(4.4) Indeterminate Forms and L'hospital's Rule

Home work $9 - 97$ (odd) . $10, 52, 56$
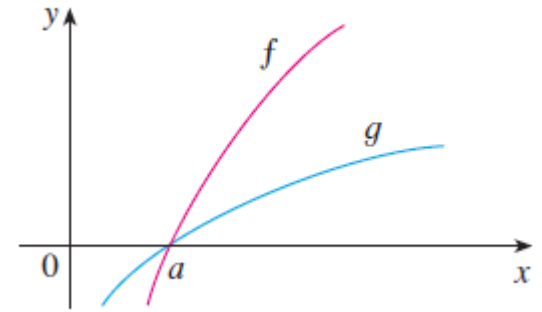
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L'Hospital's Rule   Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

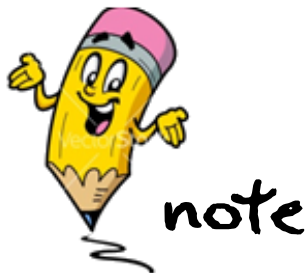Or that, $\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$

In other words, we have an indeterminate form of type

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty} \qquad \text{Then}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\infty \ or \ -\infty$

note

① L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.
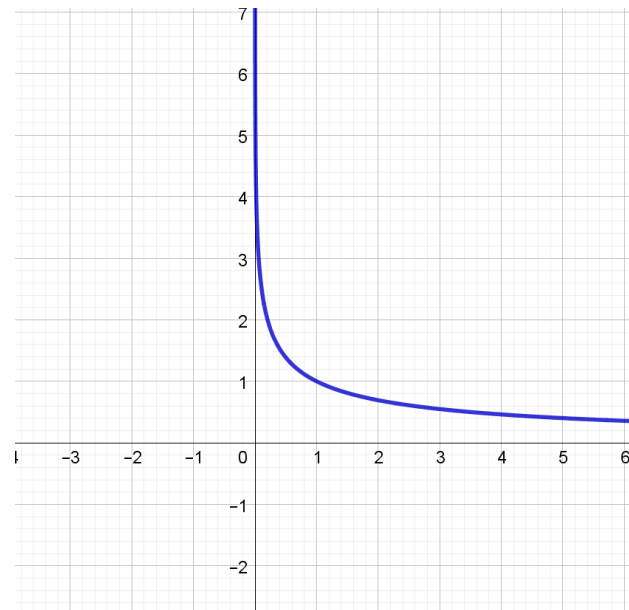
② L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is , $x \longrightarrow a$ can be replaced by any of the symbols

$$x \longrightarrow a^+, x \longrightarrow a^- , x \longrightarrow \infty, x \longrightarrow -\infty.$$

③ Notice that when using l'Hospital's Rule we differentiate the numerator and denominator separately. We do not use the Quotient Rule.
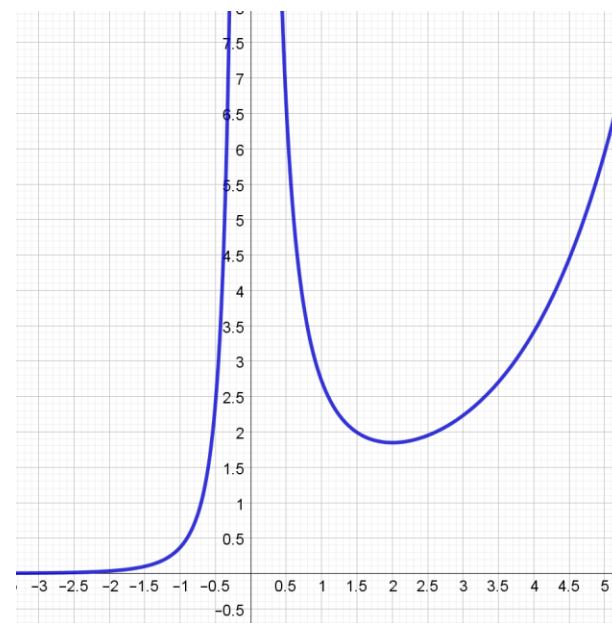
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Example 1 Find $\displaystyle\lim_{x\to 1}\frac{lnx}{x-1}$

