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Chapter 9: Center of Mass and Linear Momentum

In the previous chapters we have discussed the motion of a single mass or a single particle whose only one mass and therefore one-position vector. However this situation is a rare in the nature because real objects, like us, the chair you're sitting on, and the table in front are all ***extended***. That means they do not have a single mass concept, but a collection of masses. However, to determine the equation of motion (Newton's second law) for such a collection we present the concept of ***center of mass***. In the followings, four phenomena will be covered. These are *center of mass*, *linear momentum*, *impulse*, and *collisions*.

A- Center of Mass

For a system consisting of a collection of masses (particles) m_1, m_2, m_3 …., m_n and each mass has its position vector r_1, r_2, r_3 …., r_n, the position vector of the center of mass of the system is then defined as:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots m_n\vec{r}_n}{M}$$

Where

$$M = m_1 + m_2 + m_3 + \ldots m_n$$

Since any position vector is defined in terms of its x, y, and z components, the position vector of the center of mass can be defined as:

$$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

With

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots m_n x_n}{M}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots m_n y_n}{M}$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots m_n z_n}{M}$$

Similarly to these, the velocity of the center of mass is defined as

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \dots m_n\vec{v}_n}{M}$$

The acceleration of the center of mass is defined as

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \dots m_n\vec{a}_n}{M}$$

Using Newton's second law, we find that the force acting on the center of mass is given by

$$\vec{F} = M\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \dots m_n\vec{a}_n$$

B- Linear Momentum

For a particle with a mass m and moving with a velocity $\vec{v}$, its linear momentum is defined as

$$\vec{p} = m\vec{v}$$

It is clear from the equation above that the linear momentum is a vector quantity in the same direction of the velocity. The SI unit of the linear momentum is $kg.m/s$. For a system consisting of discrete particles, the linear momentum of the center of mass is given by

$$\vec{p}_{cm} = M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \dots m_n\vec{v}_n$$

It is also of quite importance to note that differentiation of momentum equation with respect to time gives Newton's second law.

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

Or

$$\vec{F} = \frac{d\vec{p}}{dt}$$

This is known as the *momentum-force relationship*. *If the force applied is zero, the linear momentum is constant*.

As we have seen in chapter 7 that the kinetic energy is defined as

$$K = \frac{1}{2}mv^2$$

However, we can rewrite the previous formula as

$$K = \frac{1}{2} m\vec{v}.\vec{v} = \frac{1}{2m}(m\vec{v}).(m\vec{v}) = \frac{1}{2m}[\vec{p}.\vec{p}] = \frac{p^2}{2m}$$

Or

$$\boxed{K = \frac{p^2}{2m}}$$

This is known as the *momentum-kinetic energy relationship*.

C- Impulse

The impulse for a single particle is defined as the direct product of the force and the time interval. Mathematically, it is written as

$$\boxed{\vec{I} = \vec{F}\,\Delta t}$$

For a continuous system (not single particle), the impulse is

$$\boxed{\vec{I} = \int_{t_1}^{t_2} \vec{F}.dt = \int_{t_1}^{t_2} \left(\frac{d\vec{p}}{dt}\right).dt = \int_{t_1}^{t_2} d\vec{p}}$$

However, the impulse can be written in terms of the linear momentum as

$$\boxed{\vec{I} = \Delta\vec{p}}$$

It is clear that the impulse is a vector quantity in the direction of the applied force (change of momentum). Its SI unit is $N.s$ or $Kg.m/s$.

D- Conservation Law of Linear Momentum and Collisions

For an isolated system of interacted particles, the linear momentum is conserved. This is known as the conservation law of linear momentum. Collisions provide such an example of the conservation law of linear momentum. While the total momentum is conserved for a system of isolated colliding particles, the mechanical (kinetic) energy *may* or *may not* be conserved. Depending on the conservation laws of total kinetic energy, there are two main categories for the collisions: ***elastic*** and ***inelastic***.

1- Elastic Collisions

Elastic collision of collided bodies means that the total kinetic energy before the collision ***equals*** the total kinetic energy after the collision. That means there is no energy dissipation during the collision.

