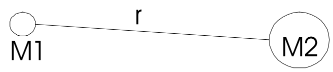
Gravity

Using the Earth's gravitational field to study its internal structure has proven to be an extremely effective means of probing structural variations at all levels depths and length scales within the Earth. To understand how to interpret variations in the Earth's gravitational field, we must first understand how gravitational potential and acceleration arise from Newton's second Law.

Gravitational Acceleration.

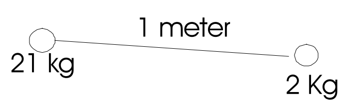
Two point masses m_1 and m_2 at a distance r apart attract one another with a force F,



$$F = G\frac{m_1 m_2}{r^2}$$

where G is the gravitational or Newtonian constant. In SI units, G has a value of 6.67 x 10^{-11} $m^3kg^{-1}s^{-2}$. This inverse square law of the gravitational attraction between two objects was deduced by Isaac Newton in 1666: the legend is that an apple falling from a tree gave him the revolutionary idea that the same force that attracted the apple downwards could also account for the moon's orbit of the Earth. Gravity is a vector, which means the force has a magnitude and a direction associated with it, just like a velocity does. The acceleration of the mass m_1 due to the presence of mass m_2 is Gm_2/r^2 and is *directed* toward m_2, while the acceleration of the mass m_2 is G_1/r^2 and is *directed* towards m_1.

Example: What is the amplitude and direction of the gravitational force which attracts two 2-Kg cantaloupes held 1 meter from each other?



F = 6.67 x 10^{-11} $m^3kg^{-1}s^{-1}$ x 2 kg x 2 kg / (1 m)^2 = 2.6 x 10^{-10} kg/m s^2, and is oriented along the line connecting the two cantaloupes. This force is equal to about a ten billionth of the gravitational force either cantaloupe would feel when being pulled by the Earth's gravitational field.

We can talk about the gravitational acceleration which is caused by a given mass. In the above problem, the force which *m1* would feel due to the presence of m2 is

$$F_m = G\frac{m_1 m_2}{r^2} = m_1 \times gravitational\_accleration$$

Canceling the mass m1, we see that the gravitational acceleration which any mass would feel in the presence of m2 is:

$$grav\_accleration = G\frac{m_2}{r^2}$$

So this is simply the acceleration which mass m2 causes in its vicinity. Notice that any object, regardless of its mass, will feel the same acceleration.

In the case of a satellite orbiting the earth, we can calculate the orbit period, or the time it takes to complete one full trip around the earth:

$$Force = GM_e m \frac{1}{r^2} = m\omega^2 r = mass \cdot acceleration \Rightarrow$$

$$\omega = \left(\frac{GM_e}{r^3}\right)^{\frac{1}{2}}$$

Since the period T = 2pi/(angular frequency) then

$$T = \left(\frac{4\pi r^3}{GM_e}\right)^{\frac{1}{2}}$$

For r = 6371 Km, G= 6.67×10^{-11} $m^3 kg^{-1} s^{-2}$, and M = 6×10^{24} Kg, we get T = 5040 Seconds, or 84 minutes. For r = 6471, meaning that the satellite is in orbit 100 Km above the Earth's surface, the Period T is 86 minutes.

We can use the above to define the concept of gravitational potential energy.

Gravitational Potential and Gravitational Potential Energy

Often in analyzing the small variations in the gravitational force which one would feel moving around the surface of the earth, it is useful to talk about gravitational potential energy. For instance, when we drop a tennis ball, the gravitational potential energy possessed by the ball is converted into kinetic energy as the ball falls, ie, the ball speeds up as it falls. To analyze this problem, we need to specify what we mean by gravitational potential energy. We define Gravitational potential V due to mass m1 as

$$V = -\frac{Gm_1}{r}$$

This describes the *gravitational potential* V caused by the mass m_1. The gravitational potential *energy* which a mass m2 would possess due to the gravitational potential from mass m1 would be

$$V = -\frac{Gm_1 m2}{r}$$

Once we define this potential, then any object of mass mx will have potential energy equal to -Gm1mx/r . We can then obtain the gravitational acceleration from the potential by taking its derivative with respect to *r*.

