(1.5) Inverse Functions and Logarithms

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CALCULUS 110

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1-1 function

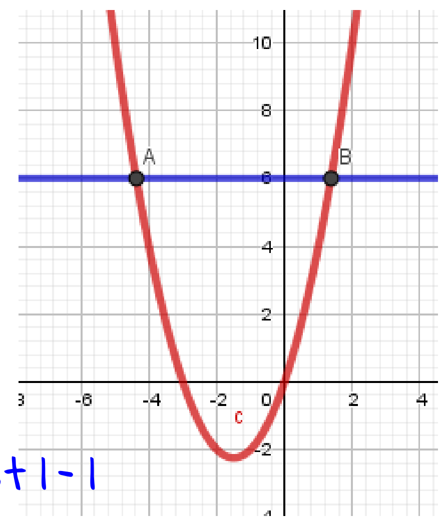
A function f is called a one to one function if it never takes on the same value twice; that is $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

i.e. if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

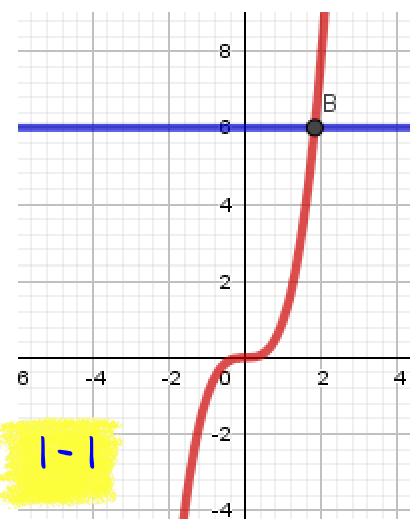
or if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Horizontal Line Test

A function is $1-1 \Leftrightarrow$ no horizontal line intersects its graph more that once.



Not 1-1



1-1

Example 1

(a) Is the function $f(x) = x^3$ one-to-one?

(b) Is $g(x) = x^2$ one $-$ to $-$ one?

solution

Example 2

Use definition to show that

$f(x) = \sqrt[3]{\frac{x+2}{2}}$ is $1-1$.

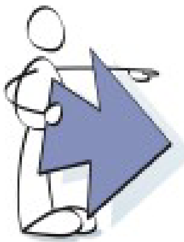
solution

Inverse Function

Let f be a one-to-one function with domain A and range B. Then the inverse function f^{-1} has the domain B and A and is denoted by:

$$f^{-1}: B \to A$$

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$



Don't mistake the (-1) in f^{-1} for an exponent. Thus:

$$f^{-1}(x) \neq \frac{1}{f(x)} = [f(x)]^{-1}$$

remark

If f is not $1-1$, then f^{-1} is not uniquely defined.

Example 3

If $f(1) = 5, f(3) = 7, f(8) = -10$ then $f^{-1}(5) = \cdots, f^{-1}(7) = \cdots, f^{-1}(10) = \ldots\ldots.$

Cancellation Equations: $f: A \to B$

① $f^{-1}(f(x)) = x \quad \forall\, x \in A \;\; i.e. \forall\, x \in D_f$

② $f(f^{-1}(x)) = x \quad \forall\, x \in B \;\; i.e. \forall\, x \in D_{f^{-1}}$

Example

$f(x) = x^3, \; f^{-1}(x) = x^{\frac{1}{3}}$

$f^{-1}(f(x)) = \ldots\ldots\ldots\ldots\ldots\ldots$

How to Find the Inverse Function of a 1 − 1 function

① Write $y = f(x)$

② Solve this equation for x in terms of y if possible.

③ Express f^{-1} as a function of x, interchange x and y.

The resulting equation is $y = f^{-1}(x)$

Example 4

Find the inverse of $f(x) = x^3 + 2$

solution

①

②

③

remark

The graph of f^{-1} is obtained by reflecting the graph of $f(x)$ about the line $y = x$

