Numerical solution for Fredholm–Volterra integral equation of the second kind by using collocation and Galerkin methods

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Received 11 March 2008; accepted 12 December 2009
Available online 23 December 2009

Abstract The Fredholm–Volterra integral equation of the second kind with continuous kernels with respect to position and time, is solved numerically, using the Collocation and Galerkin methods. Also the error, in each case, is estimated.

1. Introduction

Many problems of mathematical physics, engineering and contact problems in the theory of elasticity lead to integral equations. The following references Muskhelishvili et al. (1953), Green (1969), Atkinson et al. (1976) and Delves and Mohamed (1985), contain many different methods to solve the integral equations analytically. At the same time the numerical methods take an important place in solving the integral equations numerically. The references Linz et al. (1985), Kanwal et al. (1996), Atkinson et al. (1997) and Abdou and Mohamed (2002) contain many different methods for solving the integral equations numerically. The discussion of the Fredholm–Volterra integral equations numerically and analytically can be found in the works of Abdou and co-workers, see (Schiavane and Constanda, 2002; Abdou et al., 2003; Abdou and Salama, 2004), when the Fredholm integral term is considered in position and Volterra integral term in time. In all work of Abdou in Fredholm–Volterra integral equation when the kernel of position is continuous have not been solved.

Therefore, in this paper, we consider the Fredholm–Volterra integral equations of the second kind with continuous kernels with respect to position and time. The
existence and uniqueness of the solution, under certain conditions, will be proved in the space $L_2[a, b] \times C[0, T]$ for all $0 \leq t \leq T \leq \infty$.

A numerical method is used to represent the Fredholm–Volterra equation in the form of a linear system of Fredholm integral equations where the existence and uniqueness of the system are discussed. Also we used the Collocation and Galerkin methods to obtain a linear system of algebraic equations, which is also solved numerically. Moreover the error estimate, in each case, is calculating.

2. The existence and uniqueness of the solution

Consider the Fredholm–Volterra integral equation

\[ \mu \phi(x, t) - \lambda \int_a^b k(x, y) \phi(y, t) \, dy - \lambda \int_0^t F(t, \tau) \phi(x, \tau) \, d\tau = f(x, t) \quad (2.1) \]

where $k(x, y)$ and $F(t, \tau)$ are continuous functions which represent the kernel of Fredholm and Volterra integral terms, respectively. The known function $f(x, t)$ is called the free term of the integral equation. Here the Fredholm is considered in position, while Volterra is considered in time. The constant $\mu$ defines the kind of integral equation while the constant $\lambda$ may be complex, with a physical meaning.

In order to guarantee the existence of a unique solution of (2.1). We assume the following:

(i) The kernel of position satisfies $|k(x, y)| \leq N_1$ for all $a \leq x, y \leq b$, where $N_1$ is a constant.
(ii) The positive continuous function $F(t, \tau) \in C[0, T] \times [0, T]$ for all $0 \leq t, \tau \leq T < \infty$ and satisfies $|F(t, \tau)| \leq N_2$ where $N_2$ is a constant.
(iii) The given function and its norm is defined as:

\[ \|f(x, t)\| = \max_{0 \leq t \leq T} \int_a^b \{f^2(x, t)\}^{1/2} \, dx \leq N_3 \]

where $N_3$ is a constant.
(iv) The unknown function $\phi(x, t)$ satisfy the Lipschitz condition with respect to position $|\phi(x_1, t) - \phi(x_2, t)| \leq A(t)$ where $x_1, x_2 \in [a, b]$ and Hölder condition with respect to time $|\phi(x, t_1) - \phi(x, t_2)| \leq B(x) |t_1 - t_2|^{\alpha}$, $0 \leq \alpha \leq 1$, and its norm is defined as

\[ \|\phi(x, t)\| = \max_{0 \leq t \leq T} \int_a^b \{\phi^2(x, t)\}^{1/2} \, dx \leq N_4 \]

3. The system of Fredholm integral equations

For representing (2.1) as a system of Fredholm integral equations, we use the numerical method, see Delves and Mohamed (1985) and Atkinson et al. (1997).