$$gravitational\_accleration = -\frac{Gm_1}{r^2}$$

$$= -\frac{\partial}{\partial}\left(-\frac{Gm_1}{\quad}\right)$$

$$= \frac{\partial V}{\partial r}$$

If we generalize to three dimensions, the acceleration is simply the gradient of the gravitational potential.:

$$a = -gradV$$
$$= -\nabla V$$

Note that gravitational acceleration acts in the direction which is perpendicular to the local gravitational equipotential surface, ie, the surface over which the gravitational potential is constant.

Geoid

The geoid is simply any surface defined by constant gravitational potential, called an equipotential surface. For a spherical mass, the geoid would assume the shape of a sphere, but for any distribution of mass which deviates from a perfect, a surface of constant GP would assume a more complicated shape, which we discuss below. Typically in studying the Earth's geoid, the reference level is defined as a mean global sea level. The ocean surface is itself an equipotential surface, and the mean, or global average, sea level is in general not the same level as the sea surface at any given point on the globe. This is due to the fact that local mass anomalies under the ocean cause the ocean surface, or the geoid, to deform in their vicinity. Since the earth is neither a perfect sphere nor a perfect oblate sphereoid, geodesists use the surface of the oceans as the reference equipotential, since the liquid in the ocean follows an constant potential surface- if it didn't, the water would flow down- potential ('downhill') until the equipotential surface was created.

We can convert changes in the geoid into an equivalent geoid height anomalies by considering the work done, or energy expended, to move an object from one equipotential surface to another. Remember that work expended equals Force x Distance over which it is applied. To move an object of unit mass (m1 =1) a distance R to R+h away from object m2, we have

$$dW = F \bullet dr$$
$$= \frac{Gm_2}{r^2} \cdot dr$$
$$\Delta W = \Delta GPE = \int_R^{R+h} \frac{G\ m_2}{r^2} dr$$
$$= -\frac{Gm_2}{R+h} - \frac{Gm_2}{R}$$
$$= Gm_2 \left(\frac{-R+R+h}{(R+h)R} \right)$$

Now, since (R+h)R is approximately equal to R^2 when R >> h, then this reduces to

$$\Delta W = Gm_2 \frac{h}{R^2}$$

$$= g.h$$

and in general , the work done, or change in potential energy in moving an object of mass m1 from R to R+h equals

$$\Delta W = m_1 g.h$$

This makes sense- by lifting an object of mass m1 up a distance h, you impart to it extra GPE of the amount m1xgxh. Along the same lines, we can relate a change in just potential to a change in height. Then,

$$\Delta V = g \cdot h$$

Thus, we can describe a change in the geoid by the equivalent height an object would have to move up or down to possess the same potential.

The lateral variations in density distribution with the earth, although the result in gravity anomalies, can therefore be described in terms of a geoid height anomaly. Since the gravitational acceleration is normal to the geoid, a trough exists in the geoid whenever negative gravity anomaly, or mass deficit,, and likewise there is a bulge in the geoid wherever there is a mass excess, or positive gravity anomaly.

