CALCULUS 110

(2.3) Calculating Limits Using the Limits Laws

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Calculating Limits Using the Limits Laws

In this section we use the following properties of limits, called the Limit Laws, to calculate limits.

Limit Laws Suppose c is a constant and the limits $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist. Then:

(1) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

(2) $\lim_{x\to a} c\, f(x) = c \lim_{x\to a} f(x)$

(3) $\lim_{x\to a} f(x)\, g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

(4) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

(5) $\lim_{x\to a} \left(f(x)\right)^n = (\lim f(x))^n$

(6) $\lim_{x\to c} c = c$

(7) $\lim_{x\to a} x = a$

(8) $\lim_{x\to a} x^n = a^n$ n is +ve

(9) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ n is +ve

(10) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ n is +ve

If n is even, we assume that $\lim_{x\to n} f(x) > 0$

$$\infty + \infty = \infty$$

$$number + \infty = \infty$$

$$number - \infty = -\infty$$

$$number \cdot \infty = \pm\infty$$

$$\frac{0}{\infty} = 0$$

$$\frac{number}{0} = \pm\infty$$

$$\infty \cdot \infty = \infty$$

Indeterminate forms

Example 2

Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$

(b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

solution

(b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

solution

Direct substitution property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions with the direct substitution property are called continuous at a

Example 3

Find $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

solution

Example 4

Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1, & x \neq 1 \\ \pi, & x = 1 \end{cases}$$

solution

Example 5

Evaluate $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h}$

solution

=

Example 6

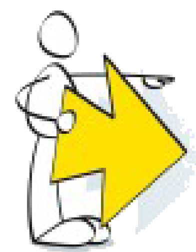
Find $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$

solution

Some limits are best calculated by first finding the left- and right-hand limits.

Theorem (1)

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$



The limit laws hold for one-sided limits.

Example 7

Show that $\lim_{x \to 0} |x| = 0.$

solution

Recall That

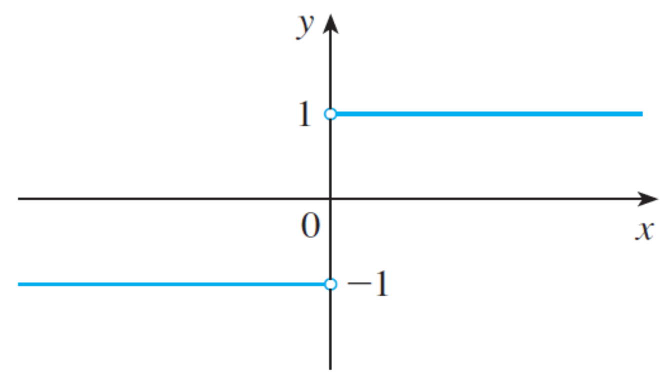
$$|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

What about $\lim_{x \to 2} |x|$??

Example 8

$\lim_{x \to 0} \frac{|x|}{x}$ doesn't exist (T or F.)

solution

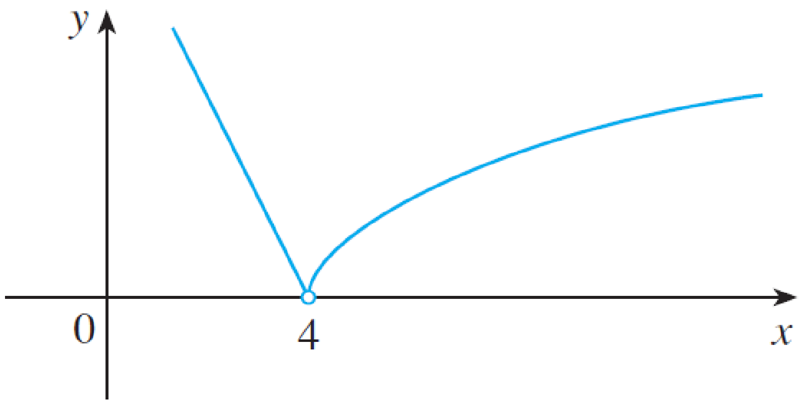


Example 9

Determine whether the following limit exists

$$\lim_{x \to 4} f(x), \quad f(x) = \begin{cases} \sqrt{x-4}, if \;\; x > 4 \\ 8 - 2x, if \;\; x < 4 \end{cases}$$

solution



$$\lim_{x \to 4^+} \sqrt{x-4} = 0$$

$$\lim_{x \to 4^-} 8 - 2x = 0$$

Thus the limit exists and $\lim_{x \to 4} f(x) = 0$.

remark

when do we have to study limit from the left and from the right?

