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On the Existence and Uniqueness of Solutions for Q-Fractional Boundary Value Problem

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Abstract

We discuss in this paper the existence and uniqueness of solutions for boundary value problem

$$ {}_cD_q^{\alpha}u(t) = f\left(t, u(t)\right), $$
$$ a\,u(0) + b\,u(T) = c, $$

in a Banach space. Under certain conditions on f, the existence of solutions is obtained by applying Banach fixed point theorem and Schaefer's fixed point theorem.

Keywords: Q-differential equation; Caputo fractional q-derivative; Fractional q-integral; Existence solution; Fixed point theorem

1. Introduction

Fractional calculus is a discipline to which many researchers are dedicating their time, perhaps because of its demonstrated applications in various fields of science and engineering [16]. In particular, the existence of solutions to fractional boundary value problems is currently under strong research[3].

The q-difference calculus or quantum calculus is an old subject that was initially developed by Jackson [9,10], Basic definitions and properties of q-difference calculus can be found in the book [11].

The fractional q-difference calculus had its origin in the works by Al-Salam [2] and Agarwal [1]. More recently, maybe due to the explosion in research within the fractional differential calculus setting, new developments in this theory of fractional q-difference calculus were made, e.g., q-analogues of the integral and differential fractional operators properties such as Mittage-Leffler function [17] , just to mention some.

Very recently some basic theory for the initial value problems of fractional differential equations involving Riemann-Liouville differential operator has been discussed by Lakshmikantham and Vatsala [12,13]. Some existence results were given for the problem (1)-(2) with $q=1$ by [14] and $q=1, \alpha=1$ by Tisdell in [19].

In this paper, we present existence results for the problem

$$ {}_cD_q^{\alpha}u(t)=f\left(t,u(t)\right), \text{ for each } \quad t\in I=\left[0,T\right], \quad 0<\alpha<1, \quad 0<q<1, \tag{1} $$

$$ a\,u(0)+b\,u(T)=c, \tag{2} $$

where ${}_cD_q^{\alpha}$ is the Caputo fractional q-derivative, $f:\left[0,T\right]\times\mathbb{R}\to\mathbb{R}$, is a continuous function, $a,b,c,$ are real constants with $a+b\neq 0.$ In Section 3, we give two results, one based on Banach fixed point theorem (Theorem 3.1) and another one based on Schaefer's fixed point theorem (Theorem 3.2).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C\left(I,\mathbb{R}\right)$ we denote the Banach space of all continuous functions from I into $\mathbb{R}$ with the norm

$$ \|u\|_{\infty}:=\sup\left\{|u(t)|:t\in I\right\}. $$

Let $q \in (0,1)$ defined by [11]

$$[a]_q = \frac{q^a - 1}{q - 1} = q^{a-1} + \ldots + 1, \quad a \in \mathbb{R}.$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}$ is

$$(a-b)^0 = 1 \quad , \qquad (a-b)^n = \prod_{k=0}^{n-1}(a - bq^k), \quad a,b \in \mathbb{R}, \quad n \in \mathbb{N}.$$

More generally, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^{\alpha} \prod_{i=0}^{\infty} \frac{(a - bq^i)}{(a - bq^{\alpha+i})}.$$

Note that, if $b = 0$ then $a^{(\alpha)} = a^{\alpha}$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}, \ 0 < q < 1,$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function $f(x)$ is here defined by

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x},$$

and q-derivatives of higher order by

$$D_q^n f(x) = \begin{cases} f(x) & if \quad n = 0, \\ D_q D_q^{n-1} f(x) & if \quad n \in \mathbb{N}. \end{cases}$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, \quad 0 \le |q| < 1, \quad x \in [0, b].$$

If $a \in [0, b]$ and f defined in the interval $[0, b]$, its integral from a to b is defined by

$$\int_a^b f(t)\,d_qt = \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt.$$

Similarly as done for derivatives, it can be defined an operator I_q^n, namely,

$$(I_q^0 f)(x) = f(x) \quad \text{and} \quad (I_q^n f)(x) = I_q(I_q^{n-1} f)(x), \quad n \in \mathbb{N}.$$

The fundamental theorem of calculus applies to these operators I_q and D_q, i.e.,

$$(D_q\, I_q\, f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q\ f)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [11]. We now point out three formulas that will be used later (${}_iD_q$ denotes the derivative with respect to variable i) [6]

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)},$$

$${}_tD_q(t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)},$$

$$\left({}_xD_q \int_0^x f(x,t)\,d_qt\right)(x) = \int_0^x {}_xD_q f(x,t)\,d_qt + f(qx,x).$$

Remark 2.1. [6] We note that if $\alpha > 0$ and $a \le b \le t$, then $(t-a)^{(\alpha)} \ge (t-b)^{(\alpha)}$.