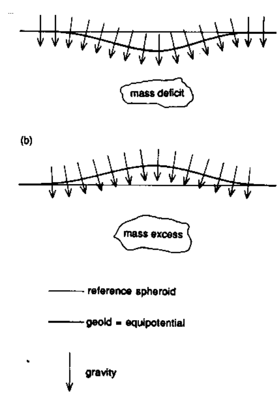
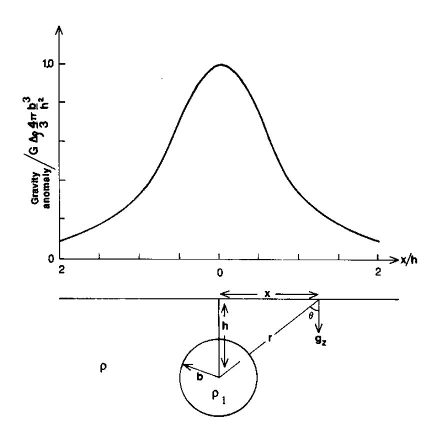


Figure 1- a) A trough in the geoid, or negative geoid height anomaly, occurs over a region with a mass deficit, such as a depression in the sea bed. A bulge in the geoid, or ositive geoid height anomaly, occurs over regions of excess mass, such as an elevated region of the sea bed.

We can calculate the gravitational attraction which will accompany a simple shape such as a buried sphere.



From above, we know that the gravitational accleration due to a sphere of mass *m* is Gm/r^2. However, this is the acceleration in the radial direction, but we want to know what the acceleration straight down. So we need some trig:

$$g_z = g\cos\theta = \frac{Gm}{r^2}\cos\theta$$
$$= \frac{Gm}{r^2}\frac{h}{r} = \frac{Gmh}{\left(x^2+h^2\right)^{\frac{3}{2}}}$$

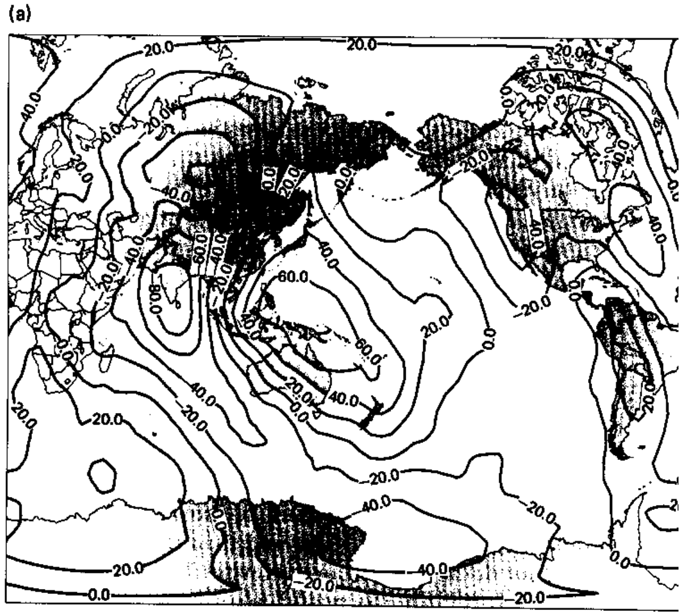
The gravity anomaly, δg_z is therefore given by

$$\delta g_z = \frac{4G\Delta\rho\pi b^3 h}{3\left(x^2+h^2\right)^{\frac{3}{2}}}$$

So the gravity anomaly due to this buried sphere is therefore symmetrical about the center of the sphere and essentially is confined to a width of about two to three times the depth of the sphere.

In the actual earth, the geoid moves about by a substantial amount, as illustrated below:

(a)



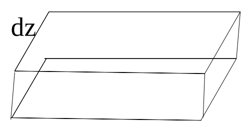
Geoid height variations, in meters. Amplitudes reach as much as +- 100 meters.

Isostasy

Isostasy is geophysical term which is an alternative statement of Archimedes principle of hydrostatic equilbrium, ie, that a floating body displaces its own weight of water. A mountain can therefore be in some sense compared to a floating iceberg or cork in water, in that it might be a relatively lighter mass of granite afloat in a sea of heavier mantle materials such as olivine or pyroxene. Isostasy therfore requires the surface layers of the earth to be rigid, and to float upon a denser substratum beneath. The rigid layer is termed the lithosphere, while the denser material below is termed the asthenosphere.

