangle x = \dfrac{b-a}{n}$. We let $a = x_0, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, x_3^*, \ldots, x_k^*, \ldots, x_n^*$ be any sample points in these subintervals, so $x_i^* \in [x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$\int\limits_a^b f(x)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\triangle x$ provided that this limit exists. If it does exist we say that f is integrable on $[a, b]$.





Note 1:

- The symbol $\int$ is called an integral sign. In the notation $\int\limits_a^b f(x)dx$ $f(x)$ is called integrand and a and b are called limits of integration, lower and upper limits respectively. The dx has no meaning by itself except indicates that the independent variable is x.

- The definite integral $\int\limits_a^b f(x)dx$ is a number, does not depend on x.

$$\int\limits_a^b f(x)dx = \int\limits_a^b f(t)dt = \int\limits_a^b f(y)dy.$$

- The sum $\sum\limits_{i=1}^{n} f(x_i^*)\triangle x$ is called a Riemann sum.

Theorem1: *[]*

If f is continuous on$[a,b]$ or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$ That is the definite integral $\int\limits_a^b f(x)dx$ exists.

Theorem2: *[]*

If f is continuous on$[a,b]$, then

$$\int\limits_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\triangle x,$$

where $\triangle x = \dfrac{b-a}{n}$ and $x_i = a + i\triangle x$

Example 1: Express $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(x_i^3 + x_i\sin x_i)\triangle x$, as an integral on $[0,\pi]$.

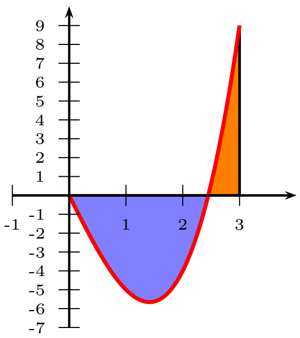
Solution:

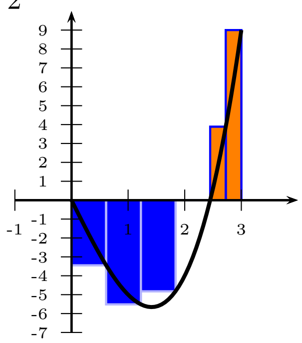
$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}(x_i^3 + x_i\sin x_i)\triangle x = \int\limits_0^{\pi}(x^3 + x\sin x)\,dx$

Example 2: Evaluate the Riemann sum for $f(x) = x^3 - 6x$ taking the sample point to be the right endpoints and $a = 0$, $b = 3$, and $n = 6$.

Solution:

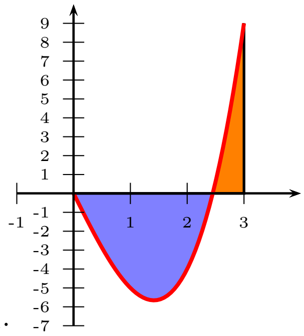
With $n = 6$, we have $\triangle x = \dfrac{3-0}{6} = \dfrac{1}{2}$ and $x_i = 0 + \dfrac{i}{2}$, $i = 1,2,3,4,5,6$. $R_6 = \sum\limits_{i=1}^{6} f(x_i)\triangle x =$ $[f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)]\dfrac{1}{2} = -3.9375$





Example 3: Evaluate $\int\limits_0^3 (x^3 - 6x)dx$.

Solution:



We have $\triangle x = \frac{3-0}{n} = \frac{3}{n}$ and $x_i = 0 + i\frac{3}{n} = \frac{3i}{n}$, $i = 1, 2, \ldots, n$. We approximate R_n.

$$\int_0^3 (x^3 - 6x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\triangle x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{27i^3}{n^3} - \frac{18i}{n}\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\frac{n^2(n+1)^2}{n^4} - \frac{54}{2}\frac{n(n+1)}{n^2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] = \frac{81}{4} - 27 = \frac{-27}{4} = -6.75$$

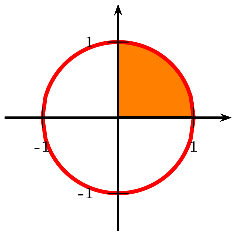
Example 4: Evaluate the following integrals by interpreting each in terms of area.

