CALCULUS 110

(2.6) Limits at Infinity ; Horizontal asymptotes

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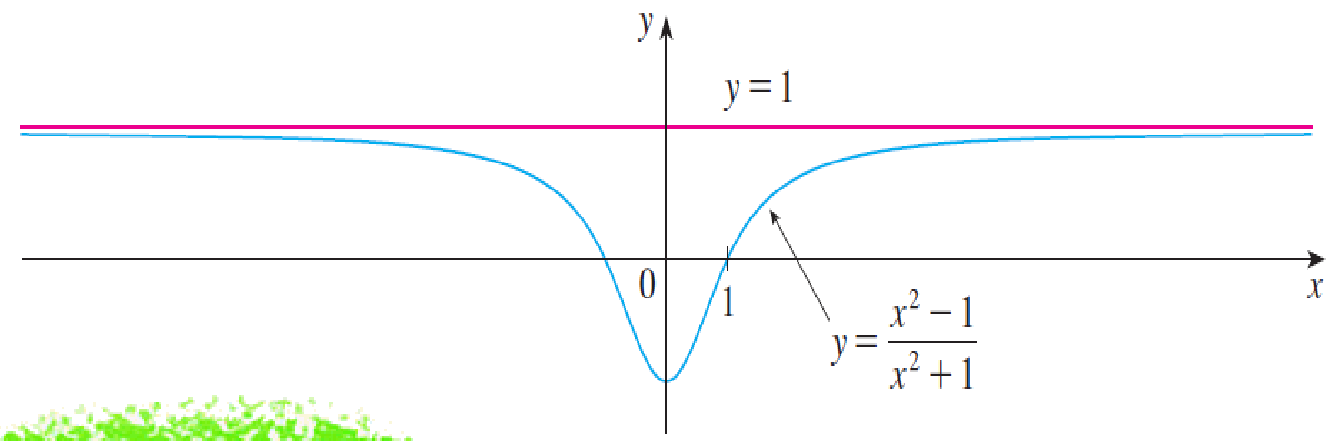
Limits at Infinity; Horizontal asymptotes

In this section we let x become arbitrarily large (positive or negative) and see what happens to y.



$f(x) = \frac{x^2-1}{x^2+1}$

$x \to \infty, f(x) \to \cdots$

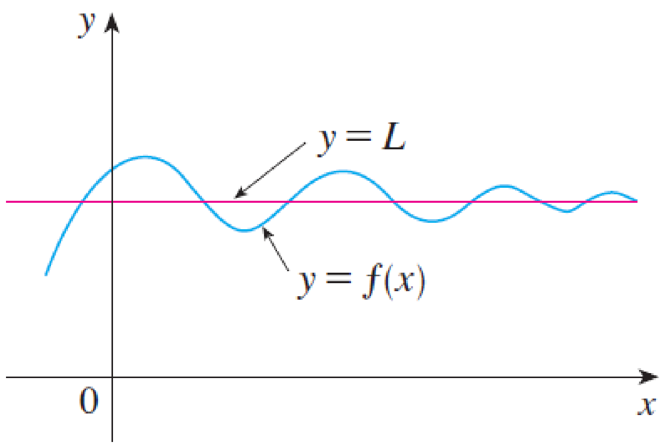
$x \to -\infty, f(x) \to \ldots$

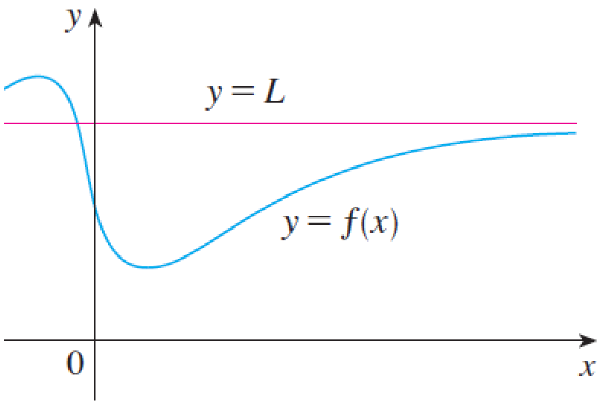
Definition of a Limit at Infinity (1)