To understand isostasy, we first have to consider how pressure builds up from the overlying rock as one moves downward through the earth. This buildup in pressure results from *hydrostatic equilibrium*, or the fact that all mass is in equilibrium, ie it is not moving. Consider a block of rock deep in the Earth, shown below, with area A, density ρ, and thickness dz. If this chunk of rock is not moving and is subject to the overburden stress of all the rock piled on top of it, we say it is in *hydrostatic equilibrium,* and all the forces on it must sum to zero.

density ρ, thickness dz, Area A.



Top Face: The force acting along the top of the cube equals Pressure x area.

Bottom Face: The force acting on the bottom of the cube equals the force along the top face *plus* the weight of the cube itself: Pressure*Area + dz*Area*ρ*g.
Thus, we have:

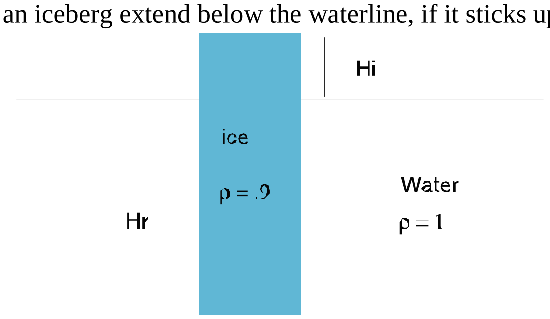
$$dForce = d\,\mathrm{Pr}essure * Area = \mathrm{Pr}essure * Area + \mathrm{dz} * \mathrm{Area} * \mathrm{r} * \gamma. - \mathrm{Pr}essure * Area$$
$$= dz * Area * \rho * g$$
$$d\,\mathrm{Pr}essure = dz * \rho * g$$
$$\frac{dP}{dz} = \rho g$$

If ρ and g depend on depth z, then to solve for the pressure at some depth R we must integrate from the surface down to depth R:

$$\mathrm{Pr}essure\_at\_R = P(R) = \int_0^R \rho(z) g(z) dz$$

The other key ingredient in understanding isostasy is Archimedes Principle- remember that a floating object displaces its mass of water. Let's calculate how far above the water surface level and iceberg should float, given that the density of ice is about 90% the density of water.

Example: How high does an iceberg extend below the waterline, if it sticks up by an amount equal to H?



Let's suppose the cross section of the ice is A. Then the mass of ice in the iceberg equals

$$Mass_{ice} = A*(Hr + Hi)*\rho = A*(Hr + Hi)*.9$$

and that mass equals the mass displaced by the water:

$$Mass_{water} = A*(Hr)*\rho = A*(Hr)*1$$

Equating these yields

$$Mass_{water} = Mass_{ice} \Rightarrow$$

$$H_{ice} = \frac{.1}{.9}H_r = .11H_r$$

So, the iceberg sticks up out of the water by an amount equal to about 10% of the amount it extends below the surface.

Airy Isostasy

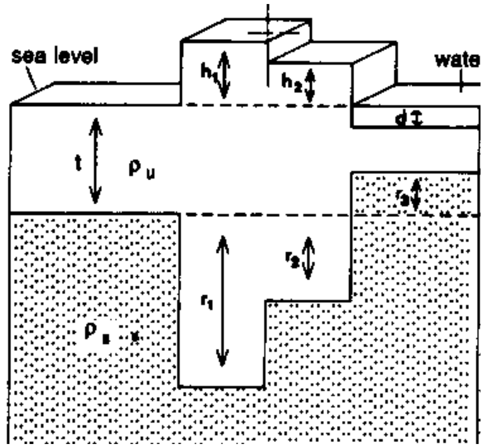
The same type of analysis can be applie to mountains, if one assumes they are in isostatic equilibrium with the surrounding host rock. We define two general types of isostasy in geophysics, the first is called **Airy Isostasy**, and this is the exact same mechanism as the iceberg above. Here, mountains are thought to have roots which extend down below the mountains, into the mantle of the earth, which is denser. The mountains, like icebergs, float in a sea of heavier rock. As in the iceberg case, we can describe how deep a mountain's root must extend by calculating it in terms of how high the mountain is above the surrounding landscape and the relative densities of the two materials. In this case, we must add up the overlying mass of rock for both the mountains and the non-mountainous areas and equate them.

