24. Let $g \in C^1[a, b]$ and p be in (a, b) with $g(p) = p$ and $|g'(p)| > 1$. Show that there exists a $\delta > 0$ such that if $0 < |p_0 - p| < \delta$, then $|p_0 - p| < |p_1 - p|$. Thus, no matter how close the initial approximation p_0 is to p, the next iterate p_1 is farther away, so the fixed-point iteration does not converge if $p_0 \neq p$.

2.3 Newton's Method

Newton's (or the *Newton-Raphson*) **method** is one of the most powerful and well-known numerical methods for solving a root-finding problem. There are many ways of introducing Newton's method. If we only want an algorithm, we can consider the technique graphically, as is often done in calculus. Another possibility is to derive Newton's method as a technique to obtain faster convergence than offered by other types of functional iteration, as is done in Section 2.4. A third means of introducing Newton's method, which is discussed next, is based on Taylor polynomials.

Suppose that $f \in C^2[a, b]$. Let $\bar{x} \in [a, b]$ be an approximation to p such that $f'(\bar{x}) \neq 0$ and $|p - \bar{x}|$ is "small." Consider the first Taylor polynomial for $f(x)$ expanded about $\bar{x}$,

$$f(x) = f(\bar{x}) + (x - \bar{x})f'(\bar{x}) + \frac{(x - \bar{x})^2}{2} f''(\xi(x)),$$

where $\xi(x)$ lies between x and $\bar{x}$. Since $f(p) = 0$, this equation with $x = p$ gives

$$0 = f(\bar{x}) + (p - \bar{x})f'(\bar{x}) + \frac{(p - \bar{x})^2}{2} f''(\xi(p)).$$

Newton's method is derived by assuming that since $|p - \bar{x}|$ is small, the term involving $(p - \bar{x})^2$ is much smaller, so

$$0 \approx f(\bar{x}) + (p - \bar{x})f'(\bar{x}).$$

Solving for p gives

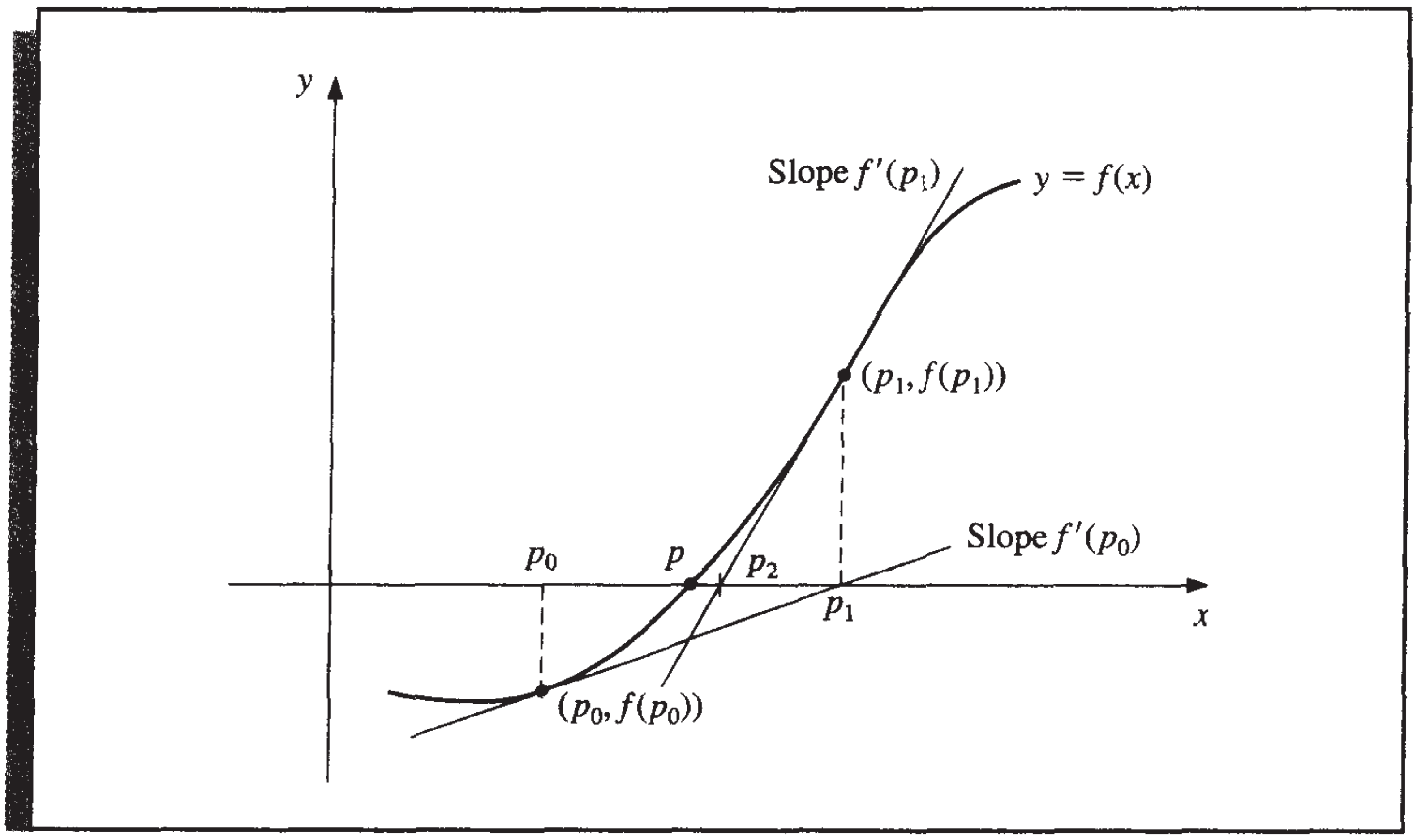
$$p \approx \bar{x} - \frac{f(\bar{x})}{f'(\bar{x})}.$$