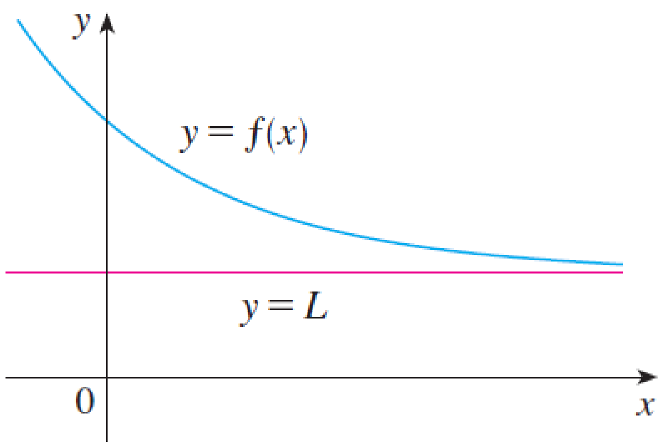
Let f be a function defined on (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.





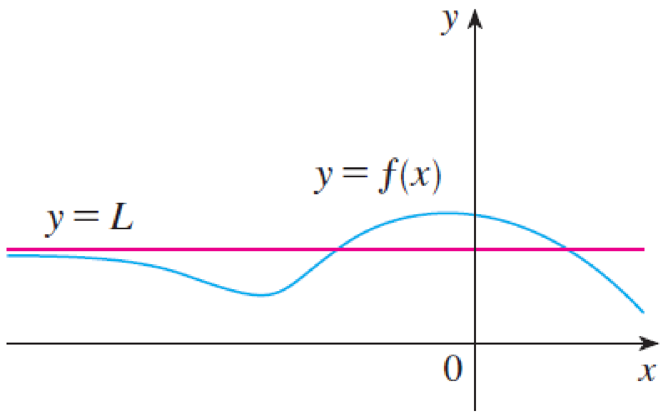


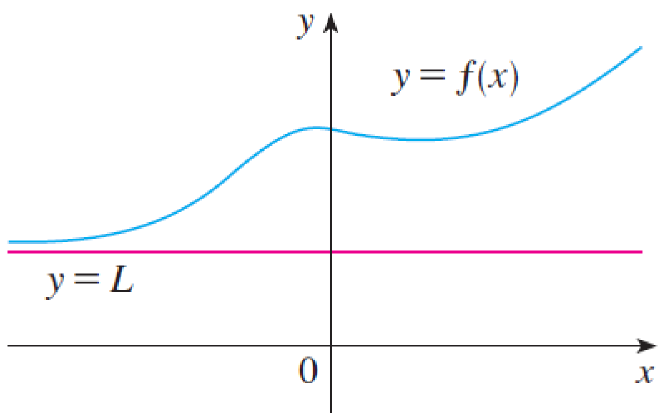
Definition (2)

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

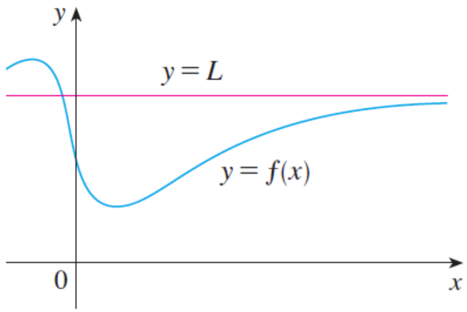
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

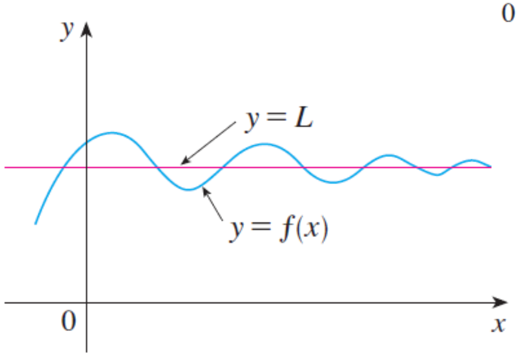


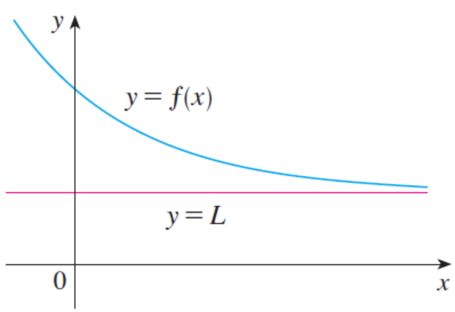


Definition (1)