Depth of Compensation

How far down do we need to go in terms of adding up the overlying mass of rock? To the bottom of the root- below this point , the densities of rocks are the same, so there's no point in adding them in. We call the bottom of the root the depth of compensation- at this depth, they hydrostatic pressures are the same regardless of whether or not we're under a mountain or not.

Calculating the depth of the Root.

To calculate the depth of the root for a mountain of height h_1 above sea level, we have



$$t\rho_u + r_1\rho_s = (h_1 + t + r_1)\rho_u \Rightarrow$$

$$r_1 = \frac{h_1\rho_u}{\rho_u - \rho_s}$$

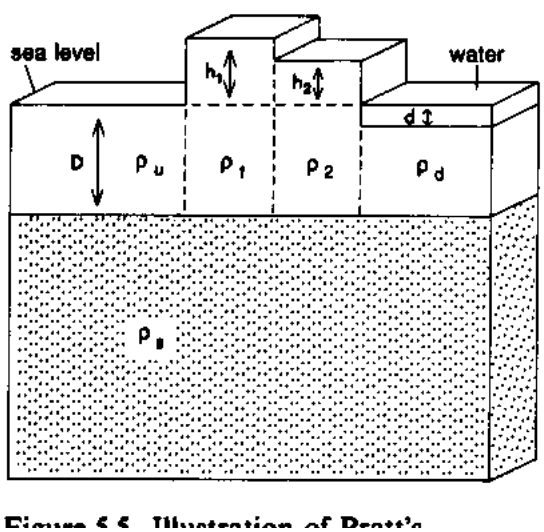
Similarly, if we we have a basin full of water, rather than a mountain, we can calculate the amount of missing light material, or the anti-root, that must exist to cause the basin to sink so low. In this case,

$$r_3 = \frac{d(\rho_w - \rho_u)}{\rho_s - \rho_u}$$

To summarize, Airy Isostasy is the mechanisms by which mountains 'float' on the mantle. Another type of isostasy is possible, however, which is known as **Pratt Isostasy.**

Pratt Isostasy

The mechanisms by which mountains are held up in Pratt isostasy is lateral variations in density, rather than deep continental roots extending down into the mantle. In this model, we assume that the height of the mountains vary simply as their density varies, much in the way that styrofoam will float higher on water than will waterlogged wood, for instance. As in Airy Isostasy, we have a depth of compensation equal to the depth of the bottom of the continent. However, in this case, we compute what the different densities must be to support the variable height mountains observed. The assumption here is that the bottom of the all the mountains exists at the same depth. Solving for the relative densities, we have



$$\rho_u D = (h_1 + D)\rho_1$$
$$= (h_2 + D)\rho_2$$
$$= (D - d)\rho_d + d\rho_w$$

Thus

$$\rho_1 = \frac{\rho_u D}{h_1 + d}$$

and

$$\rho_d = \frac{\rho_u D - d\rho_w}{D - d}$$

In general, most continental regimes consist of both Airy and Pratt Isostasy. The Sierra Nevada range, for instance, is known seismically to not have a root, so it is both lighter than the surround rock, but it also is dynamically supported by the mantle.