Definition 2.1.[18] Let $\alpha \ge 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann–Liouville type is $({}_{RL}I_q^0 f)(x) = f(x)$ and

$$({}_{RL}I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)^{(\alpha-1)} f(t)\,d_qt, \quad \alpha \in \mathbb{R}^+, x \in [0,1].$$

Definition 2.2.[18] The fractional q-derivative of the Riemann–Liouville type of order $\alpha \ge 0$ is defined by $({}_{RL}D_q^0 f)(x) = f(x)$ and

$$({}_{RL}D_q^\alpha f)(x) = (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f)(x), \quad \alpha > 0,$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Definition 2.3.[18] The fractional q-derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$({}_C D_q^{\alpha} f)(x) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(x), \quad \alpha > 0,$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Lemma 2.1.[18] Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then, the next formulas hold:

1. $(I_q^{\beta} I_q^{\alpha} f)(x) = (I_q^{\alpha+\beta} f)(x),$
2. $({}_C D_q^{\alpha} I_q^{\alpha} f)(x) = f(x).$

Theorem 2.1.[18] Let $\alpha > 0$ and p be a positive integer. Then, the following equality holds:

$$({}_{RL} I_q^{\alpha}\, {}_{RL} D_q^{p} f)(x) = (D_q^{p} I_q^{\alpha} f)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^{k} f)(0).$$

Theorem 2.2.[18] Let $x > 0$ and $\alpha \in \mathbb{R}^{+} \setminus \mathbb{N}$. Then, the following equality holds:

$$(I_q^{\alpha}\, {}_C D_q^{\alpha} f)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{x^{k}}{\Gamma_q(k+1)} (D_q^{k} f)(0).$$

3. Existence of solutions

Let us start by defining what we mean by a solution of the problem (1)-(2).

Definition 3.1.[14] A function $u \in C^{1}([0,T], \mathbb{R})$ is said to be a solution of (1)-(2) if u satisfies the equation ${}_c D_q^{\alpha} u(t) = f(t, u(t))$ on I, and the condition $a\,u(0) + b\,u(T) = c$.

For the existence of solutions for the problem (1)-(2), we need the following auxiliary lemma.

Lemma 3.1.[14] Let $0 < \alpha < 1$, $0 < q < 1$ and let $y : [0,T] \to \mathbb{R}$ be continuous. A function u is a solution of fractional q-integral equation

$$u(t) = u_0 + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} y(s)\, d_q s$$

if and only if u is a solution of the initial value problem for the fractional q-differential equation

$$ {}_cD_q^\alpha u(t) = y(t), \quad t \in [0,T],$$
$$u(0) = u_0.$$

As a consequence of lemma 3.1 we have the following result which is useful in what follows.

Lemma 3.2.[14] Let $0 < \alpha < 1$, $0 < q < 1$ and let $y : [0,T] \to \mathbb{R}$ be continuous. A function u is a solution of the fractional q-integral equation

$$u(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} y(s)\, d_q s - \frac{1}{a+b} \left[\frac{b}{\Gamma_q(\alpha)} \int_0^T (T - qs)^{(\alpha-1)} y(s)\, d_q s - c \right]$$

if and only if u is a solution of the fractional BVP

$$ {}_cD_q^\alpha u(t) = y(t) \quad , \quad t \in [0,T],$$
$$a\, u(0) + b\, u(T) = c.$$

Our first result is based on Banach fixed point theorem.

Theorem 3.1.[18] Assume that:

(H1) There exists a constant $K > 0$ such that

$\left| f(t,u_1) - f(t,u_2) \right| \le K \left| u_1 - u_2 \right|$, for each $t \in I$, and all $u_1, u_2 \in \mathbb{R}$.

If
$$\frac{KT^\alpha \left(1 + \frac{|b|}{|a+b|} \right)}{\Gamma_q(\alpha+1)} < 1, \tag{3}$$

then the BVP (1)-(2) has a unique solution on $[0,T]$.