$$\lim_{x\to\infty} f(x) = L$$







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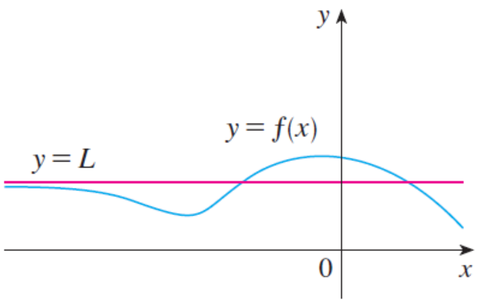
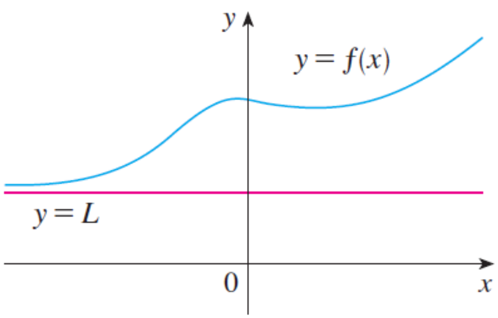




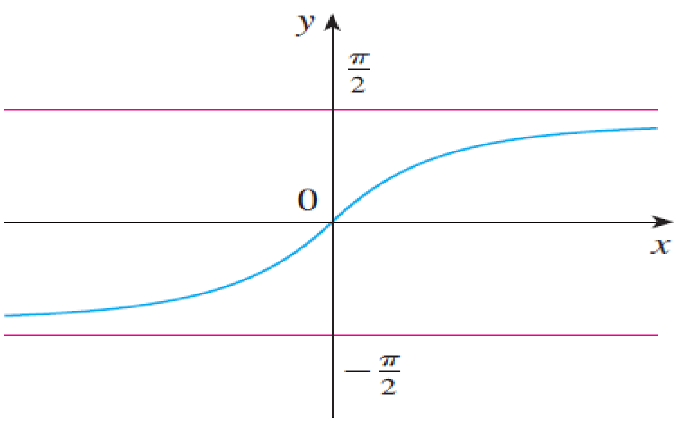
Horizontal Asymptote

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

Theorem 4

See example 6

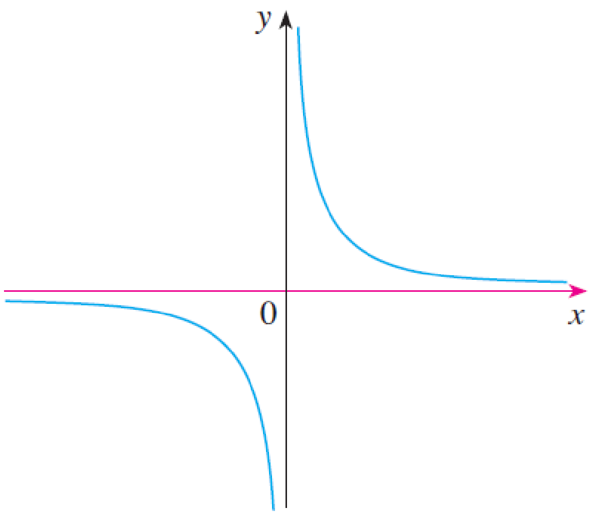
$$\lim_{x\to-\infty} tan^{-1}(x) = \qquad \lim_{x\to\infty} tan^{-1}(x) =$$



Example 2

Find $\lim_{x\to-\infty} \frac{1}{x}$, $\lim_{x\to\infty} \frac{1}{x}$

solution



Example 1

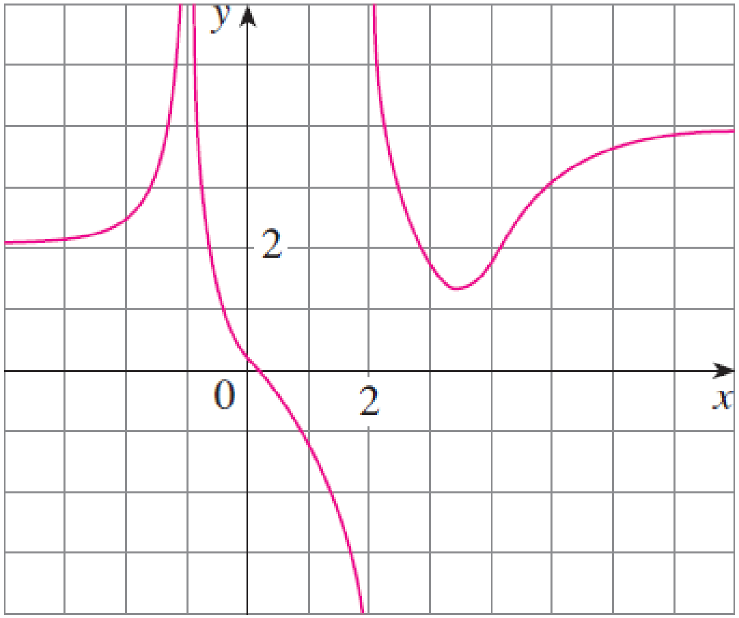
Find the infinite limits, limits at infinity, and asymptotes for the function $f(x)$ whose graph is shown in the following Figure.