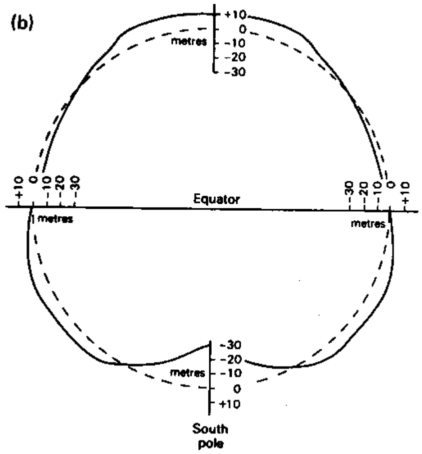
Gravity Corrections

When we go out and measure the gravity over a mountain range or basin, there are several corrections that we have to apply to what we measure before we can interpret our results in terms of the structures which produce them.

The first correction comes from the fact that the earth is not a perfect sphere, but is rather an oblate spheroid, as shown below. The *reference gravity formula* adopted by the International Association of Geodesy in 1967 is

$$g(\lambda) = g_c(1 + \alpha \sin^2 \lambda + \beta \sin^4 \lambda$$

where λ is the latitude of the point, and α and β are coefficients equal to , respectively, 5.27 x 10^{-3} and 2.34 x 10^{-5}, and g_c is the value of gravity at the equator (latitude = 0), 9.780 m/s^2.



The averaged shape of the earth, calculated by assuming that the earth is symmetric about is rotation axis (solid line), compared with a spheroid of flattening 1/298.25).

This formula for gravity as a function of latitude takes into account the fact that earth is not a sphere, and so gravity should not be uniform along it's surface. It depends on whether you're measuring at the poles, where the radius is shortest, or at the equator, where the radius is longest.

The next correction that has to be made allows for the fact that the point at which the measurement is made is typically not at sea level, which is what $g(\lambda)$ above describes. Typically, gravity is measured off on a mountain top or in an oceanic basin somewhere. So we have to take into account that we're not at sea level. This correction, known as the *free air correction*, only adjusts for the fact that we're not at sea level- it makes no allowance for any material that may exist between sea level and the point we're standing on, ie, the mountain underneath us: all such material is assumed to be air. To solve for this correction, we use the inverse square law and assume that the earth is a perfect sphere. As you might guess, the gravity at elevation h , g_h, is just the gravity at sea level, g_0, multiplied times the ratio of the two radiuses:

$$g(h) = g_0 \left(\frac{R}{R+h} \right)^2$$

where R is the radius of the earth to sea level, and h is our distance above sea level. Since $h << R$, we can approximate this with the first term of the binomial expansion (remember: $(1+a)^b = 1+ba+\ldots\ldots$, and we keep only the first two terms)

$$= g_0 \left(\frac{R+h}{R} \right)^{-2}$$

$$= g_0 \left(1 + \frac{h}{R} \right)^{-2}$$

$$g(h) \cong g_0 (1 - 2\frac{h}{R})$$

To calculate the *free air correction* , we then subtract this value from the gravity at sea level:

$$\delta g_f = g_0 - g(h) = \frac{2hg_0}{R}$$

As gravity decreases with height above the surface, points above sea level are corrected back to the value they would have at sea level by adding in this correction δg_f. The correction amounts to 3.1 x 10^{-6} m /s^2 per meter of elevation.

The *free air anomaly*, not to be confused with the *free air correction*, is then the measured value of gravity, g_{obs}. with the two corrections for the non-spherical earth and the free air applied:

$$g_f = g_{obs} - g(\lambda) + \delta g_f$$

$$= g_{obs} - g(\lambda)\left(1 - \frac{2h}{R}\right)$$

Notice the signs that each correction has: the *free air* correction is positive, because we're adding in gravity that would be there if the measuring point were at sea level, which it's not. $g(\lambda)$ is subtracted because we're subtracting away the reference earth model to obtain the free air anomaly g_f.