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator

$$F : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$$

defined by

$$F(u)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s, u(s))\, d_q s$$

$$-\frac{1}{a+b}\left[\frac{b}{\Gamma_q(\alpha)}\int_0^T (T-qs)^{(\alpha-1)} f\left(s,u(s)\right) d_q s - c\right]. \tag{4}$$

Clearly, the fixed point of the operator F are solution of the problem (1)-(2). We shall use the Banach contraction principle to prove that F defined by (4) has a fixed point. We shall show that F is a contraction.

Let $x_1, x_2 \in C\left([0,T],\mathbb{R}\right)$. Then, for each $t \in I$ we have

$$\begin{aligned}
\left|F(x_1)(t)-F(x_2)(t)\right| &\le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\left| f\left(s,x_1(s)\right)-f\left(s,x_2(s)\right)\right| d_q s \\
&\quad + \frac{|b|}{\Gamma_q(\alpha)|a+b|}\int_0^T (T-qs)^{(\alpha-1)}\left| f\left(s,x_1(s)\right)-f\left(s,x_2(s)\right)\right| d_q s \\
&\le \frac{K\| x_1-x_2\|_\infty}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} d_q s \\
&\quad + \frac{|b| K\| x_1-x_2\|_\infty}{\Gamma_q(\alpha)|a+b|}\int_0^T (T-qs)^{(\alpha-1)} d_q s \\
&\le \left[\frac{KT^{\alpha}\left(1+\frac{|b|}{|a+b|}\right)}{\Gamma_q(\alpha+1)}\right]\| x_1-x_2\|_\infty .
\end{aligned}$$

Thus

$$\| F(x_1)-F(x_2)\|_\infty \le \left[\frac{KT^{\alpha}\left(1+\frac{|b|}{|a+b|}\right)}{\Gamma_q(\alpha+1)}\right]\| x_1-x_2\|_\infty .$$

Consequently by (3) F is a contraction. As a consequence of Banach fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1)-(2).

The second result is based on Schaefer's fixed point theorem.

Theorem 3.2. Assume that:
(H2) The function $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous.
(H3) There exists a constant $M>0$ such that

$$|f(t,u)|\le M \quad \text{for each } t\in I \text{ and all } u\in\mathbb{R}.$$

Then the BVP (1)-(2) has at least one solution on $[0,T]$.

Proof. We shall use Schaefer's fixed point theorem to prove that F defined by (4) has a fixed point. The proof will be given in several steps.

Step 1. F is continuous.

Let $\{u_n\}$ be a sequence such that $u_n\to u$ in $C([0,T],\mathbb{R})$. Then for each $t\in[0,T]$

$$\begin{aligned}
|F(u_n)(t)-F(u)(t)| &\le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\left|f(s,u_n(s))-f(s,u(s))\right|d_qs \\
&+\frac{|b|}{\Gamma_q(\alpha)|a+b|}\int_0^T (T-qs)^{(\alpha-1)}\left|f(s,u_n(s))-f(s,u(s))\right|d_qs \\
&\le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\sup_{s\in[0,T]}\left|f(s,u_n(s))-f(s,u(s))\right|d_qs \\
&+\frac{|b|}{\Gamma_q(\alpha)|a+b|}\int_0^T (T-qs)^{(\alpha-1)}\sup_{s\in[0,T]}\left|f(s,u_n(s))-f(s,u(s))\right|d_qs
\end{aligned}$$

$$\le \frac{\left\|f(\cdot,u_n(\cdot))-f(\cdot,u(\cdot))\right\|_\infty}{\Gamma_q(\alpha)}\left[\int_0^t (t-qs)^{(\alpha-1)}d_qs+\frac{|b|}{|a+b|}\int_0^T (T-qs)^{(\alpha-1)}d_qs\right]$$

$$\le \frac{T^\alpha\left(1+\dfrac{|b|}{|a+b|}\right)\left\|f(\cdot,u_n(\cdot))-f(\cdot,u(\cdot))\right\|_\infty}{[\alpha]_q\,\Gamma_q(\alpha)}.$$

Since f is a continuous function, we have

$$\left\|F(u_n)-F(u)\right\|_\infty \le \frac{T^\alpha\left(1+\frac{|b|}{|a+b|}\right)\left\|f(\cdot,u_n(\cdot))-f(\cdot,u(\cdot))\right\|_\infty}{\Gamma_q(\alpha+1)} \to 0 \text{ as } n\to\infty.$$

Step 2: F maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$.