Example 2 Find $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}$

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Example 3

Evaluate $\lim\limits_{x\to\infty}\dfrac{lnx}{\sqrt{x}}$

$y = \dfrac{\ln x}{\sqrt{x}}$

2

0   10,000

−1

Example 4

Find $\lim\limits_{x\to 0}\dfrac{tanx - x}{x^3}$

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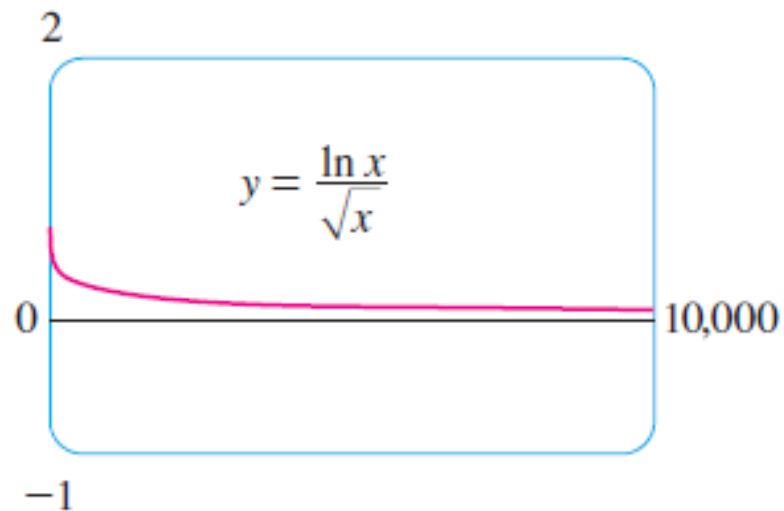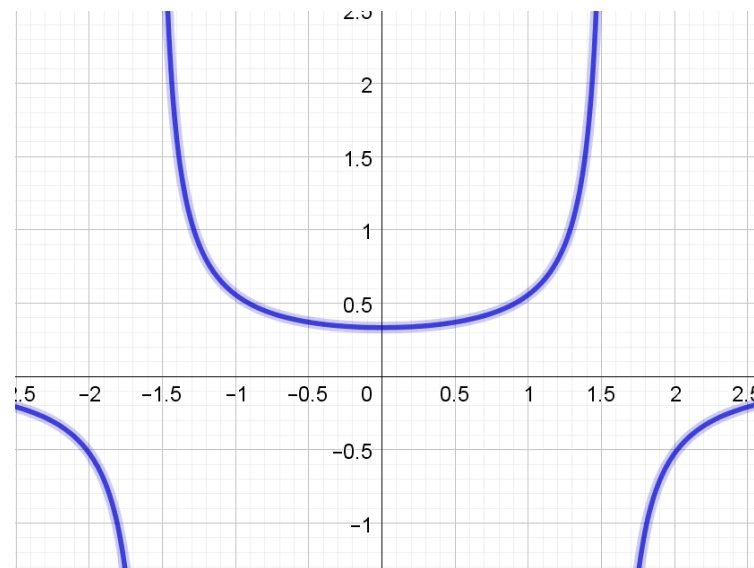
Example 5

Find $\displaystyle\lim_{x\to\pi^-}\frac{sinx}{1-cosx}$

If we blindly attempted to use l'Hospital's Rule, we would get

This is wrong! Although the numerator $\sin x \to 0$ as $x \to \pi^-$, notice that the denominator $1 - cosx \nrightarrow 0$, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous at π and the denominator is nonzero there:

If $\displaystyle\lim_{x\to a} f(x) = 0$ and $\displaystyle\lim_{x\to a} g(x) = \pm\infty$

Then $\displaystyle\lim_{x\to a} f(x)g(x) = 0.\infty$

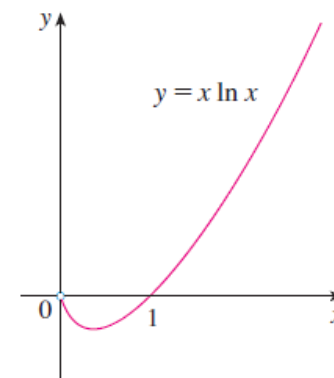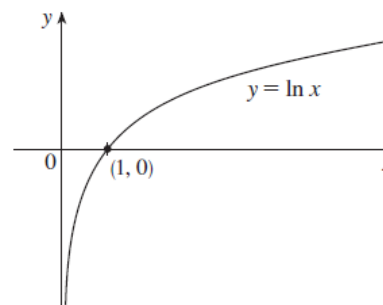is called an indeterminate form of type

$0.\infty$

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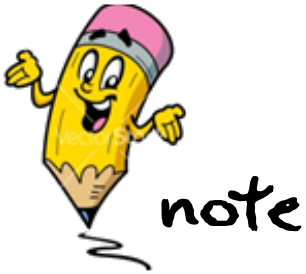We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

Example 6   Find $\displaystyle\lim_{x\to 0^+} x\,lnx$

Recall that:
The gragh of ln x

$y = \ln x$ $(1, 0)$

$y = x \ln x$

note

① We solve the limit of indeterminate product by writing

$$fg = \frac{f}{1/g} \quad \text{or} \quad \frac{g}{1/f}$$

② In the previous example if we solve it by taking $fg = \frac{x}{1/\ln x}$, it will be more complicated.