For two elastically collided bodies of masses m_1 and m_2 and initial speeds of v_{1i} and v_{2i}, their final speeds after collision are given by:

$$\boxed{\vec{v}_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{v}_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)\vec{v}_{2i}}$$

$$\boxed{\vec{v}_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)\vec{v}_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)\vec{v}_{2i}}$$

Special cases:

- ***If m_2 is initially at rest, the final velocities tend to be***

$$\boxed{\vec{v}_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)\vec{v}_{1i}}$$

$$\boxed{\vec{v}_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)\vec{v}_{1i}}$$

- ***If $m_2 >> m_1$, the final velocities tend to be***

$$\boxed{\vec{v}_{1f} \approx -\vec{v}_{1i} + 2\vec{v}_{2i}}$$

$$\boxed{\vec{v}_{2f} \approx \vec{v}_{2i}}$$

- ***If $m_1 >> m_2$, the final velocities tend to be***

$$\boxed{\vec{v}_{1f} \approx \vec{v}_{1i}}$$

$$\boxed{\vec{v}_{2f} = 2\vec{v}_{1i} - \vec{v}_{2i}}$$

- ***If $m_1 = m_2$, the final velocities tend to be***

$$\boxed{\vec{v}_{1f} = \vec{v}_{2i}}$$

$$\boxed{\vec{v}_{2f} = \vec{v}_{1i}}$$

2- Inelastic Collisions

Inelastic collision of collided bodies means that the total kinetic energy before the collision ***does not equal*** the total kinetic energy after the collision. That means some kinetic energy will be converted into other form of energy (usually heat) during the collision. In this case only the linear momentum is conserved.

For two inelastically collided bodies of masses m_1 and m_2 and initial speeds of v_{1i} and v_{2i}, their final speed after collision is given by:

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

Or

$$\boxed{\vec{v}_f = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1 + m_2}}$$

Special cases:

- ***If m_2 is initially at rest, the final velocity tends to be***

$$\boxed{\vec{v}_f = \left(\frac{m_1}{m_1 + m_2}\right)\vec{v}_{1i}}$$

- ***If v_{1i} =- v_{2i}, the final velocity tends to be***

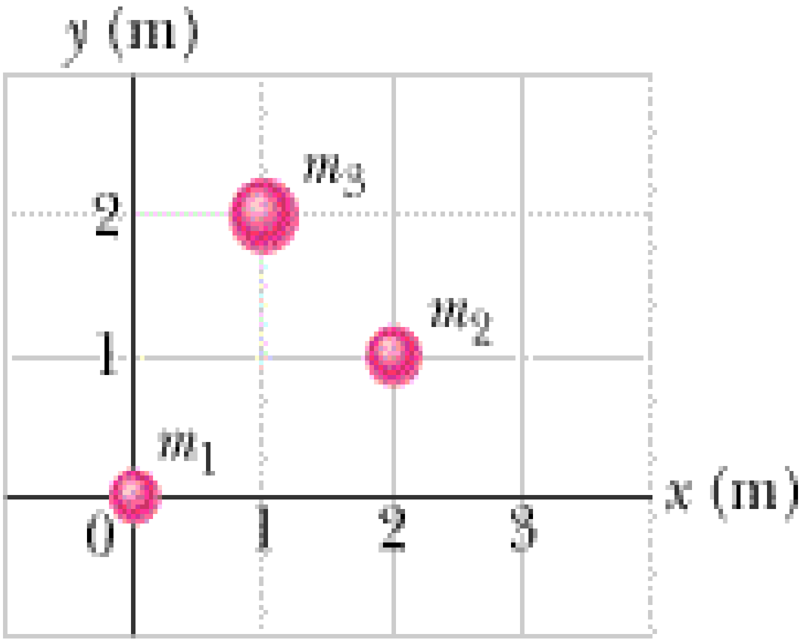
$$\boxed{\vec{v}_f = 0}$$

Remark: *If the number of collided bodies after the collision is different from those before the collision, the collision is classified as inelastic collision. Otherwise, it is elastic collision.*

Examples:

1. Three particles of masses m_1=2 kg, m_2=3 kg, and m_3=5 kg are arranged in the *xy* plane, as shown in the figure below. Find the position vector of the center of mass.

