Calculus 202

Dr. Rola Asaad Hijazi

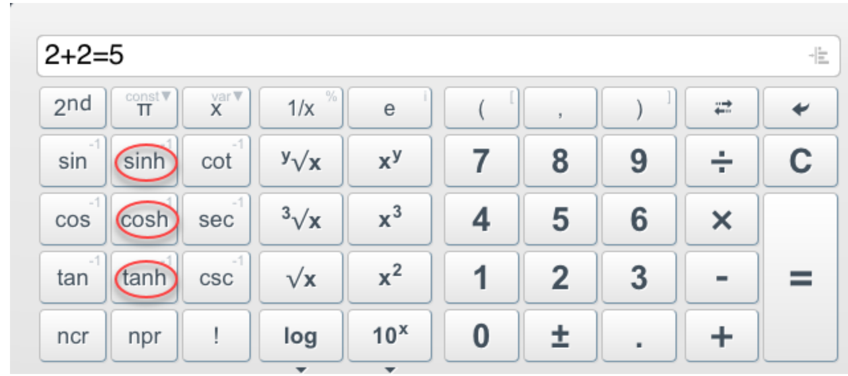
3.11 Hyperbolic Functions

(page 259, 8th edition)

HOMEWORK 1 - 21 (ODD),
31 - 43 (ODD), 40, 42.

3.11 HYPERBILIC FUNCTIONS

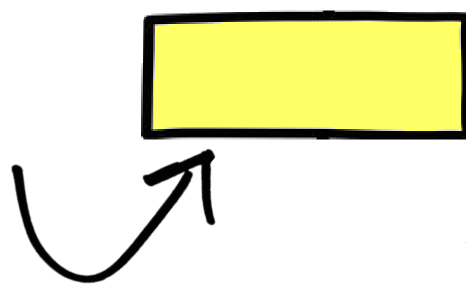
These symbols: sinh, cosh, tanh
they look like sin, cos, tan,
but with letter (h) at the end.

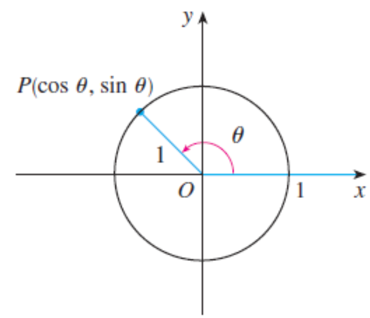


These are called ....

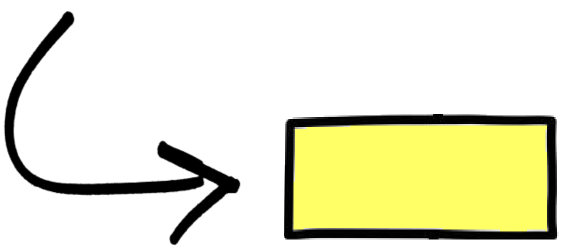
They are kind of the Trigonometric Functions except where as:

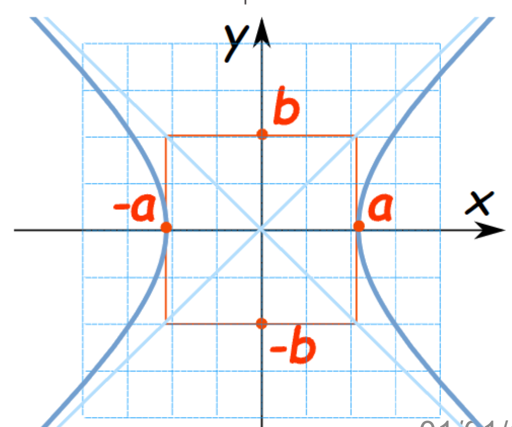
The trig. Functions are related to the





Hyperbolic functions are related to the





So how do hyperbolic functions relate to the hyperbola.

