0.2. Functions

Definition 2.1. Let A and B be any two subset of $\mathbb{R}$. A **function** f is a rule that assigns to each element x in A exactly one element y in B. In this case, we write $y = f(x)$ which is called the image of x.

The set A is called the domain of f, and denoted by D_f. The set $R_f = \{f(x) | x \in A\} \subseteq B$ is called the range of f.

Vertical Line Test. If any vertical line intersects the graph in more than one point, the curve is not a graph of a function.

Definition 2.2.

1) A polynomial is a function in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $f(x) = a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$ (the coefficients), with $a_n \neq 0, n \geq 0$ is an integer (the degree of the polynomial). The domain is $D_f = \mathbb{R}$.

2) A rational function is a function in the form

$$f(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomials, $q(x) \neq 0$. The domain is

$$D_f = \{x \in \mathbb{R} | q(x) \neq 0\} = \mathbb{R} \setminus \{\text{zeros of } q(x)\}$$

3) A radical function is a function in the form

$$f(x) = \sqrt[n]{p(x)},$$

where p is a polynomial, and $n \geq 2$ (the index), is a natural number. For the domain we have two cases.

a) If n is odd, then $D_f = \mathbb{R}$.

b) If n is even, then $D_f = \{x \in \mathbb{R} | p(x) \geq 0\}$.

Example 2.1.

The followings are polynomials:

1) $f(x) = 3$, of degree 0 (constant function).

2) $f(x) = 2x - 1$, of degree 1 (linear function).

3) $f(x) = 5x^2 - 2x + 7$, of degree 2 (quadratic function).

4) $f(x) = x^3 + 5x^2 - 2x + 7$, of degree 3 (cubic function).

5) $f(x) = 3x^4 - 2x^3 + 5x^2 - 2x + 1$, of degree 4 (quartic function).

Example 2.2.
1) Find the domain of the rational function

$$f(x)=\frac{x^2+1}{x^2-x-6}$$

2) Find the domain of the rational function

$$f(x)=\frac{x+3}{x^2+1}$$

Solution. 1) The function $f(x)=\dfrac{x^2+1}{x^2-x-6}$ is a rational function. Then

$$\begin{aligned} D_f &= \{x\in\mathbb{R} \mid x^2-x-6\neq 0\} \\ &= \mathbb{R}\setminus\{-2,3\} \\ &= (-\infty,-2)\cup(-2,3)\cup(3,\infty) \end{aligned}$$

2) The function $f(x)=\dfrac{x+3}{x^2+1}$ is a rational function. Then

$$\begin{aligned} D_f &= \{x\in\mathbb{R} \mid x^2+1\neq 0\} \\ &= \mathbb{R} \\ &= (-\infty,\infty) \end{aligned}$$

Example 2.3. 1) Find the domain of the function

$$f(x)=\sqrt[3]{x^2-4}$$

2) Find the domain of the function

$$f(x)=\sqrt{x-3}$$

3) Find the domain of the function

$$f(x)=\sqrt{x^2-4}$$

4) Find the domain of the function

$$f(x)=\sqrt{4-x^2}$$

Solution. 1) The function $f(x)=\sqrt[3]{x^2-4}$ is a radical function with odd index . Then

$$D_f=\mathbb{R}$$

2) The function $f(x)=\sqrt{x-3}$ is a radical function with even index . Then

$$D_f = \{x \in \mathbb{R} | x - 3 \geq 0\}$$
$$= \{x \in \mathbb{R} | x \geq 3\}$$
$$= [3, \infty)$$

3) The function $f(x) = \sqrt{x^2 - 4}$ is a radical function with even index. Then

$$D_f = \{x \in \mathbb{R} | x^2 - 4 \geq 0\}$$
$$= \{x \in \mathbb{R} | (x+2)(x-2) \geq 0\}$$
$$= (-\infty, -2] \cup [2, \infty)$$

	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$(x+2)$	$---$	$+++$	$+++$
$(x-2)$	$---$	$---$	$+++$
$x^2 - 4 = (x+2)(x-2)$	$+++$	$---$	$+++$

4) The function $f(x) = \sqrt{4 - x^2}$ is a radical function with even index. Then

$$D_f = \{x \in \mathbb{R} | 4 - x^2 \geq 0\}$$
$$= \{x \in \mathbb{R} | (2+x)(2-x) \geq 0\}$$
$$= [-2, 2]$$

	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
$(2+x)$	$---$	$+++$	$+++$
$(2-x)$	$+++$	$+++$	$---$
$4 - x^2 = (2+x)(2-x)$	$---$	$+++$	$---$

Example 2.4. Find the $x-$intercepts and $y-$intercepts of $f(x) = x^2 - 4x + 3$.

Solution. To find the $x-$intercepts, we solve $f(x) = 0$. Then

$$x^2 - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$

Then $x = 0$ or $y = 0$.

To find the y −intercepts, we set $x = 0$. Thus $y = 3$.

Let consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where, $a \neq 0$. Then the solution(s) is given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Theorem 2.1. A polynomial of degree n has at most n distinct zeros.