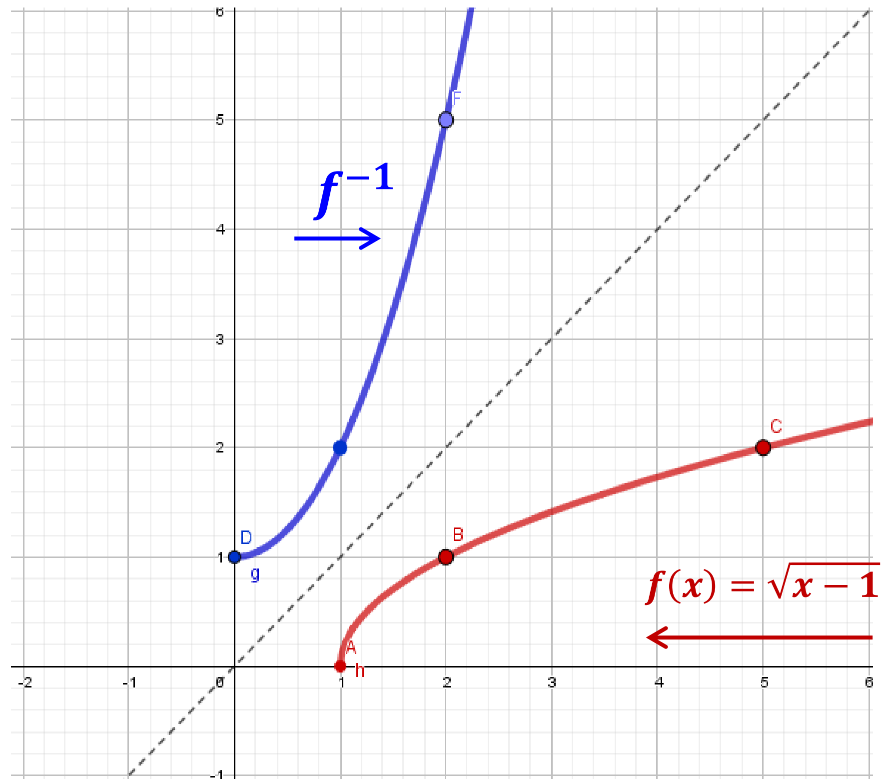
Example 5

Sketch the graph of $f(x) = \sqrt{x-1}$ and its inverse function using the same coordinate axes.

x	1	2	5
y	0	1	2
	(1,0)	(2,1)	(5,2)

The coordinate of f^{-1}:

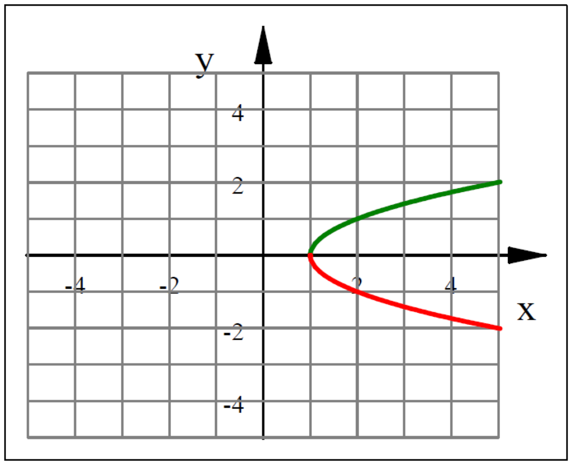
_ _ _ _ _ _ _   _ _ _ _



Example

The figure which represent a graph of a function and it's inverse at the same coordínate axis is

a)