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1. \tag{2.5}$$

Figure 2.7 illustrates how the approximations are obtained using successive tangents. (Also see Exercise 11.) Starting with the initial approximation p_0, the approximation p_1 is the x-intercept of the tangent line to the graph of f at $(p_0, f(p_0))$. The approximation p_2 is the x-intercept of the tangent line to the graph of f at $(p_1, f(p_1))$ and so on. Algorithm 2.3 follows this procedure.

Figure 2.7



ALGORITHM 2.3

Newton's

To find a solution to $f(x) = 0$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. *(Compute p_i.)*

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. *(Update p_0.)*

Step 7 OUTPUT ('The method failed after N_0 iterations, N_0 =', N_0);
(The procedure was unsuccessful.)
STOP. ■

The stopping-technique inequalities given with the Bisection method are applicable to Newton's method. That is, select a tolerance $\varepsilon > 0$, and construct $p_1, \ldots p_N$ until

$$|p_N - p_{N-1}| < \varepsilon, \tag{2.6}$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0, \tag{2.7}$$

or

$$|f(p_N)| < \varepsilon. \tag{2.8}$$

A form of Inequality (2.6) is used in Step 4 of Algorithm 2.3. Note that inequality (2.8) may not give much information about the actual error $|p_N - p|$. (See Exercise 14 in Section 2.1.)

Newton's method is a functional iteration technique of the form $p_n = g(p_{n-1})$, for which

$$g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1. \tag{2.9}$$

In fact, this is the functional iteration technique that was used to give the rapid convergence we saw in part (e) of Example 3 in Section 2.2.

It is clear from equation (2.9) that Newton's method cannot be continued if $f'(p_{n-1}) = 0$ for some n. In fact, we will see that the method is most effective when f' is bounded away from zero near p.

EXAMPLE 1 Suppose we would like to approximate a fixed point of $g(x) = \cos x$. The graph in Figure 2.8 implies that a single fixed-point p lies in $[0, \pi/2]$.