Solution



We know that the position vector of the center of mass is defined as:

$$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

The components of the coordinate of the center of mass are

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{M} = \frac{2\times 0 + 3\times 2 + 5\times 1}{2+3+5} = 1.1\, m$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{M} = \frac{2\times 0 + 3\times 1 + 5\times 2}{2+3+5} = 1.3\, m$$

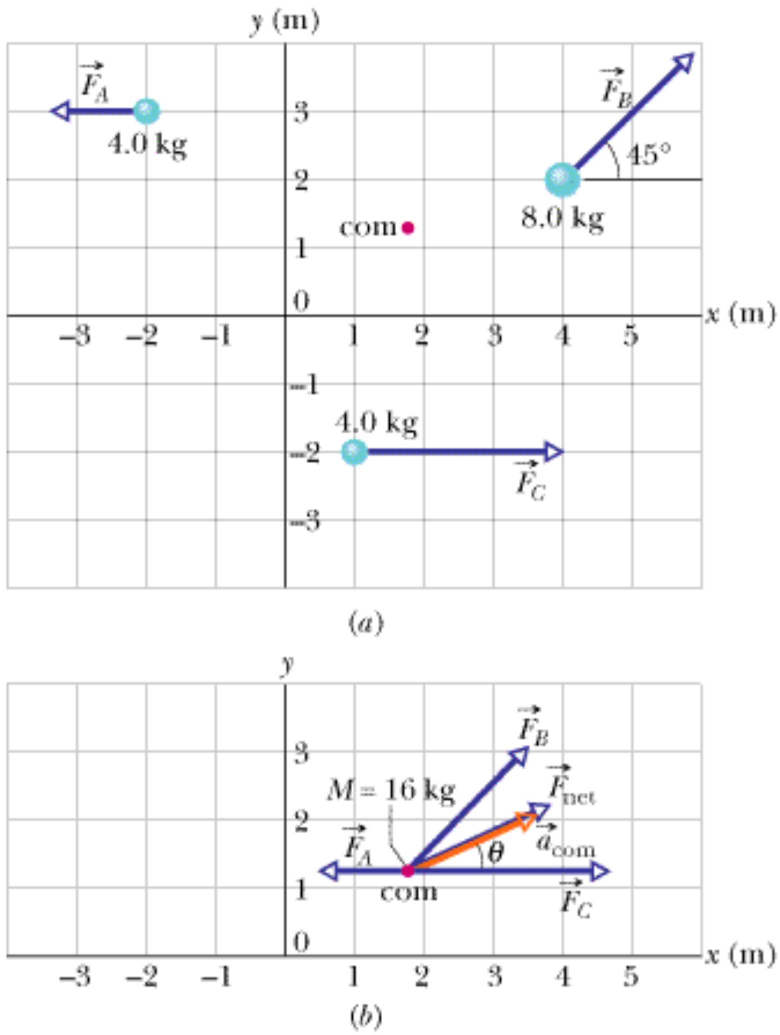
$$z_{cm} = \frac{m_1z_1 + m_2z_2 + m_3z_3}{M} = \frac{2\times 0 + 3\times 0 + 5\times 0}{2+3+5} = 0.0\, m$$

Therefore the position vector of the center of mass is defined as:

$$\vec{r}_{cm} = 1.1\,\hat{i} + 1.3\,\hat{j}$$

2. Three particles of masses m_A=4 kg, m_B=8 kg, and m_C=4 kg are affected by three forces F_A=6 N, F_B=12 N, and F_C=14 N, as shown in the figure below. What is the magnitude of the acceleration of the center of mass of the system, and its direction?

Solution



The force applied on the center of mass is defined as

$$\vec{F} = M\vec{a}_{cm}$$

Now we calculate the total force applied on the system

$$\vec{F} = \vec{F}_A + \vec{F}_B + \vec{F}_C$$

Where

$$\vec{F}_A = -6\hat{i}$$

$$\vec{F}_B = 12\cos 45\hat{i} + 12\sin 45\hat{j} = 8.5\hat{i} + 8.5\hat{j}$$

$$\vec{F}_C = 14\hat{i}$$

Therefore

$$\vec{F} = \left(-6\hat{i}\right) + \left(8.5\hat{i} + 8.5\hat{j}\right) + \left(14\hat{i}\right) = 16.5\hat{i} + 8.5\hat{j}$$

Therefore, the acceleration of the center of mass is

$$\vec{a}_{cm} = \frac{\vec{F}}{M} = \frac{16.5\hat{i} + 8.5\hat{j}}{4+8+4} = 1.03\hat{i} + 0.53\hat{j}$$

The angle that the acceleration makes with the x-axis can be determined using the tangent as

$$\tan\theta = \frac{a_y}{a_x}$$

We find

$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{0.53}{1.03}\right) = 27.3^0$$

3. Three particles of masses m_1=1 kg, m_2=2 kg, and m_3=3 kg are located in *xy* plane as (3,2), (-1,1), and (3,-2), respectively. Find the coordinate of the center of mass.