solution

(1) $\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x) = \lim_{x\to -1} f(x) =$

(2) $\lim_{x\to 2^-} f(x) =$ ---- -, $\lim_{x\to 2^+} f(x) =$--- -

(3) $\lim_{x\to -\infty} f(x) =$---- -, $\lim_{x\to \infty} f(x) =$--- -



The following Theorems are important rules for calculating limits.

Theorem 5

If $r > 0$ is a rational number, then

$$\lim_{x\to\pm\infty} \frac{1}{x^r} = ---\ -$$

Ex.: $\lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x^{2/3}} = --\ -$

Theorem 6

(a) $\lim_{x\to\infty} x^n = ---\ -$

(b) $\lim_{x\to-\infty} x^n = \begin{cases} --, & n \text{ is even} \\ --, & n \text{ is odd} \end{cases}$

$\lim_{x\to\infty} x^4 = \lim_{x\to\infty} x^8 = --\ -$

$\lim_{x\to-\infty} x^4 = --\ -,$

$\lim_{x\to-\infty} x^3 = --\ -$

Theorem 7

$$\lim_{x\to\pm\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = ----\ -$$

Example 10

$\lim_{x\to\infty} x^2 - x = \infty - \infty$

(the limit laws can't be applied to infinite limits because ∞ is not a number)However apply the above theorem:

Theorem 8

(a) $\lim_{x\to\infty} e^x = ---\ -$

(b) $\lim_{x\to-\infty} e^x = ---\ -$

See example 7

rational function $\frac{P(x)}{Q(x)}$

If $f(x)$ is a rational function $\frac{P(x)}{Q(x)}$ compare the degree of the numerator and denominator.

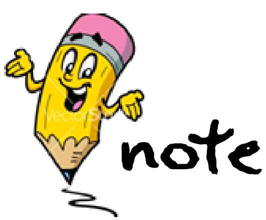
(1) If $\deg P(x) > \deg Q(x)$, then $\lim_{x\to\pm\infty} f(x) =$ - - - -

Study the sign
See example 11

(2) If $\deg P(x) < \deg Q(x)$, then $\lim_{x\to\pm\infty} f(x) =$ - - - -

(3) If $\deg P(x) = \deg Q(x)$, then $\lim_{x\to\pm\infty} f(x) =$ - - - - - - - - - - - - - - -

See example 3

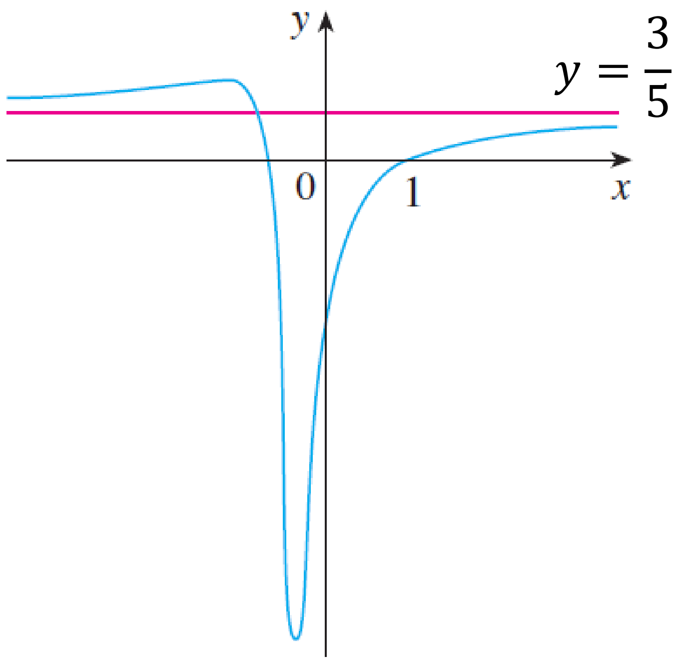


if $f(x) = \frac{H(x)}{W(x)}$, but not rational, we divide every term by the highest power of the quotient. See example 4