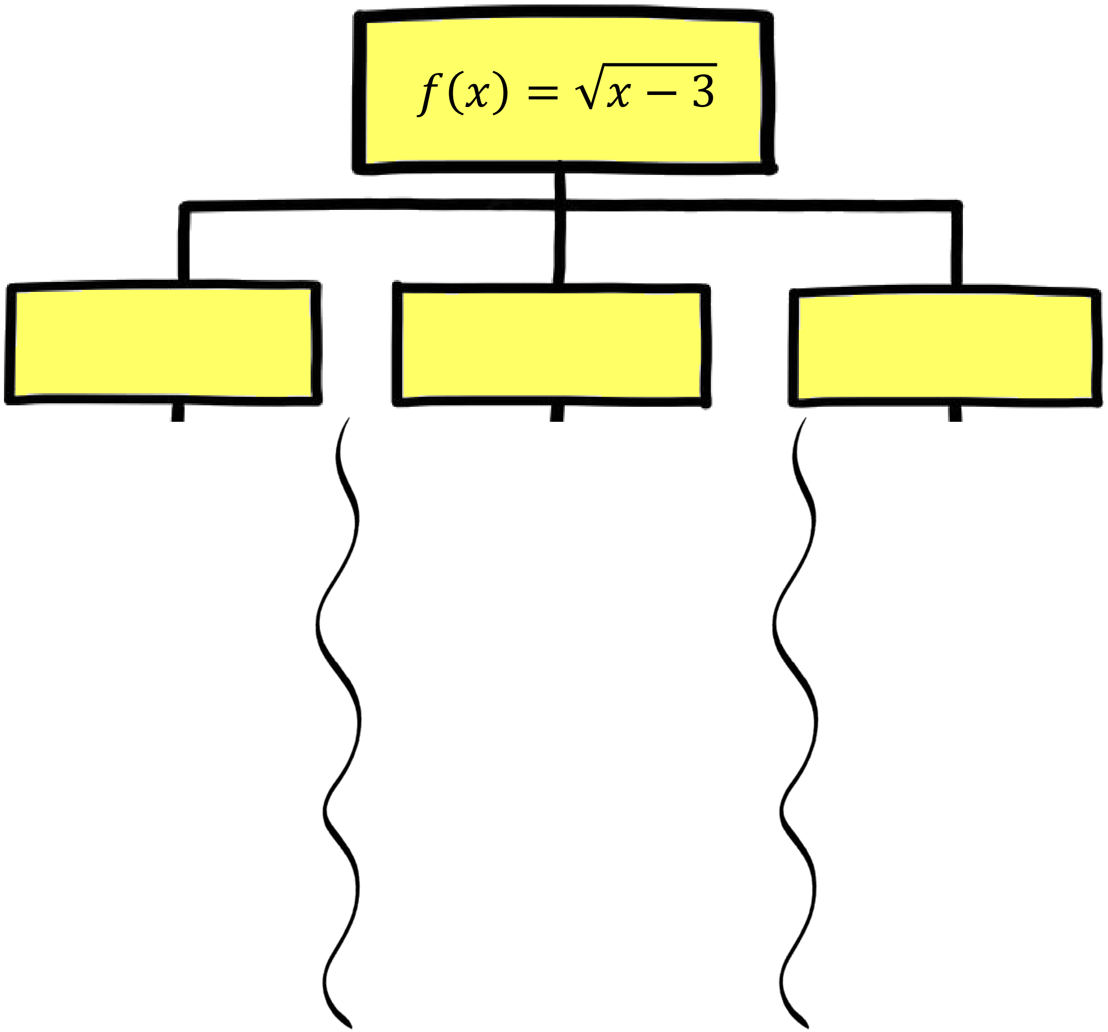
1. If the function is a piecewise defined function and the definition changes around the point. like example 9.

2. a is an end point of an interval of the domain.

Example Find $\lim_{x\to 0}\sqrt{x}$

solution

In fact, we have 3 cases for $\sqrt{f(x)}$



note In case of

$$f(x) = \begin{cases} f_1(x), & x = a \\ f_2(x), & x \neq a \end{cases}$$

We don't need to study limit from the right and from the left.

Theorem (2)

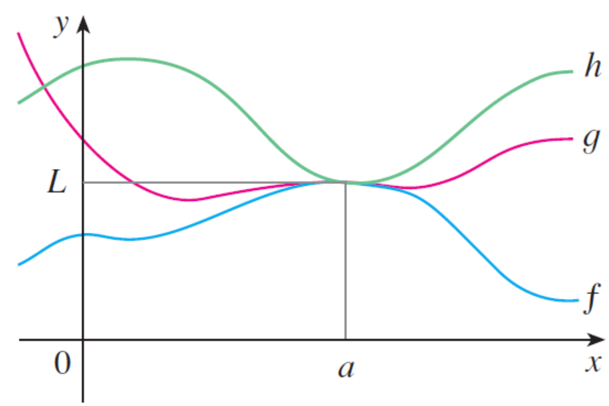
If $f(x) \leq g(x)$, where x is near to a (except possibly at a) and the limits of f and g both exist as $x \to a$, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

Theorem (3) (the squeeze theorem)

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$. Then $\lim_{x \to a} g(x) = L$