Solution

The components of the coordinate of the center of mass are

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{M} = \frac{1\times 3 + 2\times(-1) + 3\times 3}{1+2+3} = 1.67$$

$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{M} = \frac{1\times 2 + 2\times 1 + 3\times(-2)}{1+2+3} = -0.34$$

The coordinate of the center of mass is

(1.67,0.34)

4. Four particles of masses m_1=2 kg, m_2=4 kg, and m_3= m_4= 3 kg have the following velocities: $\vec{v}_1 = 3\hat{i} + 4\hat{j}$, $\vec{v}_2 = 5\hat{i} - \hat{j}$, $\vec{v}_3 = -4\hat{i}$, and $\vec{v}_4 = 2\hat{j}$, where the velocities are measured in m/s. Find the linear momentum of the center of mass of the system.

Solution

The linear momentum of the center of mass of the system is

$$\vec{p}_{cm} = M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + m_4\vec{v}_4$$

$$\vec{p}_{cm} = 2 \times \left(3\hat{i} + 4\hat{j}\right) + 4 \times \left(5\hat{i} - \hat{j}\right) + 3 \times \left(-4\hat{i}\right) + 3 \times \left(-2\hat{j}\right)$$

$$\vec{p}_{cm} = 14\hat{i} - 2\hat{j}$$

5. A motorcycle of mass 120 kg moves with a fixed speed of 15 m/s. Calculate the magnitude of its linear momentum.

Solution

The magnitude of the linear momentum is

$$p = mv = 120 \times 15 = 1800\ kg.m/s$$

6. A car is moving with a constant speed of 27 m/s. If its momentum is 21600 kg.m/s, what is its mass?

Solution

The magnitude of the linear momentum is defined as

$$p = mv$$

Therefore the mass is obtained by

$$m = \frac{p}{v} = \frac{21600}{27} = 800\ kg$$

7. A truck of mass 2000 kg has a kinetic energy of 144 kJ. Calculate (a) its linear momentum and (b) its speed.

Solution

(a) The kinetic energy and linear momentum are related through

$$K = \frac{p^2}{2m}$$

Therefore the momentum is

$$p = \sqrt{2mK} = \sqrt{2 \times 2000 \times 144000} = 24000\ kg.m/s$$

(b) The speed of the truck can be determined using either kinetic energy or linear momentum definitions. Using the linear momentum expression we find

$$v = \frac{p}{m} = \frac{24000}{2000} = 12\ m/s$$

8. A 2.00 kg particle has a velocity of $\vec{v} = 1.5\hat{i} + 2\hat{j}\,(m/s)$. Find the x and y components of its momentum and the magnitude of the total momentum.

Solution

The linear momentum is defined as

$$\vec{p} = m\vec{v}$$

Therefore the components of linear momentum are

$$p_x = mv_x = 2 \times 1.5 = 3\ kg.m/s$$

$$p_y = mv_y = 2 \times 2 = 4\ kg.m/s$$

The magnitude of the total momentum is

$$p = \sqrt{p_x^2 + p_y^2} = \sqrt{3^2 + 4^2} = 5\ kg.m/s$$

9. A car of mass 850 kg is moving with a speed of 20 m/s, suddenly the driver increases the speed to 27 m/s in 10 sec. Find (a) the magnitude of the impulse and (b) the acceleration of the car.