Let us define :

① The hyperbolic sine $\quad \sinh x =$

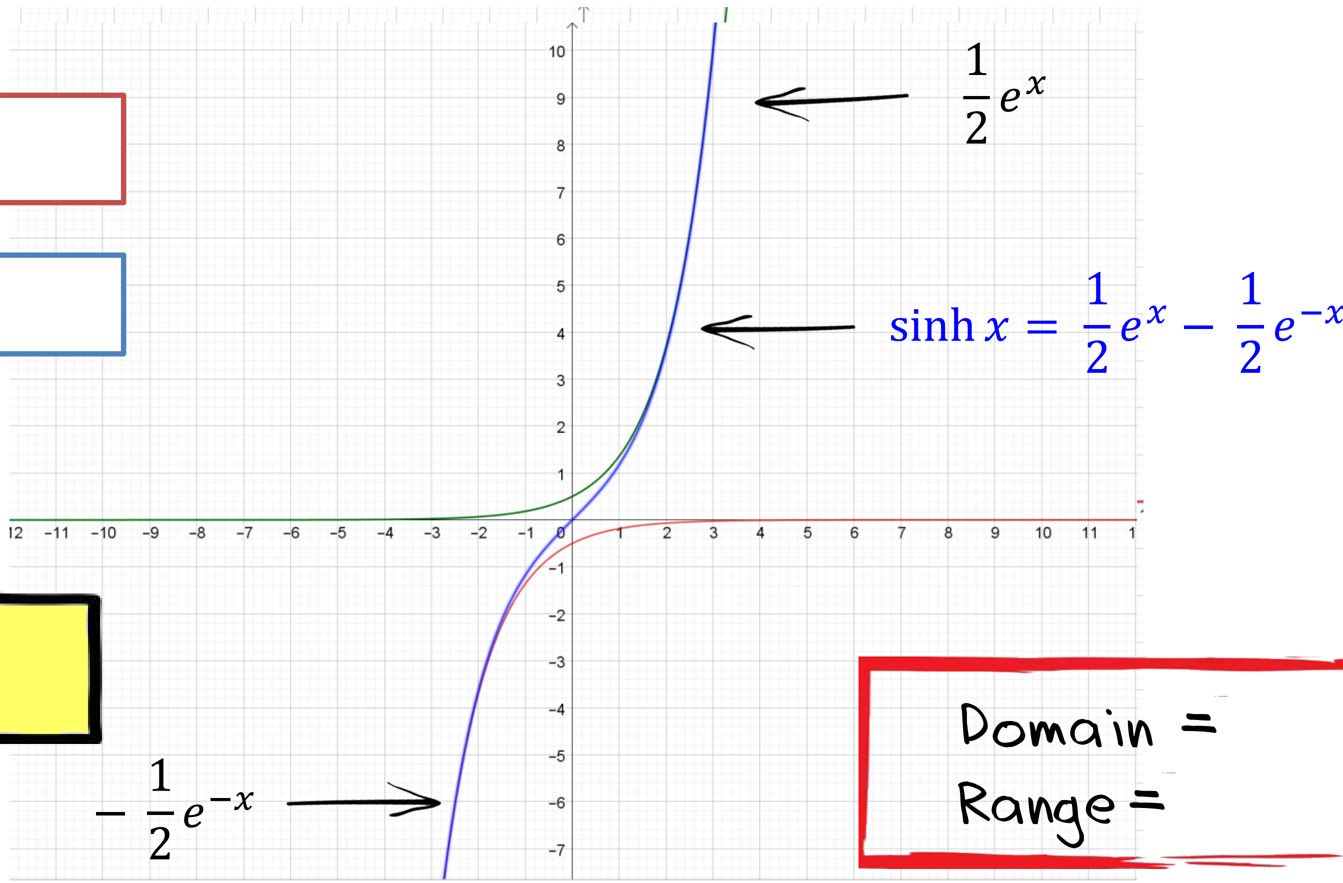
② The hyperbolic cosine $\quad \cosh x =$