Figure 2.8

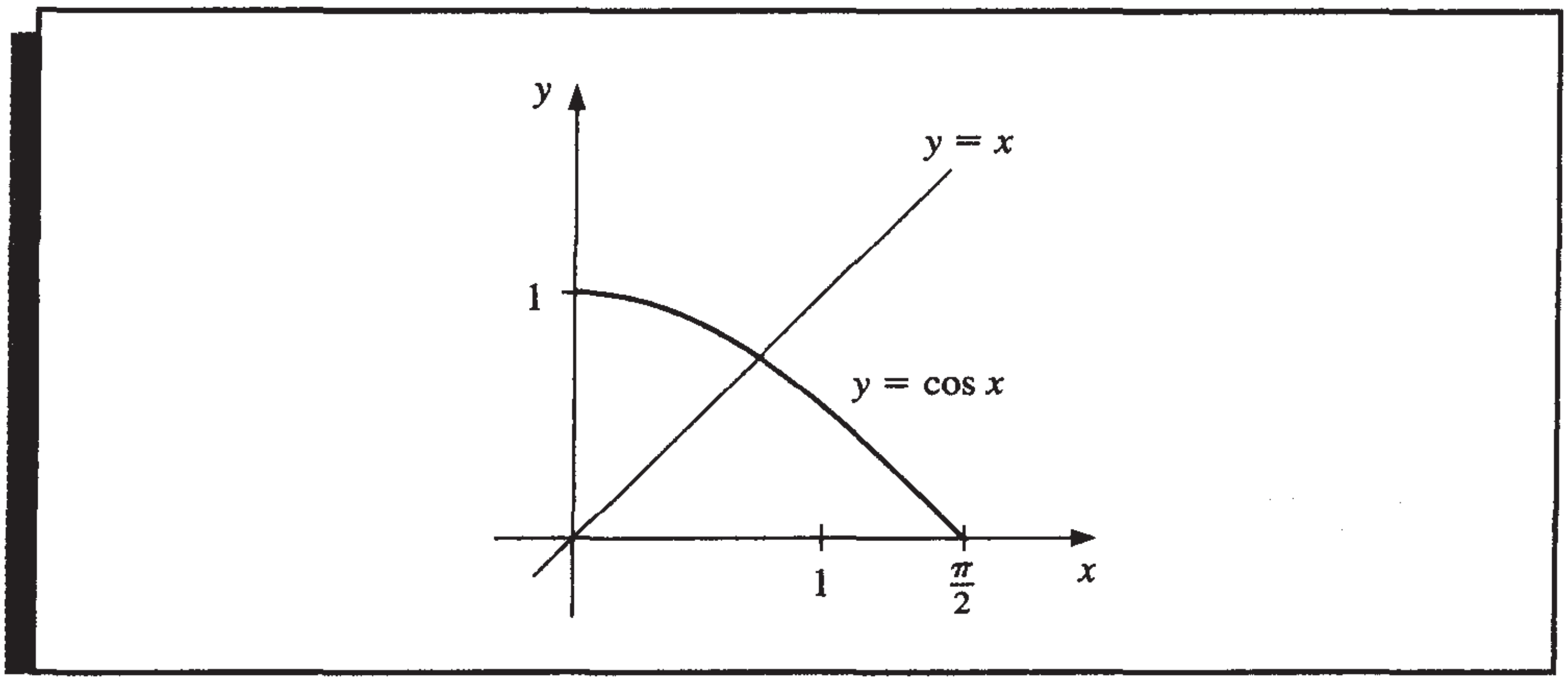


Table 2.3 shows the results of fixed-point iteration with $p_0 = \pi/4$. The best we could conclude from these results is that $p \approx 0.74$.