Solution

(a) We know that the impulse is defined as the change in the linear momentum

$$I = \Delta p = m(v_f - v_i) = 850 \times (27 - 20) = 5950 \; N.s$$

(b) Also we know that the impulse is defined as

$$I = F \; \Delta t$$

Therefore the magnitude of the force is

$$F = \frac{I}{\Delta t} = \frac{5950}{10} = 595 \; N$$

From Newton's second law we have

$$F = ma$$

Therefore the acceleration is

$$a = \frac{F}{m} = \frac{595}{850} = 0.7 \; m/s^2$$

10. A baseball (m= 0.12 kg) has an initial velocity of v_i= 30 m/s as it approaches a bat. The ball departs from the bat with a final velocity of v_f= 45 m/s. (a) Determine the impulse applied to the ball by the bat. (b) Assuming that the time of contact is Δt=1.0 × 10^{-3} s, find the average force exerted on the ball by the bat.

Solution

(a) We know that the impulse is defined as the change in the linear momentum

$$I = \Delta p = m\left(v_f - v_i\right)$$

Here you need to be very careful about the velocity, the initial velocity is opposite to the final velocity. If we consider the positive direction is in the same direction of the final velocity, we will have

$$I = \Delta p = 0.12 \times \left(45 - (-30)\right) = 9\ N.m$$

(b) We know that the impulse is also defined as

$$I = F\ \Delta t$$

Therefore the average force is

$$F = \frac{I}{\Delta t} = \frac{9}{1 \times 10^{-3}} = 9000\ N$$

11. In a perfectly inelastic collision, a car (900 kg) moving with a speed of 25 m/s collides with another stationary car of mass 1100 kg. If they move together after the collision, find their velocity.

Solution

We know that the linear momentum is conserved for the system. Since the second car is initially at rest, the final velocity is given by

$$v_f = \left(\frac{m_1}{m_1 + m_2}\right) v_{1i} = \left(\frac{900}{900 + 1100}\right) \times 25 = 11.25\ m/s$$

We note that the final velocity is in the same direction of the initial velocity of the first car.

12. Toyota hilux (1200 kg) moving with a speed of 30 m/s collides with Hyundai accent of mass 800 kg, which is moving in the same direction with a speed of 15 m/s. If collision is completely inelastic, find their final speed.

Solution

We know that the linear momentum is conserved for the system. The final speed is

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} = \frac{1200 \times 30 + 800 \times 15}{1200 + 800} = 24\ m/s$$

13. A 0.05 kg bullet moving at 400 m/s strikes a wooden block of mass 1.95 kg that is initially at rest. If the bullet embeds the block and the whole system moves together, find their final speed.

Solution

This is an inelastic collision and because of the block was initially at rest; the final speed of the system is given by

$$v_f = \left(\frac{m_1}{m_1 + m_2}\right) v_{1i} = \left(\frac{0.05}{0.05 + 1.95}\right) \times 400 = 10\ m/s$$

14. A 20.0 g bullet is stopped in a block of wood (m = 4.98 kg). The speed of the system (bullet and wood combination) immediately after the collision is 0.5 m/s. What was the original speed of the bullet if the block was initially at rest?

Solution

We know that for the inelastic collision, the final speed is

$$v_f = \left(\frac{m_1}{m_1 + m_2}\right) v_{1i}$$

Therefore

$$v_{1i} = \left(\frac{m_1 + m_2}{m_1}\right) v_f = \left(\frac{0.02 + 4.98}{0.02}\right) \times 0.5 = 125\ m/s$$

15. A car (of mass 1500 kg) moving with a speed of 20 m/s makes a head-on-collision with another car having a mass of 1200 kg and a speed of 30 m/s. If the collision is perfectly elastic, find their velocities after the collision.

Solution

It is quite important to note that in head-on-collision, the initial speeds of the collided cars are opposite to each other. So, if you consider the velocity of the first car is positive (along the positive axis), you have to assume the velocity of the second car to be negative (along the negative axis). From these, we find the final velocity of the first car is

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)v_{2i} = \left(\frac{1500 - 1200}{1500 + 1200}\right) \times 20 + \left(\frac{2 \times 1200}{1500 + 1200}\right) \times (-30) = -24.5\ m/s$$

The minus sign indicates that the first car will move back (opposite to its original direction). The final velocity of the second car is

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)v_{2i} = \left(\frac{2 \times 1500}{1500 + 1200}\right) \times 20 + \left(\frac{1500 - 1200}{1500 + 1200}\right) \times (-30) = 18.9\ m/s$$