Example 3

Evaluate $\lim\limits_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$

solution



Example 11

Evaluate

(a) $\lim\limits_{x\to\infty} \dfrac{x^2 + x}{3 - x}$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2 + x}{3 + x}$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2 + x}{x^5 - 2x^2}$

(d) $\lim\limits_{x\to\infty} \dfrac{3x^5 + x}{4x^5 + 2x^2}$

solution

(a)

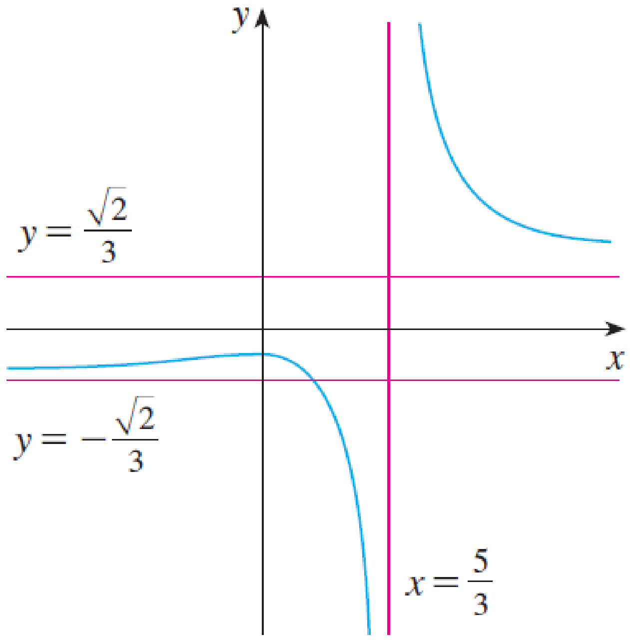
(b)

(c)

(d)

Example 4** Find the horizontal and vertical asymptotes of $f(x) = \dfrac{\sqrt{x^2+1}}{3x-5}$