Divide the interval $[0, T]$ as

\[ 0 = t_0 < t_1 < \ldots < t_k < \ldots < t_N = T, \quad i.e. t = t_k, \quad k = 0, 1, 2, \ldots, N. \]

Then, using the quadrature formula, Te Volterra term in (2.1) becomes

\[ \int_0^t \phi(x, \tau) \, F(t_k, \tau) \, d\tau = \sum_{j=0}^{k} u_j F(t_k, t_j) \phi(x, t_j) + O(h_k^{p-1}), \]

\[ h_k = \max_{0 \leq j < k} h_j, \quad h_j = t_{j+1} - t_j \]

where $h_k \to 0, \quad p > 0$ \quad (3.1)

The values of $k$ and the constant $p$ depend on the number of derivatives of $F(t, \tau)$, for all $\tau \in [0, T]$, w.r.t. $t$, and $u_0 = \frac{1}{2} h_0, u_k = \frac{1}{2} h_k, u_t = h_t, (t \neq 0, k)$.

Using (3.1) in (2.1) after letting $t = t_k, k = 1, 2, \ldots, N$, We have

\[ \mu \phi(x, t_k) = f(x, t_k) + \lambda \int_a^b k(x, y) \phi(y, t) \, dy + \lambda \sum_{j=0}^{k} u_j F(t_k, t_j) \phi(x, t_j) \]

or,

\[ \mu \phi_k(x) = f_k(x) + \lambda \int_a^b k(x, y) \phi_k(y) \, dy + \lambda \sum_{j=0}^{k} u_j F_k, j = F(t_k, t_j), \]

the formula (3.3) become

\[ \mu \phi_k(x) = g_k(x) + \lambda \int_a^b k(x, y) \phi_k(y) \, dy \]

where $\mu_k = \mu - \lambda u_k F_k$, $\gamma_k = \lambda u_k F_k$.

\[ g_k(x) = f_k(x) + \lambda \sum_{j=0}^{k} u_j F_k, j = F(t_k, t_j), \quad n = 0, 1, \ldots, N \]

The formula (3.4) represents a linear system of Fredholm integral equations of the second kind, where $\gamma_k F_k \neq \mu_k$.

Now, we will solve the linear system (3.4) using the Collocation method and Galerkin method.

3.1. Collocation method

Collocation method is based on approximating the solution $\phi(x, t)$ by a partial sum:

\[ S(x, t_k) = \sum_{i=1}^{N} c_i(t_k) \psi_i(x) \]

of $N$ linearly independent functions $\psi_1(x), \psi_2(x), \ldots, \psi_N(x)$ on the interval $(a, b)$. Therefore we have

\[ \mu S(x) - \lambda \int_a^b k(x, y) S(y) \, dy \approx f(x, t) + \lambda \sum_{j=0}^{k-1} w_j F_j S(x) + \epsilon(x, c_1(t), c_2(t), \ldots, c_N(t)) + R(h_k^{p-1}) \]

(3.6)

Of course, if the approximate solution (3.5) is to be substituted into (3.4) for $\phi(x, t)$, there will be an error $\epsilon(x, c_1(t), c_2(t), \ldots, c_N(t))$. This error depends on $x, t$ and the way for which the coefficients $c_1(t), c_2(t), \ldots, c_N(t)$ are chosen in the formula (3.6). Let $t = t_i, \quad i = 0, 1, 2, \ldots, N$. Then using the quadrature formula, we have

\[ \mu_i S(x_m) - \lambda \int_a^b k(x, y) S(y) \, dy \approx f(x, t_i) + \lambda \sum_{j=0}^{k-1} w_j F_j S(x_m), \quad i = 0, 1, 2, \ldots, N \]

(3.7)

For determining the coefficients $c_1(t), c_2(t), \ldots, c_N(t)$ of the approximate solution $S(x_m)$ from (3.5), in terms of the given $N$ linearly independent functions $\psi_1(x), \psi_2(x), \ldots, \psi_{N-1}(x)$, from (3.5), in terms of the given $N$ linearly independent functions $\psi_1(x), \psi_2(x), \ldots, \psi_{N-1}(x)$,
ψ_N(x), perform the integration, then substitute x = x_1, x_2, ..., x_N for which the error ε(x, c_1(t), c_2(t), ..., c_N(t)) vanishes.