③ The hyperbolic tan $\quad \tanh x =$

① The Hyperbolic Sine

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

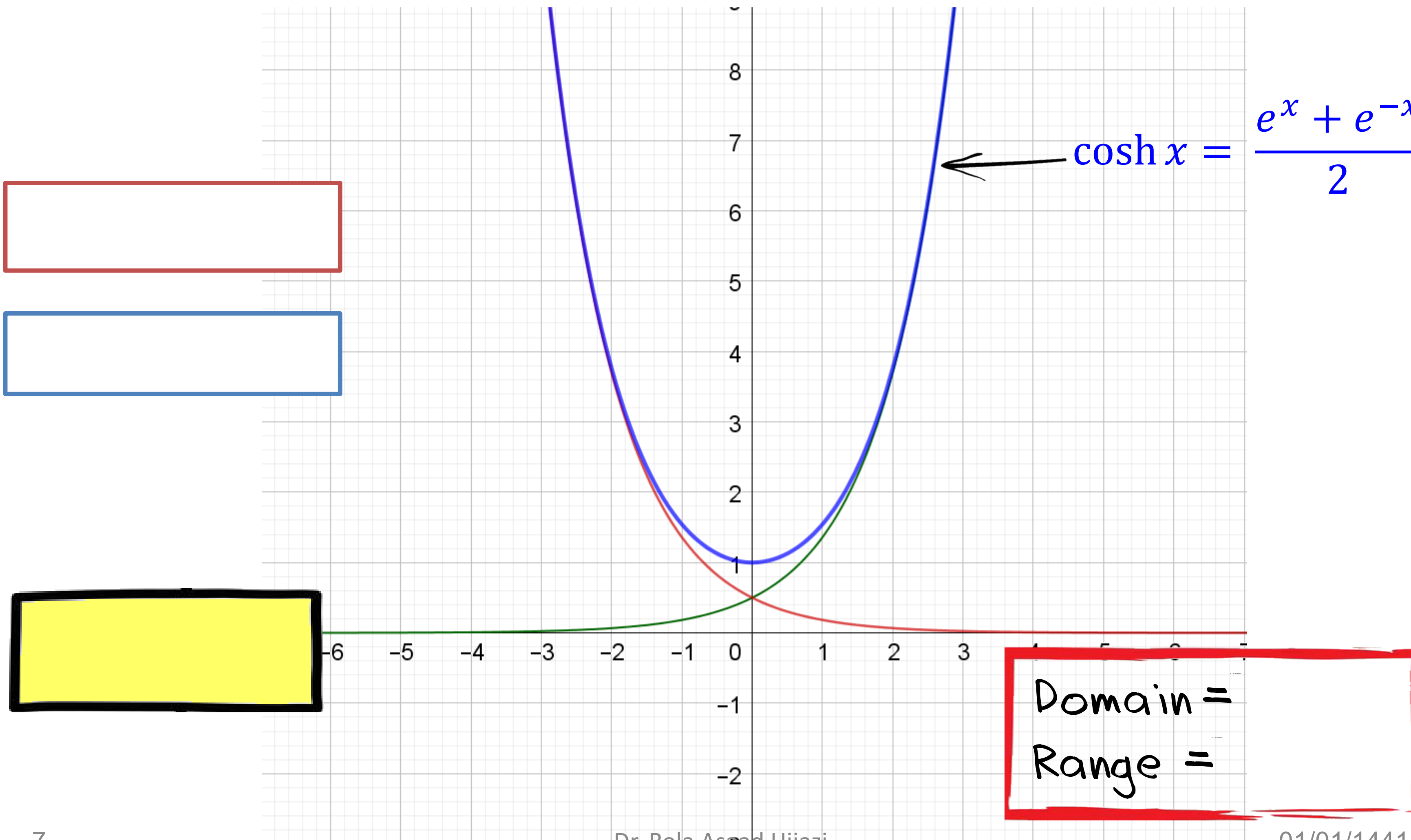
What does this mean graphically?