③ When we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

Example 7

$$\lim_{x \to 1^+} \frac{1}{\ln x} - \frac{1}{x-1}$$

$$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \infty$$

Then $\quad \lim_{x \to a} f(x) - g(x) = \infty - \infty$
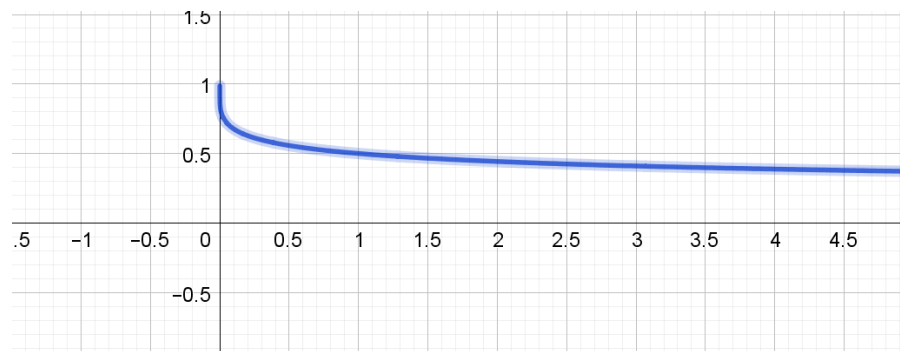
is called an indeterminate form of type

$$\infty - \infty$$

To find out, we try to convert the difference into a quotient by using a common denominator, or rationalization, or factoring out a common factor, so that we have an indeterminate form of type

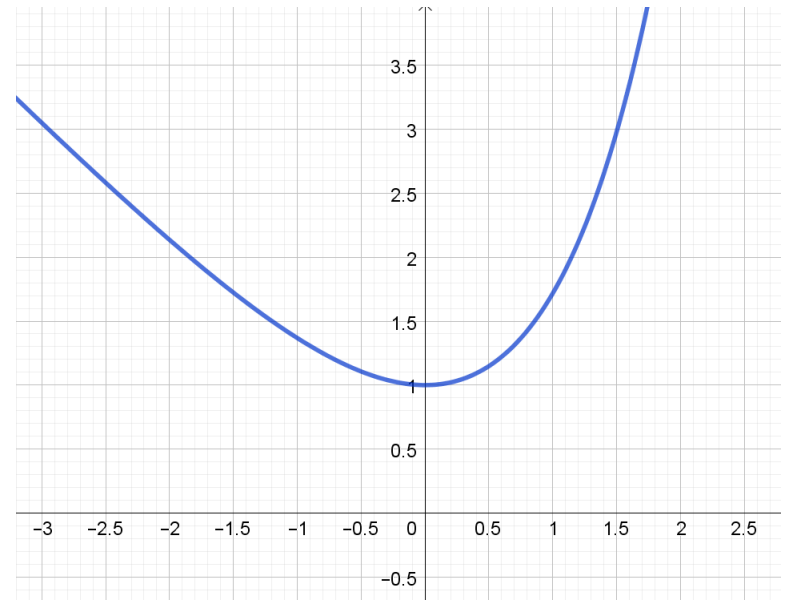$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

Example 8

Find $\displaystyle \lim_{x \to \infty} e^x - x$

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Indeterminate Powers

Several indeterminate forms arise from the limit $\lim\limits_{x \to a} = f(x)^{g(x)}$

These are called an indeterminate form of type 0^0, ∞^0, 1^∞

① If $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ type 0^0

② If $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = 0$ type ∞^0

③ If $\lim\limits_{x \to a} f(x) = 1$ and $\lim\limits_{x \to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

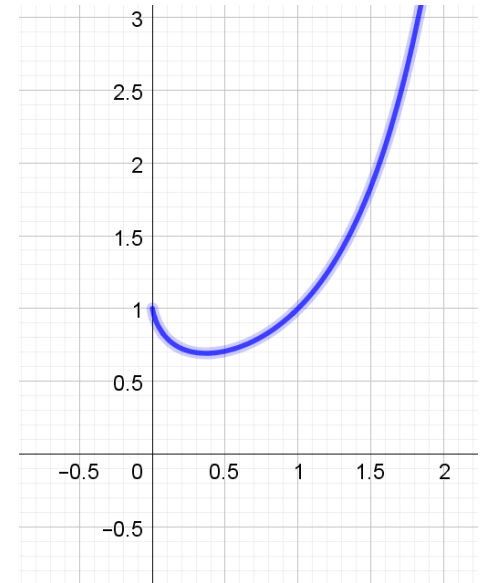$$y = f(x)^{g(x)} \quad \text{then} \quad lny = g(x)\ln(f(x))$$

or by writing the function as an exponential: $y = f(x)^{g(x)} = e^{g(x)lnf(x)}$

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Example 9 Evaluate

$$\lim_{x \to 0^+} (1 + sin4x)^{cotx}$$

Example 10 Find $\lim_{x \to 0^+} x^x$

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Find $\lim\limits_{x \to -2} \dfrac{x^3 + 8}{x + 2}$

$\lim\limits_{x \to 0} \dfrac{\sqrt{1 + 2x} - \sqrt{1 - 4x}}{x}$

$\lim\limits_{x \to 0} \csc x - \cot x$

$\lim\limits_{x \to 0^+} (4x + 1)^{\cot x}$

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