solution

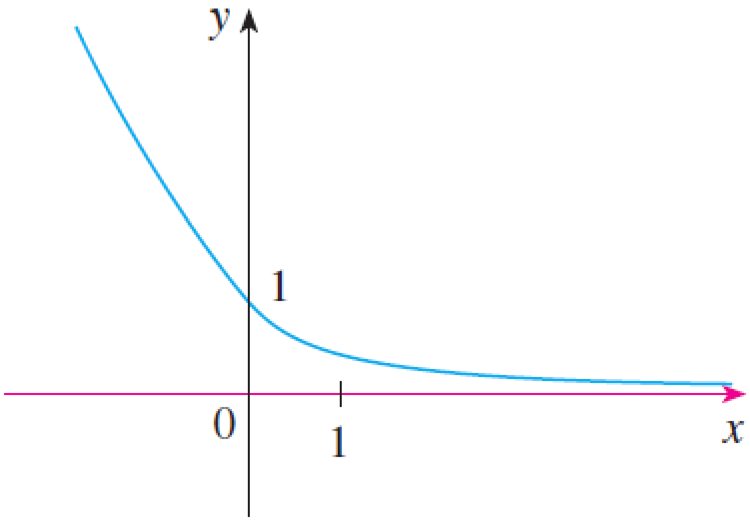


Example 5

Compute $\lim_{x\to\infty} \sqrt{x^2+1} - x$

solution

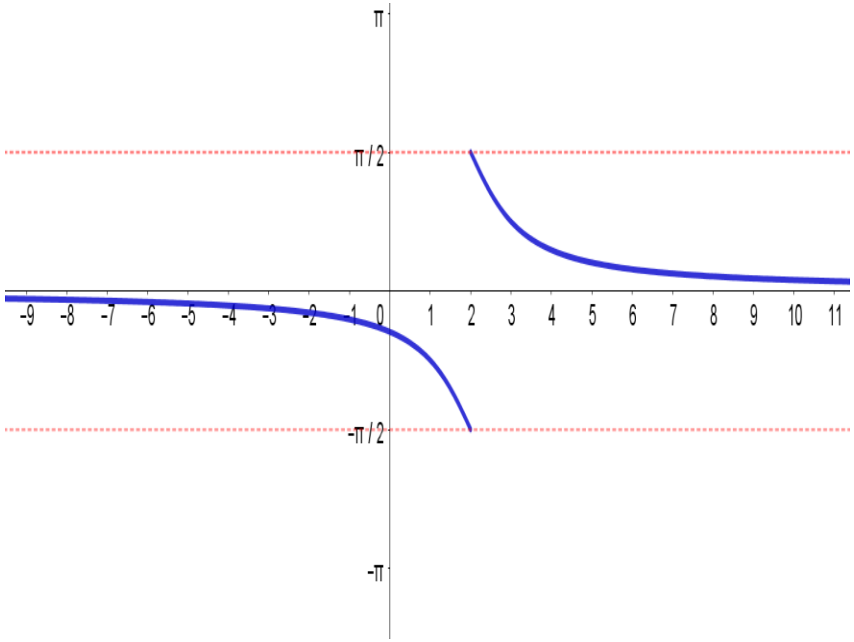
$= \infty - \infty$



Example 6

Evaluate $\lim_{x\to 2^+} arctan\left(\frac{1}{x-2}\right)$

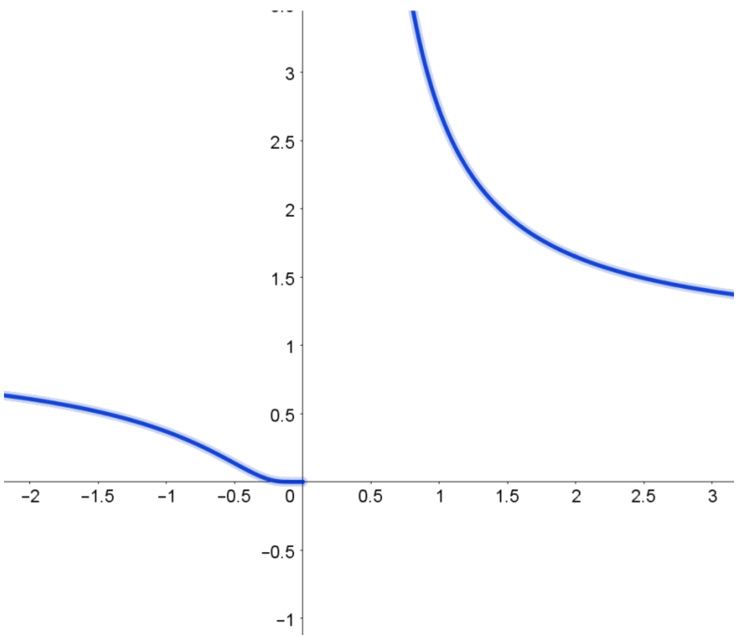
solution



Example7

Evaluate $\lim\limits_{x\to 0^-} e^{1/x}$

solution



Example 8

Find $\lim\limits_{x\to\infty} \sin x$

solution

Infinite Limits at infinity

ⓐ $\lim_{x \to \infty} f(x) = \infty$