Indeed, it is enough to show that for any $\mu>0$, there exist a positive constant r such that for each $u\in B_\mu=\left\{u\in C([0,T],\mathbb{R}):\|u\|_\infty\le\mu\right\}$, we have $\|F(u)\|_\infty\le r$.

By (H3) we have for each $t\in[0,T]$,

$$\begin{aligned}|F(u)(t)| &\le \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)}\left|f(s,u(s))\right| d_qs \\ &\quad+\frac{|b|}{\Gamma_q(\alpha)|a+b|}\int_0^T (T-qs)^{(\alpha-1)}\left|f(s,u(s))\right| d_qs+\frac{|c|}{|a+b|}\end{aligned}$$

$$\begin{aligned}&\le \frac{M}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} d_qs+\frac{|b|M}{\Gamma_q(\alpha)|a+b|}\int_0^T (T-qs)^{(\alpha-1)} d_qs+\frac{|c|}{|a+b|} \\ &\le \frac{M}{[\alpha]_q\,\Gamma_q(\alpha)}T^\alpha+\frac{M|b|}{[\alpha]_q\,\Gamma_q(\alpha)|a+b|}T^\alpha+\frac{|c|}{|a+b|}.\end{aligned}$$

Thus

$$\|F(u)\|_\infty \le \frac{M}{\Gamma_q(\alpha+1)}T^\alpha+\frac{M|b|}{\Gamma_q(\alpha+1)|a+b|}T^\alpha+\frac{|c|}{|a+b|} \quad := r.$$

Step 3. F maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.

Let $t_1,t_2\in(0,T], t_1<t_2, B_\mu$ be bounded set of $C([0,T],\mathbb{R})$ as in step 2, and let $u\in B_\mu$. Then

$$\begin{aligned}|F(u)(t_2)-F(u)(t_1)| &= \left|\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left[(t_2-qs)^{(\alpha-1)}-(t_1-qs)^{(\alpha-1)}\right] f(s,u(s))\,d_qs\right. \\ &\quad\left.+\frac{1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}(t_2-qs)^{(\alpha-1)} f(s,u(s))\,d_qs\right|\end{aligned}$$

$$\leq \frac{M}{\Gamma_q(\alpha)} \int_0^{t_1} \left[(t_2 - qs)^{(\alpha-1)} - (t_1 - qs)^{(\alpha-1)} \right] d_q s$$

$$+ \frac{M}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)} d_q s$$

$$\leq \frac{M}{\Gamma_q(\alpha+1)} (t_2^{\alpha} - t_1^{\alpha}).$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. As a consequence of Step 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $F : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$\varepsilon = \{u \in C(I,\mathbb{R}) : u = \lambda F(u) \quad \text{for some} \quad 0 < \lambda < 1\}$$

is bounded.

Let $u \in \varepsilon$, then $u = \lambda F(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in I$ we have

$$u(t) = \lambda \left[\frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s, u(s)) d_q s \right.$$

$$\left. - \frac{1}{a+b} \left(\frac{b}{\Gamma_q(\alpha)} \int_0^T (T - qs)^{(\alpha-1)} f(s, u(s)) d_q s - c \right) \right].$$

This implies by (H3) that for $t \in I$ we have

$$|F(u)(t)| \leq \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} |f(s, u(s))| d_q s$$

$$+ \frac{|b|}{\Gamma_q(\alpha)|a+b|} \int_0^T (T - qs)^{(\alpha-1)} |f(s, u(s))| d_q s + \frac{|c|}{|a+b|}$$

$$\leq \frac{M}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} d_q s$$

$$+ \frac{|b| M}{\Gamma_q(\alpha)|a+b|} \int_0^T (T - qs)^{(\alpha-1)} d_q s + \frac{|c|}{|a+b|}$$

$$\leq \frac{M}{\Gamma_q\left(\alpha+1\right)}T^{\alpha} + \frac{M\,|b|}{\Gamma_q\left(\alpha+1\right)|a+b|}T^{\alpha} + \frac{|c|}{|a+b|}.$$

Thus for every $t \in \left[0,T\right]$, we have

$$\|F(u)\|_{\infty} \leq \frac{M}{\Gamma_q\left(\alpha+1\right)}T^{\alpha} + \frac{M\,|b|}{\Gamma_q\left(\alpha+1\right)|a+b|}T^{\alpha} + \frac{|c|}{|a+b|} \quad := \mathbb{R}.$$

This shows that the set ε is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1)-(2).

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