Domain =
Range =

② The Hyperbolic Cosine

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

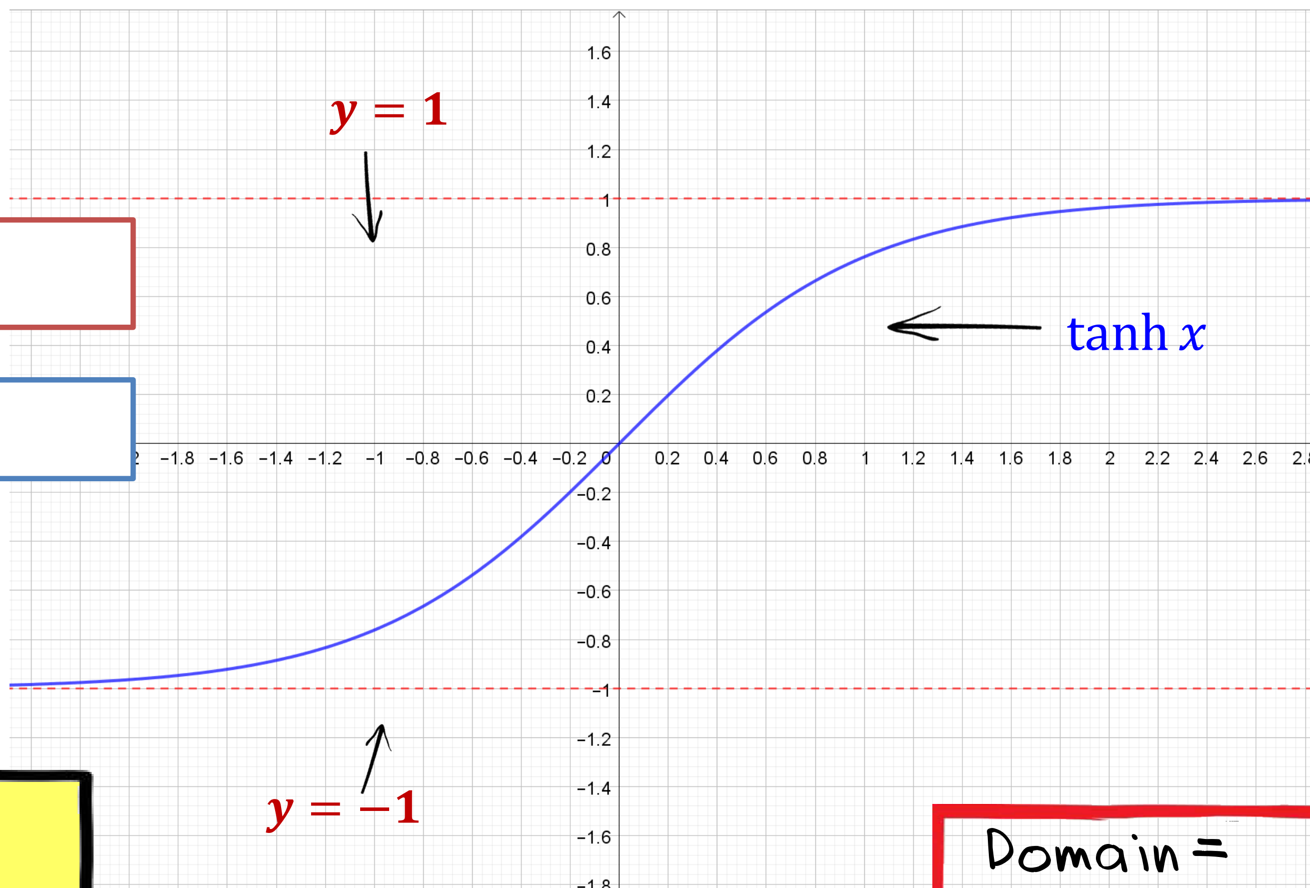


Domain =

Range =

③ The Hyperbolic Tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



Domain =

Range =

Now we can define the hyperbolic cosecant, secant, and cotangent which are just a reciprocals of sinh, cosh, tanh.