1. $\int_0^1 \sqrt{1-x^2}\,dx.$
2. $\int_0^3 (x-1)dx.$

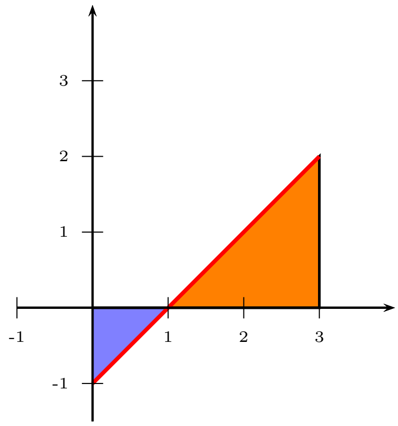
Solution:

(1) Since $f(x) = \sqrt{1-x^2} \geq 0$, then the integral is the area under the curve $y = \sqrt{1-x^2}$ for $x = 0$ to $x = 1$. Now, since $y^2 = (\sqrt{1-x^2})^2 = 1 - x^2$ then $x^2 + y^2 = 1$ for $0 \leq x \leq 1$, and $y \geq 0$. This is quarter-circle with radius 1.

Therefore $\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}(\pi(1)^2) = \frac{\pi}{4}.$



(2) The integral represent the net area under the curve $y = x - 1$ for $0 \leq x \leq 3$. The area is the deference of the areas of the two triangle.



$$\int_0^3 (x-1)dx = A_1 - A_2 = \frac{1}{2}(2.2) - \frac{1}{2}(1.1) = \frac{3}{2}.$$

5.2 Properties of The Integral

1. $\int_a^b c\,dx = c(b-a)$ where c is any constant
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ where c is any constant
4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

5.3 Properties of The Integral

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a \le c \le b$

6. If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge 0$

7. If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

8. If $m \ge f(x) \le M$ for $a \le x \le b$, then $m(b-a) \ge \int_a^b f(x)\,dx \le M(b-a)$

Example 5: Use the properties of integrals to evaluate $\int_0^1 (4+3x^2)\,dx$.

Solution:

Note that $\int_0^1 4\,dx = 4(1-0) = 4$, and $\int_0^1 x^2\,dx = \frac{1}{3}$.

$\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx = 4 + 3\frac{1}{3} = 4+1 = 5.$

Example 6: If it is known that $\int_0^{10} f(x)\,dx = 17$ and $\int_0^8 f(x)\,dx = 12$, find $\int_8^{10} f(x)\,dx$. ***Solution:***

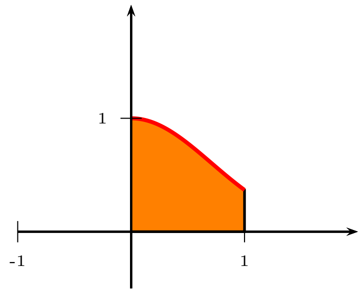
Note that $\int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx$

Hence $\int_8^{10} f(x)\,dx = \int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = 17 - 12 = 5.$.

Example 7: Use the properties of integrals to estimate $\int_0^1 e^{-x^2}\,dx$. ***Solution:***

Since $f(x) = e^{-x^2}$ is continuous on $[0,1]$ and differentiable on $(0,1)$, we can find the absolute maximum and absolute minimum. $f'(x) = -2xe^{-x^2} \Rightarrow f'(x) = 0 \Leftrightarrow x = 0$. Now, $f(0) = 1$, and $f(1) = \frac{1}{e}$, hence $\frac{1}{e} \le e^{-x^2} \le 1$, for $0 \le x \le 1$.

Then $\frac{1}{e}(1-0) \le \int_0^1 e^{-x^2}\,dx \le 1(1-0)$. Thus $\frac{1}{e} \le \int_0^1 e^{-x^2}\,dx \le 1$.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