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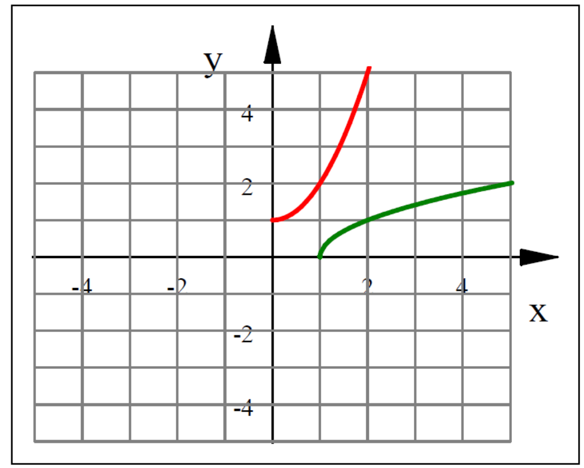
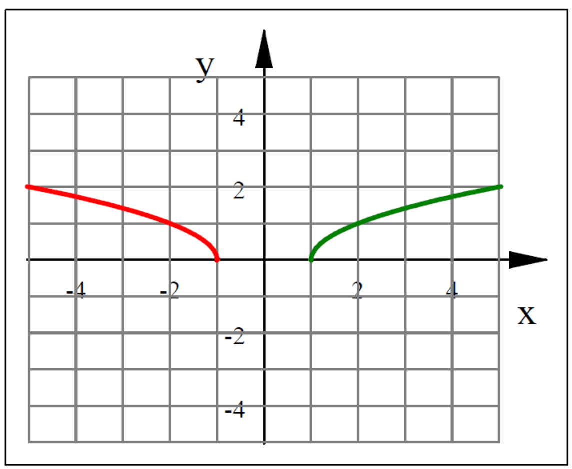
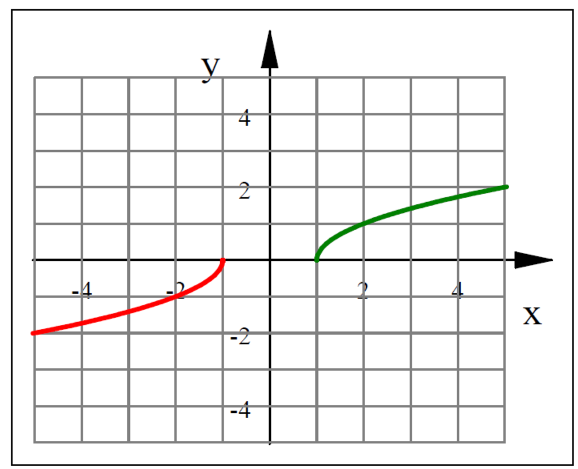


c)



d)



Logarithmic Functions

If $a > 0$, $a \neq 1$, a^x $\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$

and so it is $1-1$ by Horizontal Line Test.

So it has an inverse function f^{-1} which is called the logarithmic function with base a and is denoted by log_a

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$
$$log_a(x) = y \Leftrightarrow x = a^y$$

note

$log_a(x)$: is the exponent to which the base a must be raised to give x.

$log_2 8 = --$

Cancellation Equations:-

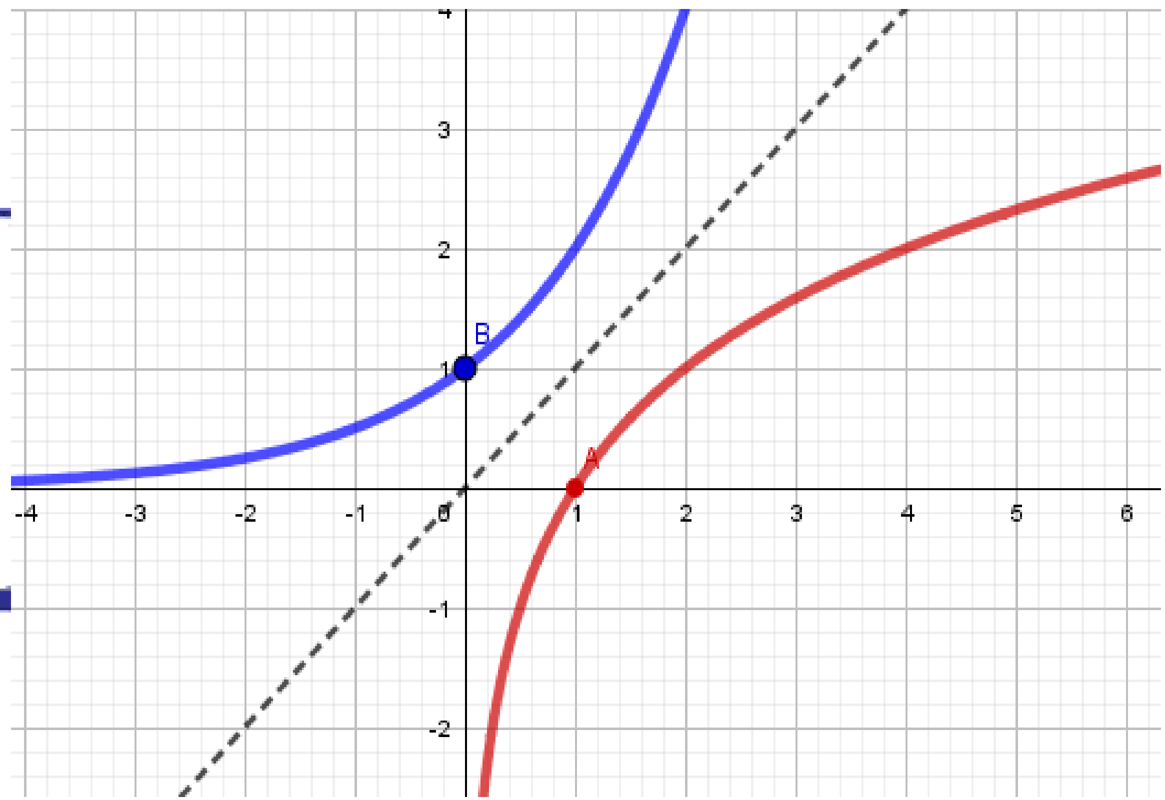
1. $log_a(a^x) = x \quad \forall \quad x \in \mathbb{R}$
2. $a^{log_a x} = x \quad \forall \quad x > 0$