There are two other corrections we are routinely applied to gravity measurements, and these, as you might guess, take into account the fact that it's typically *not* air which lies between the measuring point at altitude h and sea level, but rock. The first is called the *Bouguer correction*, which allows for the gravitational attraction of a flat, infinite plane of rock of thickness h lying between the measuring point and sea level. This correction is given by

$$\delta g_B = 2\pi G\rho h$$

where G is the gravitational constant, ρ is the density of the infinite plate of rock between height h and sea level. Taking G = 6.67x10^{-11} m^{3}kg^{-1}s^{-2} and assuming a crustal density of 2.7 g/cm^3, the Bouguer correction is 1.1 x10^{-6} m/s^2 per meter of elevation.

The second correction is much smaller, and simply accounts for the fact that typically mountains are not infinite planes of rock, as is the assumption above, but rather have peaks and valleys. This correction is known as the *terrain correction*, δg_t, and is calculated numerically using a set of templates or a Digital Elevation Model which shows the topography. Except in regions of exceptional topography, this term can usually be ignored.

The *Bouguer anomaly*, not to be confused with the *Bouguer correction* above, is the free air anomaly with these two extra corrections applied:

$$g_B = g_f - \delta g_B + \delta g_t$$

$$= g_{obs} - g(\lambda) + \delta g_f - \delta g_B + \delta g_T$$

Notice again the sign conventions. This Bouguer anomaly is the observed value of gravity minus the reference value at the latitude of measurement plus the correction needed to adjust the altitude of measurement back to sea level minus the gravitational acceleration caused by the rock between the altitude of measurement and sea level. Since we have allowed for the attraction of the rock above sea level***, the Bouguer anomaly represents the gravitational attraction of the material below sea level.***

The free air anomaly is usually used for gravity measurements at sea. It is comparable to the Bouguer anomaly over continents, since the measurements are then all corrected to the sea level datum. If a Bouguer anomaly is required for oceanic gravity measurements, it must be calculated by replacing the seawater with rocks of average crustal density. A terrain model must then also be applied to account for topography along the seabed.

We can use gravity measurements to determine whether an area is in isostatic equilibrium. If a region is in isostatic equilibrium, there should be no gravity anomaly and hence no excess or lack of massabove the compensation depth. However, in practice, interpreting gravity measurements is a convoluted process. As an example, take the mountains shown above which are in 100% isostatic compensation of the Airy type. The Bouguer anomaly across these mountains is negative, since below sea level there is a mass deficit under the mountains, ie, the low density root is holding the overlying mountains up. The Bouguer anomaly reflects the fact that the overlying mountains have been removed from the correction, which leaves only the mass deficit at depth unaccounted for, which causes the negative Bouguer anomaly.

The free air anomaly, on the other hand, will be slightly positive, since this anomaly only takes into account the fact that we're above sea level in our measurements and doesn't take into account the *distribution* of mass below us. The slight positive reading comes from the fact that the overlying mountain is closer to us and and our point of measurement than is the compensating low density material at depth, and since graviational acceleration drops off as $1/\text{distance}^2$, the closer, mountain attraction io stronger than the more distant lack of attraction due to the mass deficit in the root, which results in a slight positive free air anomaly.

The simplest way to determine whether a large-scale structure such as a mountain chain is in isostatic equilibrium is to use the free air anomaly. If a structure is totally compensated, away from the edges of the