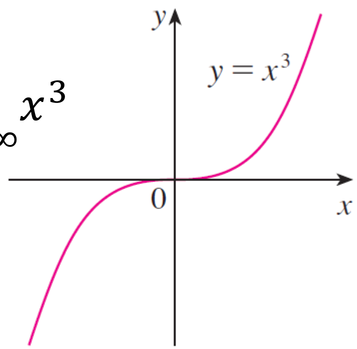
ⓑ $\lim_{x \to \infty} f(x) = -\infty$

ⓒ $\lim_{x \to -\infty} f(x) = \infty$

ⓓ $\lim_{x \to -\infty} f(x) = -\infty$

Example 9

Find $\lim_{x \to \infty} x^3$, $\lim_{x \to -\infty} x^3$

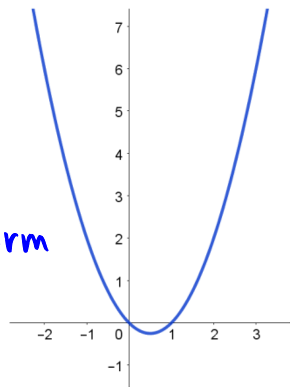


solution

Example 10

$\lim_{x \to \infty} x^2 - x = \infty - \infty$

Indeterminate form

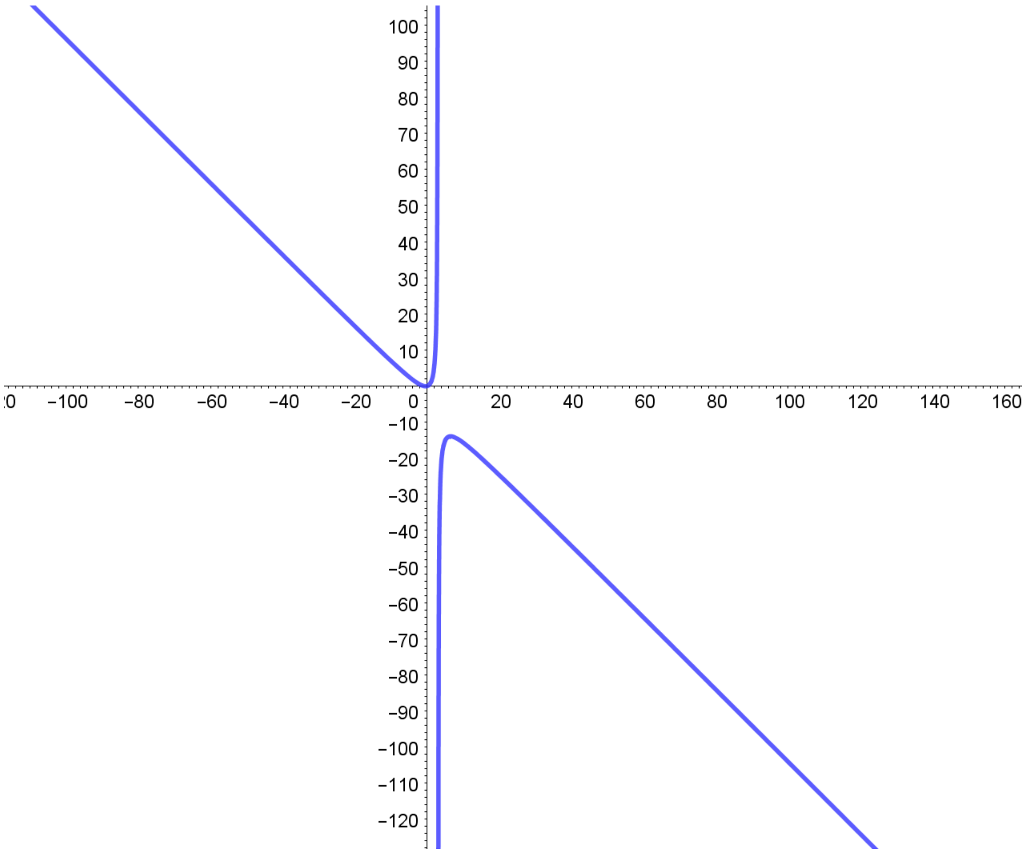


solution

Example 11

Find $\lim_{x\to\infty} \dfrac{x^2+x}{3-x} \quad = \dfrac{\infty}{-\infty}$