Definition of the Hyperbolic Functions

① $$\sinh x = \frac{e^x - e^{-x}}{2}$$

② $$\cosh x = \frac{e^x + e^{-x}}{2}$$

③ $$\tanh x = \frac{\sinh x}{\cosh x}$$

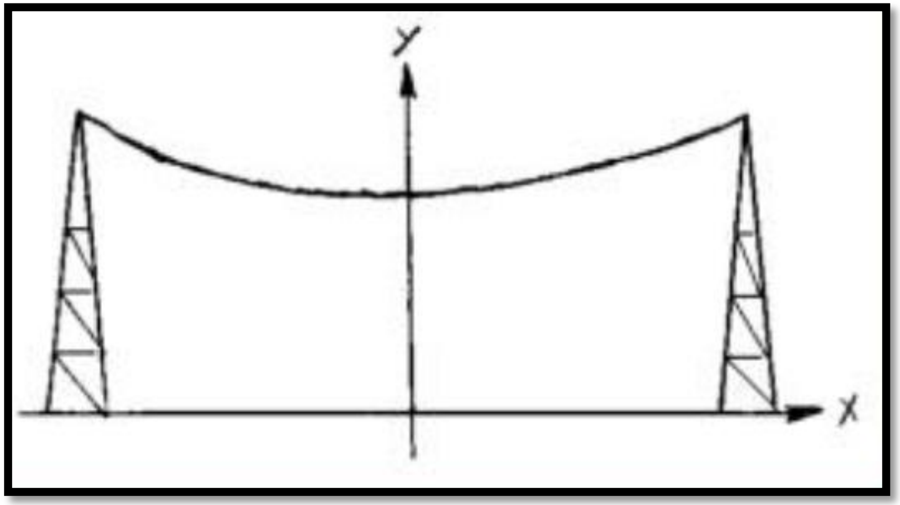
④ $$\operatorname{csch} x = \frac{1}{\sinh x}$$ hyperbolic cosecant

⑤ $$\operatorname{sech} x = \frac{1}{\cosh x}$$ hyperbolic secant

⑥ $$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$$ hyperbolic cotangent

Applications of Hyperbolic Functions

A hanging cable make a shape called a catenary.



This is the most famous application of the hyperbolic function is to use the hyperbolic cosine to describe the shape of hanging wire. It can be proved that if a heavy flexible cable such as telephone or power line is suspended between two posts at the same height, then it takes the shape of a curve with equation:

$$y = c + a\cosh\frac{x}{a}$$

Which is called a catenary (means chain).

Hyperbolic Identities

1. $\sinh(-x) = -\sinh(x)$

2. $\cosh(-x) = \cosh(x)$

3. $\cosh^2(x) - \sinh^2(x) = 1$

4. $1 - \tanh^2(x) = \operatorname{sech}^2(x)$

5. $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

6. $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

Example (1):

Prove that: (a) $\cosh^2(x) - \sinh^2(x) = 1$

Solution:

$(b)\ 1 - \tanh^2(x) = \text{sech}^2(x)$

Solution:

Exercise (11): Prove the identity:

$$sinh(x + y) = sinh(x)\,cosh(y) + cosh(x)\,sinh(y)$$

Solution:

Derivatives of Hyperbolic Functions:

① $\frac{d}{dx}\sinh x = \cosh x$

② $\frac{d}{dx}\cosh x = \sinh x$

③ $\frac{d}{dx}\tanh x = \text{sech}^2(x)$

④ $\frac{d}{dx}\text{csch}\, x = -\text{csch}(x)\coth(x)$

⑤ $\frac{d}{dx}\text{sech}\, x = -\text{sech}x \tanh x$

⑥ $\frac{d}{dx}\coth(x) = -\text{csch}^2(x)$

Remark :

Any of these differentiation rules can be combined with the Chain Rule.

Recall that: The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f \circ g(x) = f(g(x))$ is differentiable at x and:

$$F'(x) = f'(g(x)) \cdot g'(x)$$

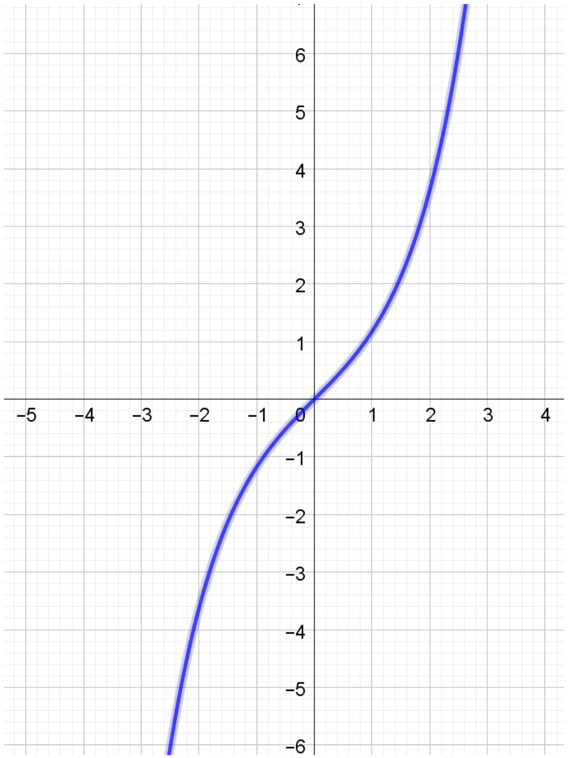
Example (2):

Let $y = cosh(\sqrt{x})$. Find $\frac{dy}{dx}$

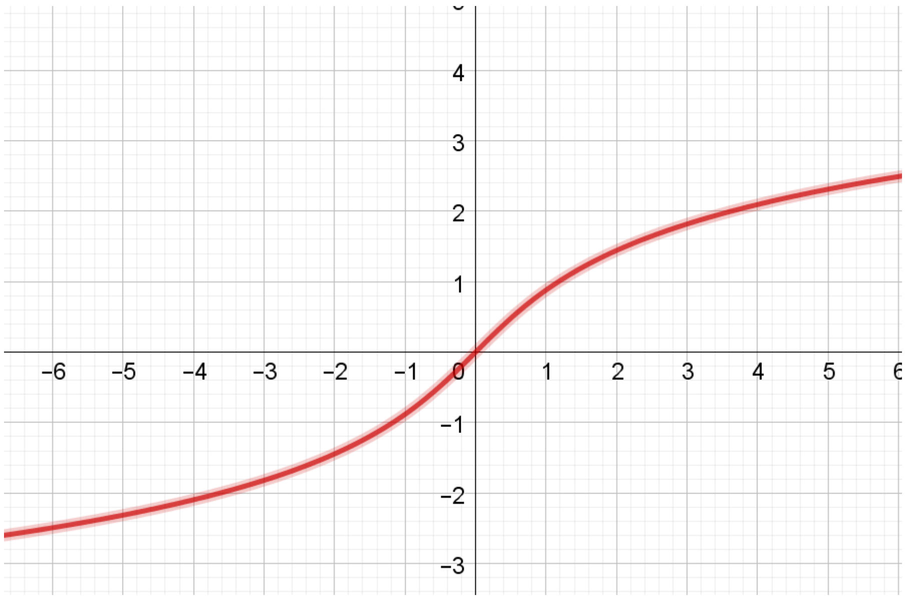
Inverse Hyperbolic Functions:

① $y = \sinh^{-1}(x) \Leftrightarrow sinh\, y = x$

$y = \sinh x$



$y = \sinh^{-1}(x)$



Domain =

Range =

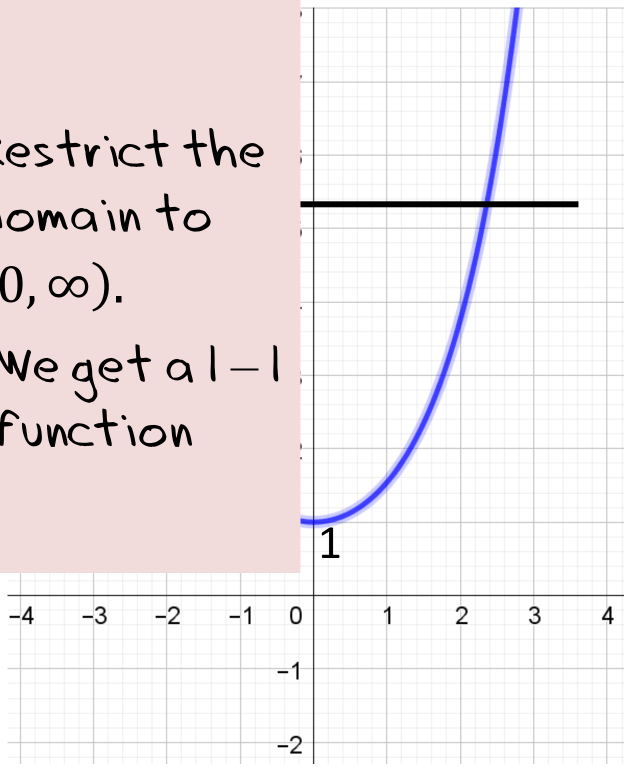
Domain =

Range =

Inverse Hyperbolic Functions:

(2) $y = \cosh^{-1}(x) \Leftrightarrow \cosh y = x \quad y \geq 0$

$y = \cosh x$



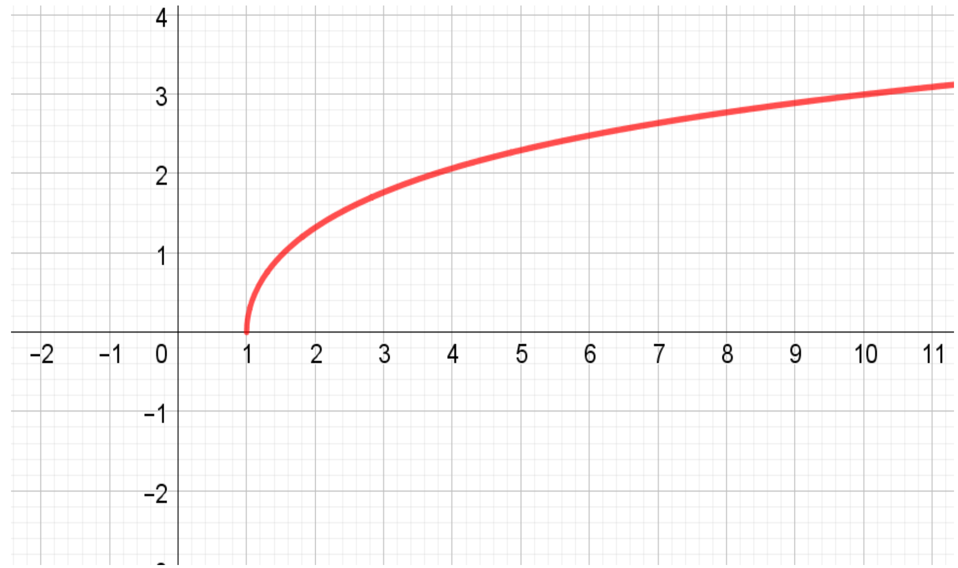
Restrict the domain to $[0, \infty)$.

We get a 1−1 function

Domain $= [0, \infty)$

Range $= [1, \infty)$

$y = \cosh^{-1}(x)$



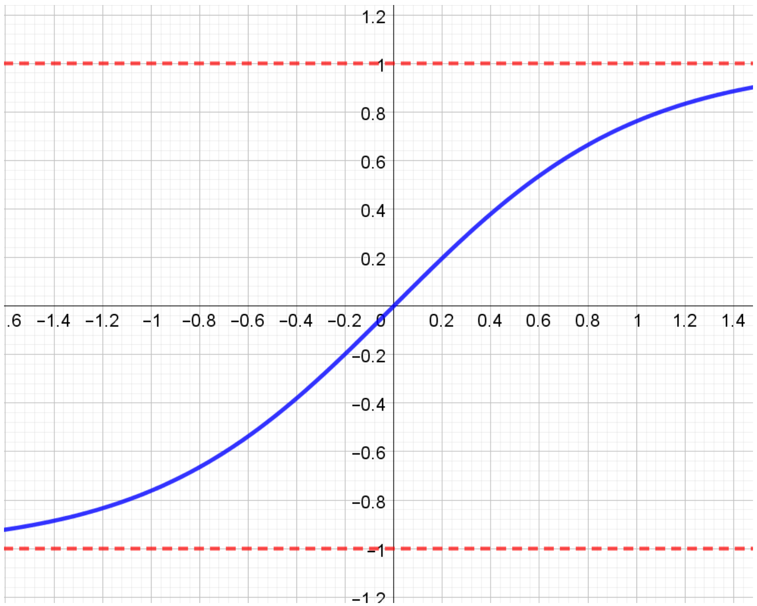
Domain =

Range =

Inverse Hyperbolic Functions:

③ $y = \tanh^{-1}(x) \Leftrightarrow \tanh y = x$

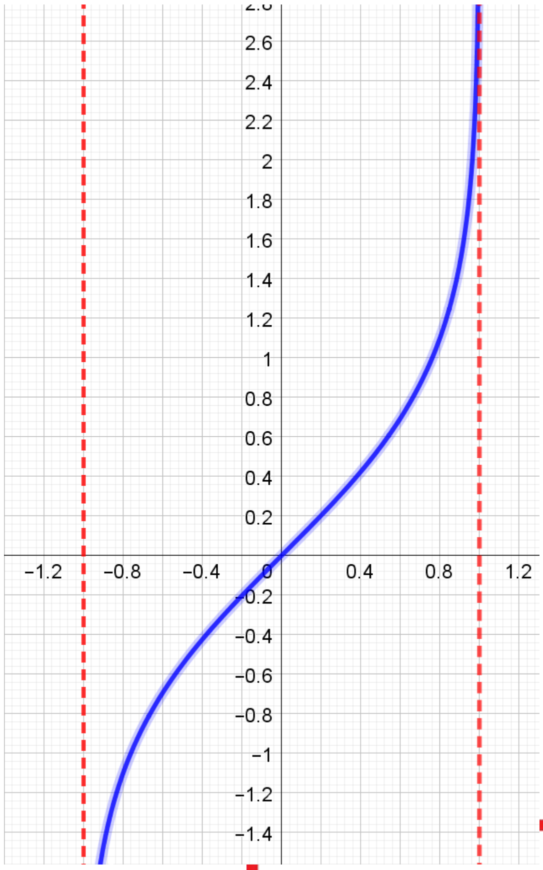
$y = \tanh x$



Domain = $\mathbb{R}$

Range = $(-1,1)$

$y = \tanh^{-1}(x)$



Domain =

Range =

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to see that the inverse hyperbolic functions can be expressed in terms of logarithms. So we have:

① $$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right) \quad : \quad x \in \mathbb{R}$$

② $$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right) \qquad x \in \mathbb{R}$$

③ $$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1$$

Derivatives of Inverse Hyperbolic Functions

① $\frac{d}{dx}\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$

② $\frac{d}{dx}\cosh^{-1}(x) = \frac{1}{\sqrt{x^2-1}}$

③ $\frac{d}{dx}\tanh^{-1}(x) = \frac{1}{1-x^2}$

④ $\frac{d}{dx}\operatorname{csch}^{-1}(x) = -\frac{1}{|x|\sqrt{1+x^2}}$

⑤ $\frac{d}{dx}\operatorname{sech}^{-1}(x) = \frac{-1}{x\sqrt{1-x^2}}$

⑥ $\frac{d}{dx}\coth^{-1}(x) = \frac{1}{1-x^2}$

Example (3):

Show that $sinh^{-1}(x) = ln\left(x + \sqrt{x^2 + 1}\right)$

Solution:

Example (4):

Prove that $\dfrac{d}{dx}\ \sinh^{-1}(x) = \dfrac{1}{\sqrt{1+x^2}}$

Solution:

Recall that:

$$y = \ln(u) \ \Rightarrow \ y' = \frac{1}{\ln(u)}.u'$$

$$y = \sqrt{u} \ \Rightarrow \ y' = \frac{1}{2\sqrt{u}}.u'$$

Example (5):

Find $\frac{d}{dx} \quad tanh^{-1}(sin\ x)$

Solution:

Exercise (41):

Find $\frac{d}{dx} \quad cosh^{-1}(\sqrt{x})$

Solution:

Exercise (42): Find the derivative:

$$y = x \tanh^{-1}(x) + \ln\left(\sqrt{1 - x^2}\right)$$

Solution:

Recall that:

(a) $sin^2(x) + cos^2(x) = 1$

(b) $y = \sqrt{x} \quad \Rightarrow y' = \frac{1}{2\sqrt{x}}$