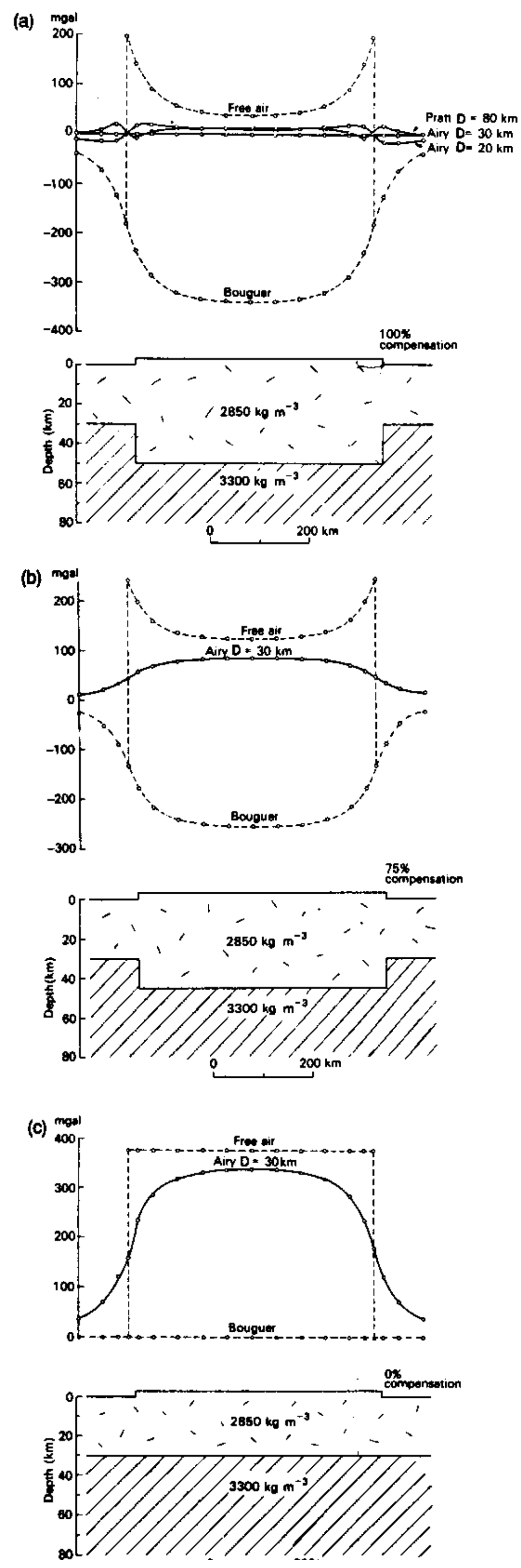
structure the free air anomaly will be very small. Near the edges is difficult to discern. If the structure is only partially compensated, the or not a t all, then the free air anomaly will be strongly positive, up to several hundred millgals, while the Bouguer anomaly will be about zero. Free air anomalies are always almost isostatic anomalies. They do not tell you what type of compensation is ocurring (ie, Pratt versus Airy), but if compensation of any mechanism is complete, then the free air anomaly will be nearly zero.

Gravity Profile Examples (below)

The examples below show several gravity anomaly profiles across isostatic and non-isostatically compensated mountains. In the example, (a), compensation is complete. The free air anomaly is nearly zero away from the edges, but otherwise slightly positive, because the mass excess of the mountain is compensated by the mass excess at depth, but the closer mass excess of the mountain produces a slightly stronger positive gravity anomaly then the negative anomaly caused by the deficit at depth. The Bouguer anomaly, on the other hand, is strongly negative, since it accounts for the gravity anomaly produced by the mountain but not the lack of gravity produced by the mass deficit at depth.

Skipping (b) for now, in (c), which is totally uncompensated, the Bouguer anomaly is zero since the there is no root to cause the negative anomaly. The free air anomaly is extremely positive, since we've only corrected for the altitude by assuming everything between sea level and our the altitude is air, when in fact it is rock and has a strong gravity signal associated with it.

In (b), the free air correction is moderately positive, while the Bouguer anomaly is moderately negative.



(a)
mgal
Free air
Pratt D = 80 km
Airy D= 30 km
Airy D= 20 km
Bouguer
100%
compensation
2850 kg m^{-3}
3300 kg m^{-3}
Depth (km)
0
200 km
(b)
mgal
Free air
Airy D = 30 km
Bouguer
75%
compensation
2850 kg m^{-3}
3300 kg m^{-3}
Depth (km)
0
200 km
(c)
mgal
Free air
Airy D = 30km
Bouguer
0%
compensation
2850 kg m^{-3}
3300 kg m^{-3}
Depth (km)