Graph of $log_a(x)$ when $a > 1$

$D =$

$R =$

Reflect about



Law of Logarithms

If x and y are positive numbers, then:

1. $log_a(xy) = log_a(x) + log_a(y)$
2. $log_{10}(x) = log(x)$
3. $log_a 1 = 0$ $\quad$ $log_a a = 1$
4. $log_a x^r = r\, log_a x$ $\quad$ $\boxed{log_a a^r = r}$
5. $log_a \frac{x}{y} = log_a(x) - log_a(y)$

Examples

1. $log_2 2^2 =$ __ , $\quad$ $log_4 16 =$ ___
2. $log_3 \frac{1}{9} =$ __
3. $log_{10} 1000 =$ ----
4. $log_{10} 0.001 =$ -----
5. $log_9 3 =$ ----

Example Evaluate $log_2 80 - log_2 5$

Natural Logarithms $ln(x)$

The logarithm with base e is called natural logarithm and it has a special notation

$$y = log_e x = ln\, x \quad ,$$
$$y = ln\, x \;\Rightarrow\; x = e^y$$

Cancelation Equations

1. $ln(e^x) = x \quad \forall \quad x \in \mathbb{R}$
2. $e^{ln\, x} = x \quad \forall \quad x > 0$

$ln\, e = --$
$ln\, 1 = --$

Example 7 Find x if $ln\, x = 5$

Example 8

Solve the equation $e^{5-3x} = 10$

solution

Example 9

Express $ln\, a + \frac{1}{2} ln\, b$ as a single logarithm.

solution

Change of Base Formula

For any positive number a $(a \neq 1)$ we have:

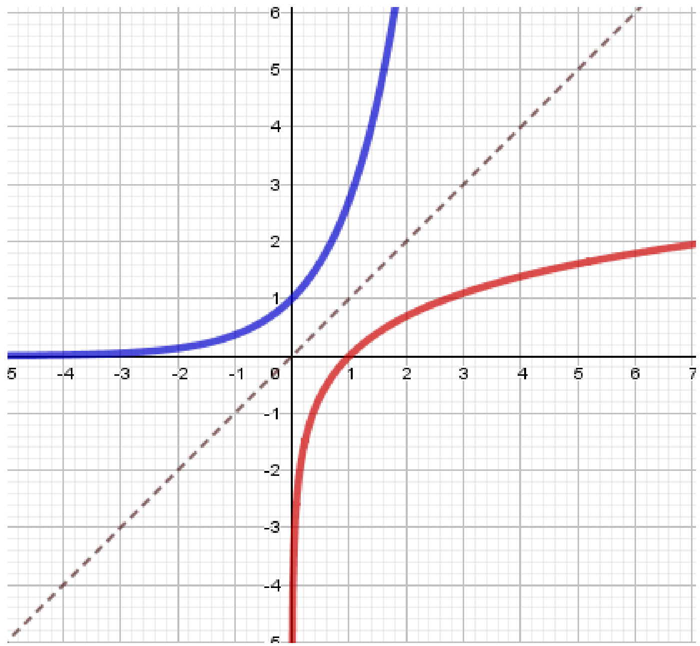
$$log_a x = \frac{lnx}{lna}$$

Example 10

Evaluate $log_8 5$

solution

Graph and growth of the natural logarithm $ln\, x$



$D =$

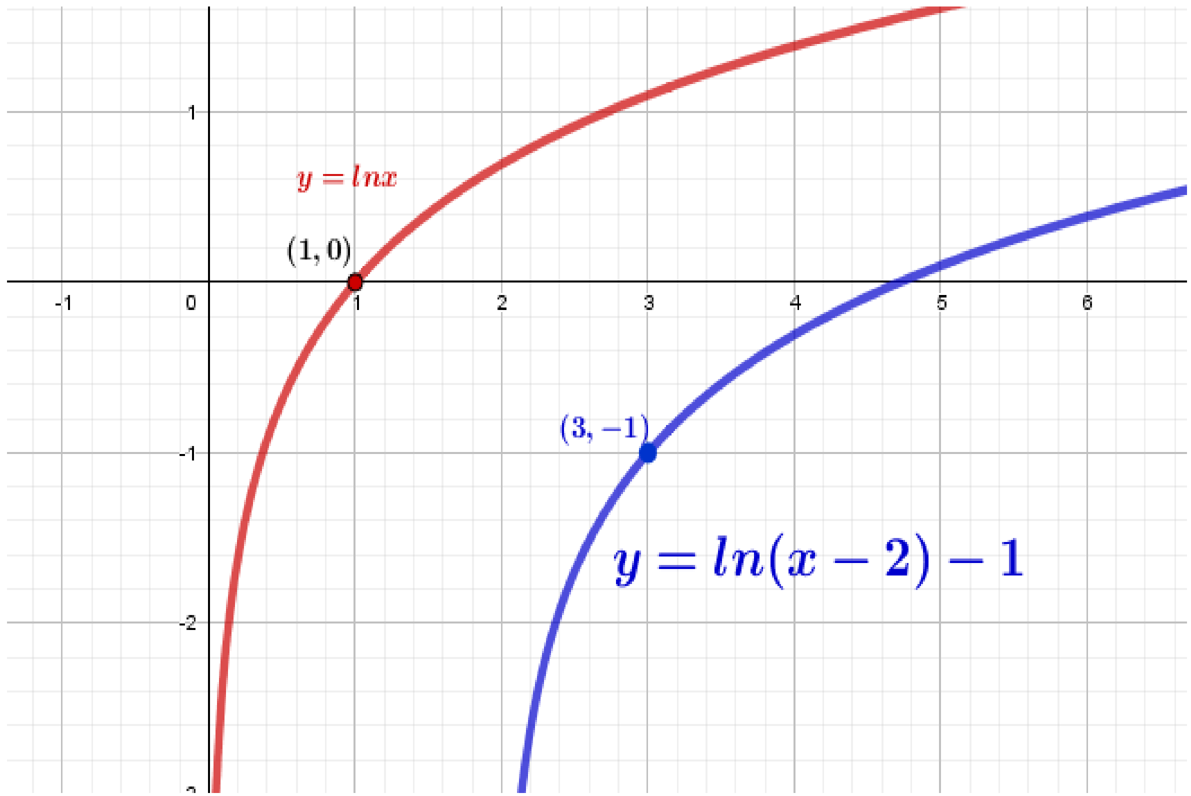
$R =$

- - - - - - function.

Example ll

Sketch the graph of $ln(x-2)-1$

solution



Domain=

Range =

Inverse Trigonometric Functions

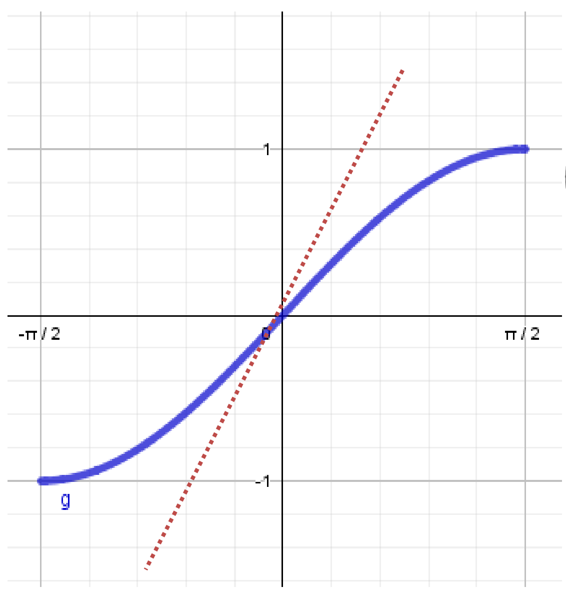
① $f(x) = sin\, x$

From the graph, $sin\, x$ is not $1-1$ function by the horizontal line test, but if we restrict the domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then it will be $1-1$ and we can define an inverse function $sin^{-1}\, x$.