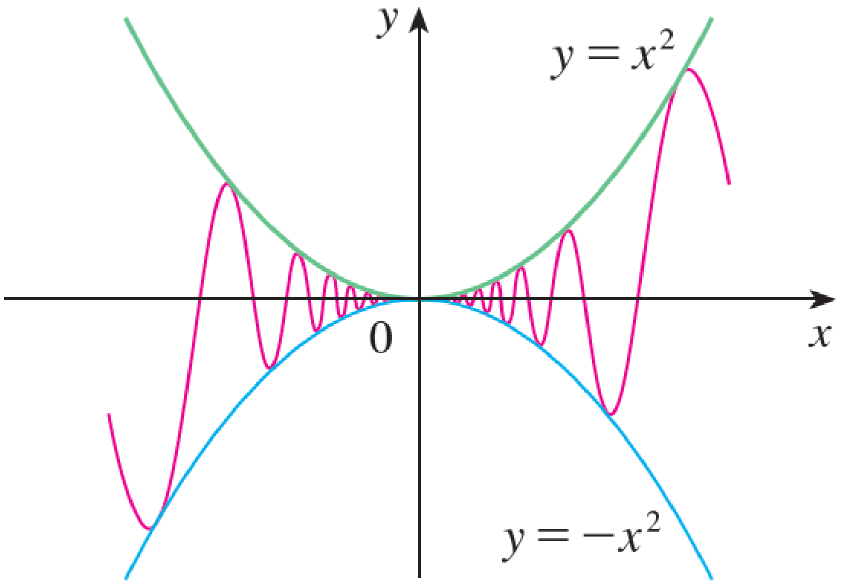


Sandwich Theorem

Example 11

Show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

solution



Evaluate the limit, if it exists.

Exercise 15

$$\lim_{t \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$$

solution

Exercise 23

$$\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$$

solution

Evaluate the limit, if it exists.

Exercise 24

$$\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h}$$

solution

Exercise 29

$$\lim_{t\to 0} \frac{1}{t\sqrt{1+t}} - \frac{1}{t}$$

solution

Exercise 59

If $\lim_{x\to 1} \dfrac{f(x)-8}{x-1} = 10$ .Find $\lim_{x\to 1} f(x)$

solution

(3.3) Limits Of Trigonometric Functions

(1) $\lim_{\theta \to 0} \frac{sin\theta}{\theta} =$ $\lim_{\theta \to 0} \frac{\theta}{sin\theta} =$

(2) $\lim_{\theta \to 0} \frac{tan\theta}{\theta} =$ $\lim_{\theta \to 0} \frac{\theta}{tan\theta} =$

(3) $\lim_{\theta \to 0} \frac{\sin(n\theta)}{m\theta} =$ $\lim_{\theta \to 0} \frac{n\theta}{\sin(m\theta)} =$

(4) $\lim_{\theta \to 0} \frac{\tan(n\theta)}{m\theta} =$ $\lim_{\theta \to 0} \frac{n\theta}{\tan(m\theta)} =$

(5) $\lim_{\theta \to 0} \frac{\sin(n\theta)}{\sin(m\theta)} =$ $\lim_{\theta \to 0} \frac{\tan(n\theta)}{\tan(m\theta)} =$

(6) $\lim_{\theta \to 0} \frac{\sin(n\theta)}{\tan(m\theta)} =$ $\lim_{\theta \to 0} \frac{\tan(n\theta)}{\sin(m\theta)} =$

(7) $\left(\lim_{\theta \to 0} \frac{\sin(n\theta)}{m\theta}\right)^p =$

(8) $\lim_{\theta \to 0} \frac{\tan^p(n\theta)}{\tan^p(m\theta)} =$

(9) $\lim_{\theta \to 0} \frac{cos\theta - 1}{\theta} =$

(10) $\lim_{x \to a} \frac{\sin(x-a)}{(x-a)} =$

Example 5

$$\lim_{x\to 0}\frac{\sin 7x}{4x} = \frac{7}{4}$$

Example 6

Calculate $\lim_{x\to 0} x \cot x$

solution

$$\lim_{x\to 0} x \cot x = 0\cdot\infty$$

$$= \lim_{x\to 0}\frac{x\cos x}{\sin x}$$

$$= \lim_{x\to 0}\frac{x}{\sin x} \quad \lim_{x\to 0}\cos x = 1\cdot 1 = 1$$

OR

$$\lim_{x\to 0} x \cot x = \lim_{x\to 0}\frac{x}{\tan x} = 1$$

Exercise 41

$$\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = \frac{6}{2} = 3$$

Exercise 42

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \frac{0}{0}$$

solution

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0}\frac{\frac{\cos\theta - 1}{\theta}}{\frac{\sin\theta}{\theta}} = \frac{0}{1}$$

Exercise 45

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta + \tan\theta}$$

solution

Exercise 48

$$\lim_{x \to 0} \frac{\sin(x^2)}{x}$$

solution

Exercise 45

$$\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\sin x-\cos x}=\frac{0}{0}$$

solution

Exercise 48

$$\lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2}=\frac{0}{0}$$

solution

homework

12, 19, 20, 22, 25, 27, 31, 32, 35 - 37

Page197: 39