Theorem 2.1. For a polynomial f , $f(a) = 0$ if and only if $(x - a)$ is a factor of $f(x)$.

Example 2.5. Find the zeros of

1) $f(x) = x^2 - 5x - 12$.

2) $f(x) = x^3 - x^2 - 2x + 2$.

Solution. 1) We have $a = 1, b = -5, c = -12$. Then

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-12)}}{2(1)} \\ &= \frac{5 \pm \sqrt{25 + 48}}{2} \\ &= \frac{5 \pm \sqrt{73}}{2} \end{aligned}$$

Thus $x = \dfrac{5 + \sqrt{73}}{2} = 6.772$, or $x = \dfrac{5 - \sqrt{73}}{2} = -1.772$.

2) By calculating $f(1)$, we have $(x - 1)$ is a factor of

$$f(x) = x^3 - x^2 - 2x + 2.$$

Then

$$\begin{aligned} f(x) &= x^3 - x^2 - 2x + 2 \\ &= (x - 1)(x^2 - 2) \end{aligned}$$

Here, we have

$$x^3 - x^2 - 2x + 2 = 0$$

$$(x-1)(x^2-2) = 0$$

$$(x-1)(x-\sqrt{2})(x+\sqrt{2}) = 0$$

Then the solution is $x = 1, x = \sqrt{2},$ or $x = -\sqrt{2}.$

Example 2.6. Find the points of intersection of the parabola $y = x^2 - x - 5$ and the line $y = x + 3$.

Solution. We set both equations equal. Then

$$x^2 - x - 5 = x + 3$$

Hence,

$$x^2 - x - 5 - x - 3 = 0$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

The solution is $x = -2,$ or $x = 4$.

Exercises 0.1 I) Identify the type of the function.

1) $x^3 - 4x + 1$ Sol: Polynomial (cubic).

2) $\dfrac{x^2 + 2x + 1}{x + 1}$ Sol: rational function.

3) $\sqrt{x^2 + 1}$ Sol: radical function.

II) Find the domain of:

4) $f(x) = x^2 + 3x - 4$ Sol: $\mathbb{R}$.

5) $f(x) = \sqrt{x + 2}$ Sol: $D_f = [-2, \infty)$.

6) $f(x) = \sqrt{x^2 - 25}$ Sol: $D_f = (-\infty, -5] \cup [5, \infty)$.

7) $f(x) = \dfrac{x + 2}{\sqrt{x^2 - 25}}$ Sol: $D_f = (-\infty, -5) \cup (5, \infty)$.

8) $f(x) = \sqrt{25 - x^2}$ Sol: $D_f = [-5, 5]$.

9) $f(x) = \dfrac{x + 2}{\sqrt{25 - x^2}}$ Sol: $D_f = (-5, 5)$.

9) $f(x) = \dfrac{4}{x^2 - 1}$ Sol: $D_f = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

10) $f(x) = \sqrt[3]{x - 1}$ Sol: $\mathbb{R}$.

III) Find the x −intercepts and y −intercepts of $y = f(x)$:

11) $y = x^2 - 2x - 8$. Sol: $x = -2, 4$ and $y = -8$.

12) $y = x^3 - 8$ Sol: $(x-2)(x^2+2x+4) = 0$ and $y = -8$.

10) $(-1,-2)$ & $(3,-2)$

Sol: 4

11) $y = \dfrac{x^2 - 4}{x+1}$ Sol: $x = \pm 2$ and $y = -4$.

IV) Find the zeros of

12) $f(x) = x^2 - 5x + 6$ Sol: $x = 2, 3$.

13) $f(x) = x^3 - 3x^2 + 2x$ Sol: $x = 0, 1, 2$.

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Workshop 2: Functions

1) If $f(x) = x^2 - 9$, then the domain is

A $D_f = \mathbb{R}$ B $D_f = (-\infty,-3] \cup [3,\infty)$ C $D_f = [-3,3]$ D $D_f = (-3,3)$

2) If $f(x) = \sqrt[3]{x-2}$, then the domain is

A $D_f = [2,\infty)$ B $D_f = \mathbb{R}$ C $D_f = (-\infty,2]$ D $D_f = (2,\infty)$

3) If $f(x) = \sqrt{x^2 - 9}$, then the domain is

A $D_f = (-\infty,-3) \cup (3,\infty)$ B $D_f = (-3,3)$

C $D_f = [-3,3]$ D $D_f = (-\infty,-3] \cup [3,\infty)$

4) If $f(x) = \dfrac{x+5}{\sqrt{9-x^2}}$, then the domain is

A $D_f = (-\infty,-3) \cup (3,\infty)$ B $D_f = (-3,3)$

C $D_f = [-3,3]$ D $D_f = (-\infty,-3] \cup [3,\infty)$

5) If $f(x) = \dfrac{x+7}{x^2 - 5x + 6}$, then the domain is

A $D_f = (-2,-3)$ B $D_f = (2,3)$

C $D_f = \mathbb{R} \setminus \{2,3\}$ D $D_f = \mathbb{R} \setminus \{-2,-3\}$

6) The function $f(x) = \sqrt{x^2 + x - 1}$ is

A Linear B Cubic C Radical D Rational

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