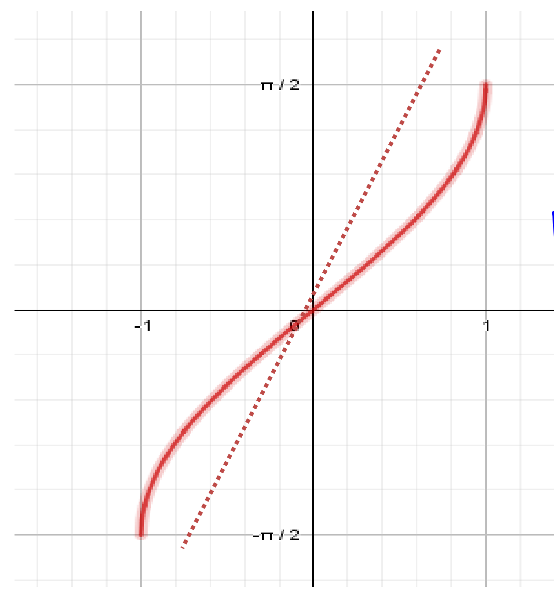
$\boldsymbol{sin\, x}$

Domain=

Range=

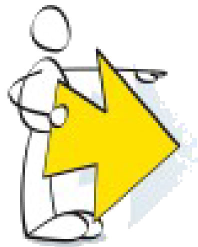


$\boldsymbol{sin^{-1}\, x}$



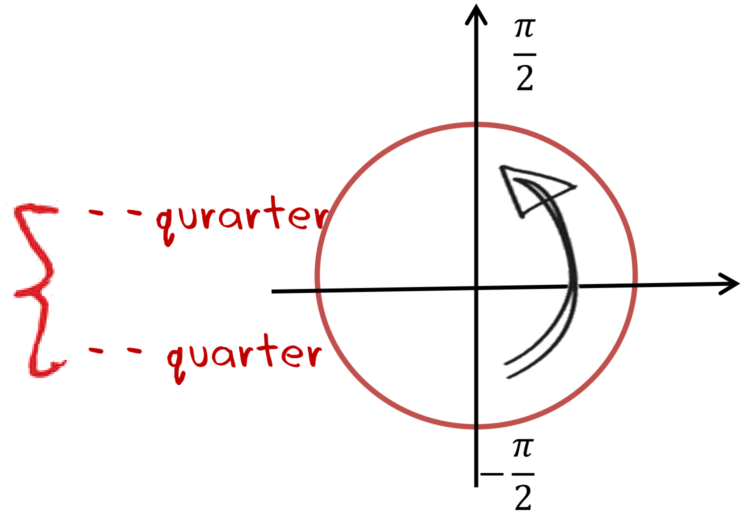
Domain=

Range=



$$\sin^{-1} x \neq \frac{1}{\sin x}$$

$-\frac{\pi}{2} \leq \sin^{-1}(x) \leq \frac{\pi}{2}$, therefore $\sin^{-1} x$ is either in the



Cancelation Equations

① $\sin^{-1}(\sin x) = x \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

② $\sin(\sin^{-1} x) = x \qquad -1 \leq x \leq 1$

$$\sin^{-1} x = y \iff \sin y = x$$

$$-1 \leq x \leq 1 \ , \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Example

Find D_f , $f(x) = \sin^{-1}(x-1)$

solution

Example 12

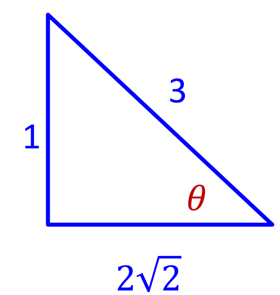
Evaluate the following:

(a) $sin^{-1}\left(\frac{1}{2}\right)$

solution

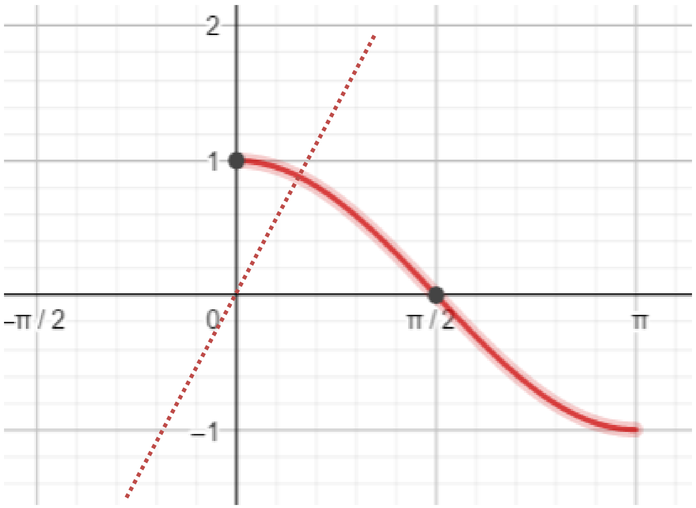
(b) $sin^{-1}\left(-\frac{1}{2}\right)$

(c) $tan\left(sin^{-1}\frac{1}{3}\right) =$



② $f(x) = \cos x$

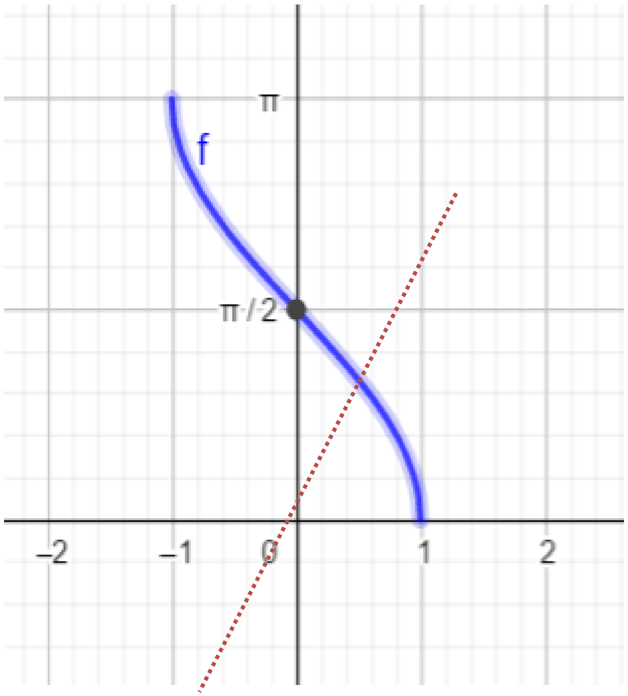
$\cos x$



$D =$

$R =$

$\cos^{-1} x$

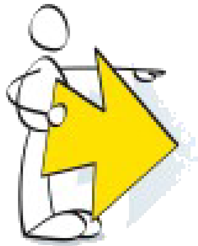


$D =$

$R =$

$$\cos^{-1} x = y \Leftrightarrow \cos y = x$$

$$-1 \le x \le 1,\ 0 \le y \le \pi$$



$$\cos^{-1} x \neq \frac{1}{\cos x}$$

$\cos^{-1} x$ is either in the { - - qurarter / - - quarter

Cancelation Equations

1. $\cos^{-1}(\cos x) = x \quad , \quad 0 \leq x \leq \pi$
2. $\cos(\cos^{-1} x) = x \quad , \quad -1 \leq x \leq 1$