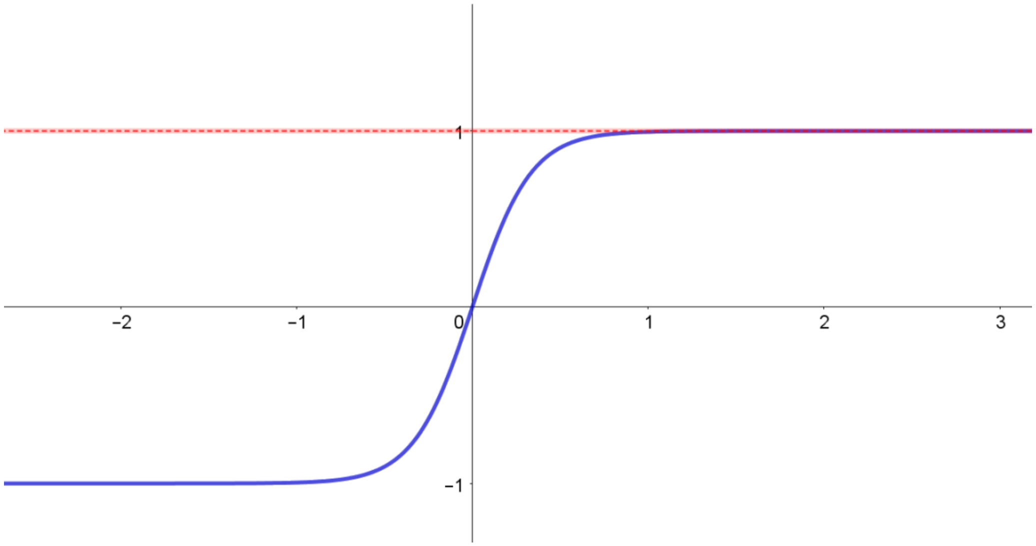
solution



Exercise 36

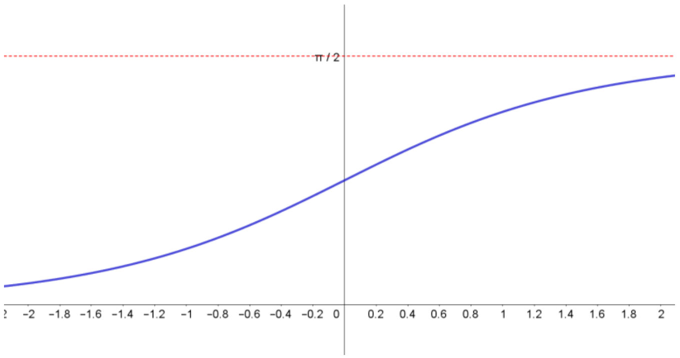
Find the limit or show that it does not exist. $\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \frac{\infty}{\infty}$

solution



Exercise 35

$\lim_{x\to\infty} tan^{-1}(e^{x})$



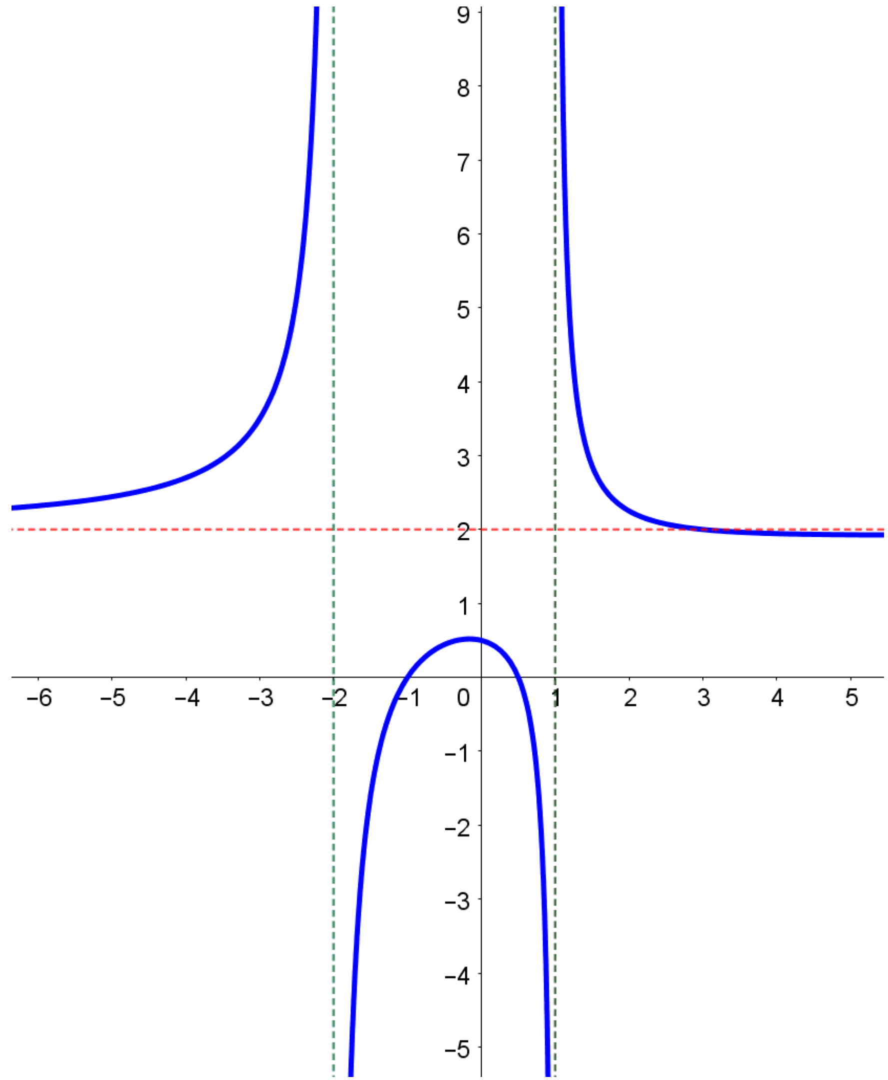
Exercise 49

Find the Horizontal and Vertical Asymptotes of $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$

solution

①

②



19, 30, 35, 37, 50