Substituting from (3.5) in (3.6), we get:

$$
\mu_i \sum_{k=1}^N c_k(t) \psi_k(x_m) - \lambda \sum_{k=1}^N c_k(t) \int_a^b k(x_m, y) \psi_k(y) dy = f_{ext} + \lambda \sum_{j=0}^{i-1} \sum_{k=1}^N w_j F_{ij} c_k(t) \psi_k(x_m) \tag{3.8}
$$

3.2. Galerkin method

This method establishes the N conditions necessary for the determination of the N coefficients in Eq. (3.5):

By making the error ε(x, c_1(t), c_2(t), ..., c_N(t)) in (3.6) orthogonal to N given linearly independent functions ψ_1(x), ψ_2(x), ..., ψ_N(x) on the interval (a, b), i.e.

$$
\int_a^b \psi_j(x) \varepsilon(x, c_1(t), c_2(t), ..., c_N(t)) dx = 0 \tag{3.9}
$$

Then from (3.6), we have

$$
\int_a^b \psi_j(x) [\mu_i S(x_m) - \lambda \int_a^b k(x_m, y) S(y) dy - \sum_{j=0}^{i-1} \sum_{k=1}^N w_j F_{ij} S(x_m) - R(h_{ji}^{(i)})] dx = 0 \tag{3.10}
$$

Eq. (3.10) can be written in the form:

$$
\int_a^b \psi_j(x) [\mu_i S(x_m) - \lambda \int_a^b k(x_m, y) S(y) dy - \sum_{j=0}^{i-1} \sum_{k=1}^N w_j F_{ij} S(x_m) - R(h_{ji}^{(i)})] dx = 0 \tag{3.11}
$$

where R(h_{ji}^{(i)}) is the error from dividing the time and h_i = max_{0<\tau<i} h_{i-1} h_{j-1} = t_j - t_i. The values of i and the constant p depend on the derivatives of F(t, τ), for all τ ∈ [0, T], with respect to t. Substituting from (3.5) into (3.11) we get

$$
\int_a^b \psi_j(x) [\mu_i \sum_{k=1}^N c_k(t) \psi_k(x_m) - \lambda \int_a^b k(x_m, y) dy - \sum_{j=0}^{i-1} \sum_{k=1}^N w_j F_{ij} \sum_{k=1}^N c_k(t) \psi_k(x_m)] dx = 0 \tag{3.12}
$$

4. Examples

Example 4.1. Consider the integral equation:

$$
\phi(x, t) = f(x, t) + \lambda \int_0^t t^2 \phi(x, \tau) d\tau + \lambda \int_a^b e^{x+y} \phi(y, t) dy
$$

where the exact solution ϕ(x, t) = t^2 e^{x^2}.

4.1. Using collocation method

In Eq. (4.1) we shall take N = 2, a = 0, b = 1,

$$
f(x, t) = \frac{3}{2} t^2 e^{x^2} - \frac{1}{5} t^2 e^{x^2} - \frac{1}{7} t^2 e^{x^2}
$$

Let the approximate solution has the form of Eq. (3.5), the three independent functions are ψ_1(x) = 1, ψ_2(x) = x, ψ_3(x) = x^2. Substituting these values in Eq. (3.8), then solving the equations formulas when x = 0, 0.5, 1.0, in this case R = 0. We get:

$$
c_1(t_0) = 0, c_2(t_0) = 0, c_3(t_0) = 0.
$$

$$
c_1(t_1) = 0.0005130967957, c_2(t_1) = 0.000432839493, \tag{4.1}
$$

$$
c_3(t_1) = 0.000415594769.
$$

$$
c_1(t_2) = 0.002104378751, c_2(t_2) = 0.001731361786,\tag{4.2}
$$

$$
c_3(t_2) = 0.001662382707.
$$

So, the solution, for t ∈ [0, 0.03], takes the form (see Table 1):

$$
S(x, t_0) = 0
$$

$$
S(x, t_1) = 0.0005130967957 + 0.000432839493 x
$$

$$
-0.000415594769 x^2 \tag{4.3}
$$

$$
S(x, t_2) = 0.002104378751 + 0.001731361786 x
$$

$$
+0.001662382707 x^2
$$

4.2. Using Galerkin method

As in collocation method, using (3.9) in (3.12), Choose three independent functions ψ_1(x) = 1, ψ_2(x) = x, ψ_3(x) = x^2 and three points x = 0, 0.5, 1.0, when we assume t ∈ [0, 0.03], then we have:

$$
c_1(t_0) = 0, c_2(t_0) = 0, c_3(t_0) = 0.
$$

$$
c_1(t_1) = 0.00051949274, c_2(t_1) = 0.00042025907,\tag{4.4}
$$

$$
c_3(t_1) = 0.00041436290.
$$

$$
c_1(t_2) = 0.00212991168, c_2(t_2) = 0.0016810402,\tag{4.5}
$$

$$
c_3(t_2) = 0.0016574556.
$$

<table>
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<tr>
<th>x</th>
<th>ϕ(x, t)</th>
<th>E^C</th>
<th>E^G</th>
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<td>0</td>
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<td>0</td>
</tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
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<td>1.0</td>
<td>0.002440453645</td>
<td>0.003051669518</td>
<td>0.003021953955</td>
</tr>
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</table>
In Eq. (4.6) we shall take
\[ S(x, t_0) = 0, \]
\[ S(x, t_1) = 0.00051949274 + 0.00042025907x + 0.00041436290x^2, \tag{4.5} \]
\[ S(x, t_2) = 0.0021299116 + 0.0016810402x + 0.0016574556x^2. \]

So, the solution is taken from:

Example 4.2. Consider the integral equation:
\[ \phi(x, t) = f(x, t) + \lambda \int_0^t \phi(x, \tau) d\tau + \lambda \int_0^b e^{-\gamma t} \phi(y, t) dy, \tag{4.6} \]
where the exact solution \( \phi(x, t) = te^{-x} \).

4.3. Using collocation method

In Eq. (4.6) we shall take \( N = 2, a = 0, b = 1, \)
\[ f(x, t) = -0.00432t + e^{-\gamma t} - 0.333e^{-x}t^2. \]

Let the approximate solution in the form of Eq. (3.5), then solving the equations formulas, when \( x = 0, 0.5, 1.0 \), we get:
\[ c_1(t_0) = 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0, \]
\[ c_1(t_1) = 0.00854193634, \quad c_2(t_1) = -0.01412633660, \]
\[ c_3(t_1) = 0.04644538564, \quad c_2(t_1) = 0.01708389624, \]
\[ c_3(t_2) = 0.02825268914, \quad c_1(t_2) = 0.009289082405. \tag{4.7} \]

So, the solution, for \( t \in [0, 0.03] \) takes the form:
\[ S(x, t_0) = 0, \]
\[ S(x, t_1) = 0.00854193634 - 0.01412633660x + 0.004644538564x^2, \]
\[ S(x, t_2) = 0.01708389624 - 0.02825268914x + 0.009289082405x^2. \tag{4.8} \]

4.4. Using Galerkin method

We choose three independent functions \( \psi_1(x) = 1, \psi_2(x) = x, \psi_3(x) = x^2 \) and three points \( x = 0, 0.5, 1.00 \), when we assume \( t \in [0, 0.03] \), we have:
\[ c_1(t_0) = 0, \quad c_2(t_0) = 0, \quad c_3(t_0) = 0, \]
\[ c_1(t_1) = 0.008459192683, \quad c_2(t_1) = -0.01395821152, \]
\[ c_3(t_1) = 0.00463077286. \]
\[ c_1(t_2) = 0.01691840427, \quad c_2(t_2) = -0.02791644232, \]
\[ c_3(t_2) = 0.00926155092. \tag{4.9} \]

So, the solution is taken the form (see Table 2):
\[ S(x, t_0) = 0, \]
\[ S(x, t_1) = 0.008459192683 - 0.01395821152x + 0.00463077286x^2, \tag{4.10} \]
\[ S(x, t_2) = 0.01691840427 - 0.02791644232x + 0.00926155092x^2. \]

Table 2 Values of the error \( E^C, E^G \) using collocation and Galerkin methods.

<table>
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<tr>
<th>( x )</th>
<th>( \phi(x, t) )</th>
<th>( E^C )</th>
<th>( E^G )</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1.00</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>t = 0.0150000000</td>
<td>0.015</td>
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</tr>
<tr>
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<td>0.005518191618</td>
<td>0.006458050020</td>
<td>0.006386437595</td>
</tr>
<tr>
<td>t = 0.0300000000</td>
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Acknowledgements

The authors would like to thank professor M.A. Abdou (University of Alexandria, Egypt) for his helpful remarks and suggestions.

References