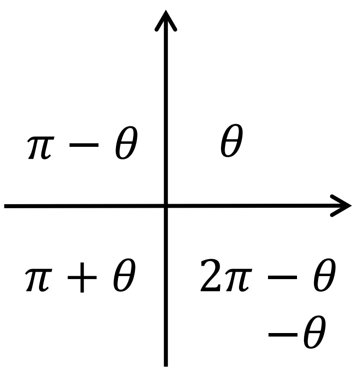
Example

(a) $\cos^{-1}(\frac{1}{2})$

solution

(b) $\cos^{-1}(-\frac{1}{2})$

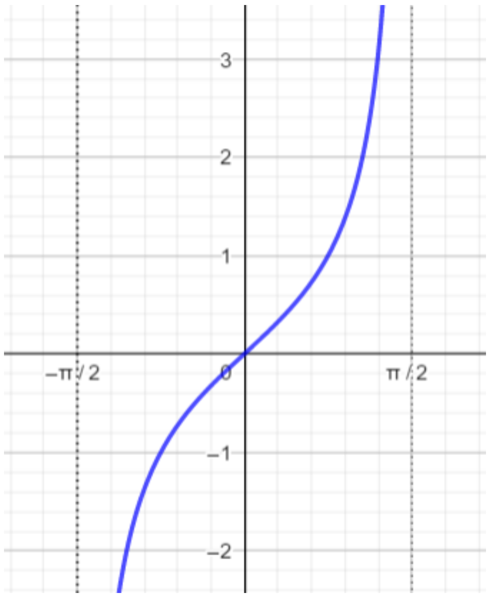
solution

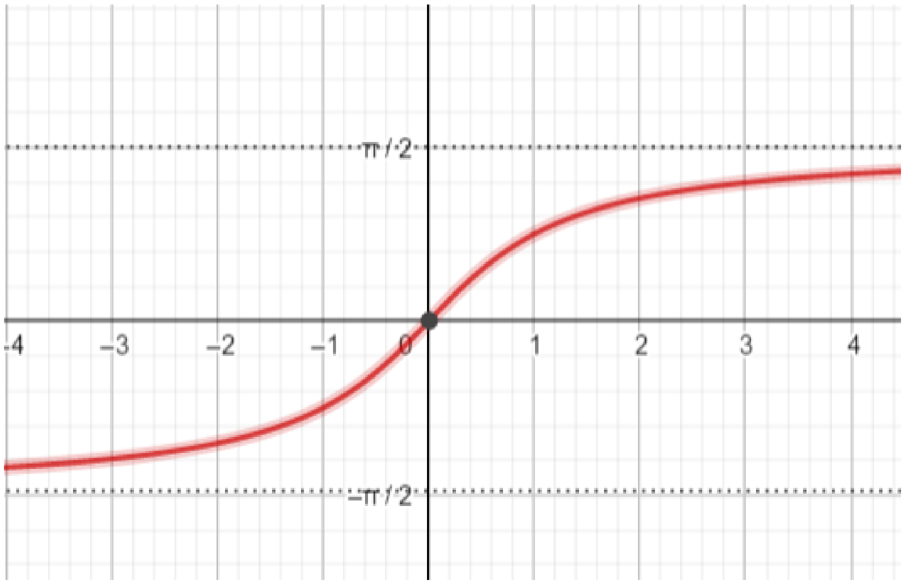


③ $f(x) = tan\, x$

$\boldsymbol{tan\, x}$ is not $1-1$, so we restrict the domain to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\boldsymbol{tan^{-1}\, x}$





$D =$

$R =$

$D =$

$R =$

$$y = tan^{-1} x \Leftrightarrow tan\, y = x$$

$$x \in (-\infty, \infty) \;,\; -\frac{\pi}{2} < y < \frac{\pi}{2}$$

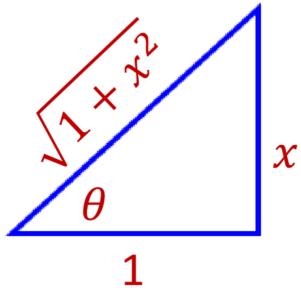
- - qurarter

- - quarter

Example

Simplify the expression $\cos(\tan^{-1} x)$.

solution



$tan^{-1}(1)$=

$tan^{-1}(-1) =$

Cancelation Equations

① $tan^{-1}(tanx) = x \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

② $tan(tan^{-1}x) = x \quad x \in \mathbb{R}$

$tan^{-1}x$ is either in the { 1st qurarter / 4th quarter

Exercise 22

Find the formula for the inverse of the function

$$f(x) = \frac{4x - 1}{2x + 3}$$

Exercise 23

$$f(x) = e^{2x-1}$$

Exercise 37(b)

Find the exact value of

$$log_8 60 \; - log_8 3 \; - log_8 5$$

Exercise 40

Express $ln(\,b) + 2\ln(c) - 3lnd$ as a single logarithm.

Exercise 48(a)

Sktech $\ln(-x)$

Exercise 53(a)

$2^{x-5} = 3$

Exercise 51

Solve each equation for x:

(a) $e^{7-4x} = 6$

(b) $log_5(3x - 10) = 2$

(c) $\ln(3x - 10) = 2$

Solution

Exercise 57

(a) Find the domain of $\ln(e^x - 3)$

(b) Find f^{-1} and its domain

Exercise 64

Find the exact value of each expression

(a) $tan^{-1}(\sqrt{3})$

(b) $tan^{-1}(-1)$

Exercise

(a) $sin^{-1}(1)$

$cos^{-1}(-1)$

(b) $cos^{-1}(1)$

$cos^{-1}(0)$

homework

21 - 26(odd) , 35 - 41(odd), 52