To approach this problem differently, define $f(x) = \cos x - x$ and apply Newton's method. Since $f'(x) = -\sin x - 1$, the sequence is generated by

$$p_n = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}, \quad \text{for } n \geq 1.$$

With $p_0 = \pi/4$, the approximations in Table 2.4 are generated. An excellent approximation is obtained with $n = 3$. We would expect this result to be accurate to the places listed because of the agreement of p_3 and p_4. ■

Table 2.3

n	p_n
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

Table 2.4

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

The Taylor series derivation of Newton's method at the beginning of the section points out the importance of an accurate initial approximation. The crucial assumption is that the term involving $(p - \overline{x})^2$ is, by comparison with $|p - \overline{x}|$, so small that it can be deleted. This will clearly be false unless $\overline{x}$ is a good approximation to p. If p_0 is not sufficiently close to the actual root, there is little reason to suspect that Newton's method will converge to the root. However, in some instances, even poor initial approximations will produce convergence. (Exercises 12 and 16 illustrate some of these possibilities.)

The following convergence theorem for Newton's method illustrates the theoretical importance of the choice of p_0.

Theorem 2.5 Let $f \in C^2[a, b]$. If $p \in [a, b]$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$. ∎

Proof The proof is based on analyzing Newton's method as the functional iteration scheme $p_n = g(p_{n-1})$, for $n \geq 1$, with

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Let k be in $(0, 1)$. We first find an interval $[p - \delta, p + \delta]$ that g maps into itself and for which $|g'(x)| \leq k$, for all $x \in (p - \delta, p + \delta)$.

Since f' is continuous and $f'(p) \neq 0$, part **(a)** of Exercise 27 in Section 1.1 implies that there exists a $\delta_1 > 0$, such that $f'(x) \neq 0$ for $x \in [p - \delta_1, p + \delta_1] \subseteq [a, b]$. Thus, g is defined and continuous on $[p - \delta_1, p + \delta_1]$. Also,

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2} = \frac{f(x)f''(x)}{[f'(x)]^2},$$

for $x \in [p - \delta_1, p + \delta_1]$, and, since $f \in C^2[a, b]$, we have $g \in C^1[p - \delta_1, p + \delta_1]$.

By assumption, $f(p) = 0$, so

$$g'(p) = \frac{f(p)f''(p)}{[f'(p)]^2} = 0.$$

Since g' is continuous and $0 < k < 1$, part **(b)** of Exercise 27 in Section 1.1 implies that there exists a δ, with $0 < \delta < \delta_1$, and

$$|g'(x)| \leq k, \qquad \text{for all } x \in [p - \delta, p + \delta].$$

It remains to show that g maps $[p-\delta, p+\delta]$ into $[p-\delta, p+\delta]$. If $x \in [p-\delta, p+\delta]$, the Mean Value Theorem implies that for some number ξ between x and p, $|g(x) - g(p)| = |g'(\xi)||x - p|$. So

$$|g(x) - p| = |g(x) - g(p)| = |g'(\xi)||x - p| \leq k|x - p| < |x - p|.$$

Since $x \in [p - \delta, p + \delta]$, it follows that $|x - p| < \delta$ and that $|g(x) - p| < \delta$. Hence, g maps $[p - \delta, p + \delta]$ into $[p - \delta, p + \delta]$.

All the hypotheses of the Fixed-Point Theorem are now satisfied, so the sequence $\{p_n\}_{n=1}^{\infty}$, defined by

$$p_n = g(p_{n-1}) = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \qquad \text{for } n \geq 1,$$

converges to p for any $p_0 \in [p - \delta, p + \delta]$. ■ ■ ■

Theorem 2.5 states that, under reasonable assumptions, Newton's method converges provided a sufficiently accurate initial approximation is chosen. It also implies that the constant k that bounds the derivative of g, and, consequently, indicates the speed of convergence of the method, decreases to 0 as the procedure continues. This result is important for the theory of Newton's method, but it is seldom applied in practice since it does not tell us how to determine δ. In a practical application, an initial approximation is selected, and successive approximations are generated by Newton's method. These will generally either converge quickly to the root, or it will be clear that convergence is unlikely.

Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of f at each approximation. Frequently, $f'(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.

To circumvent the problem of the derivative evaluation in Newton's method, we introduce a slight variation. By definition,

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

Letting $x = p_{n-2}$, we have

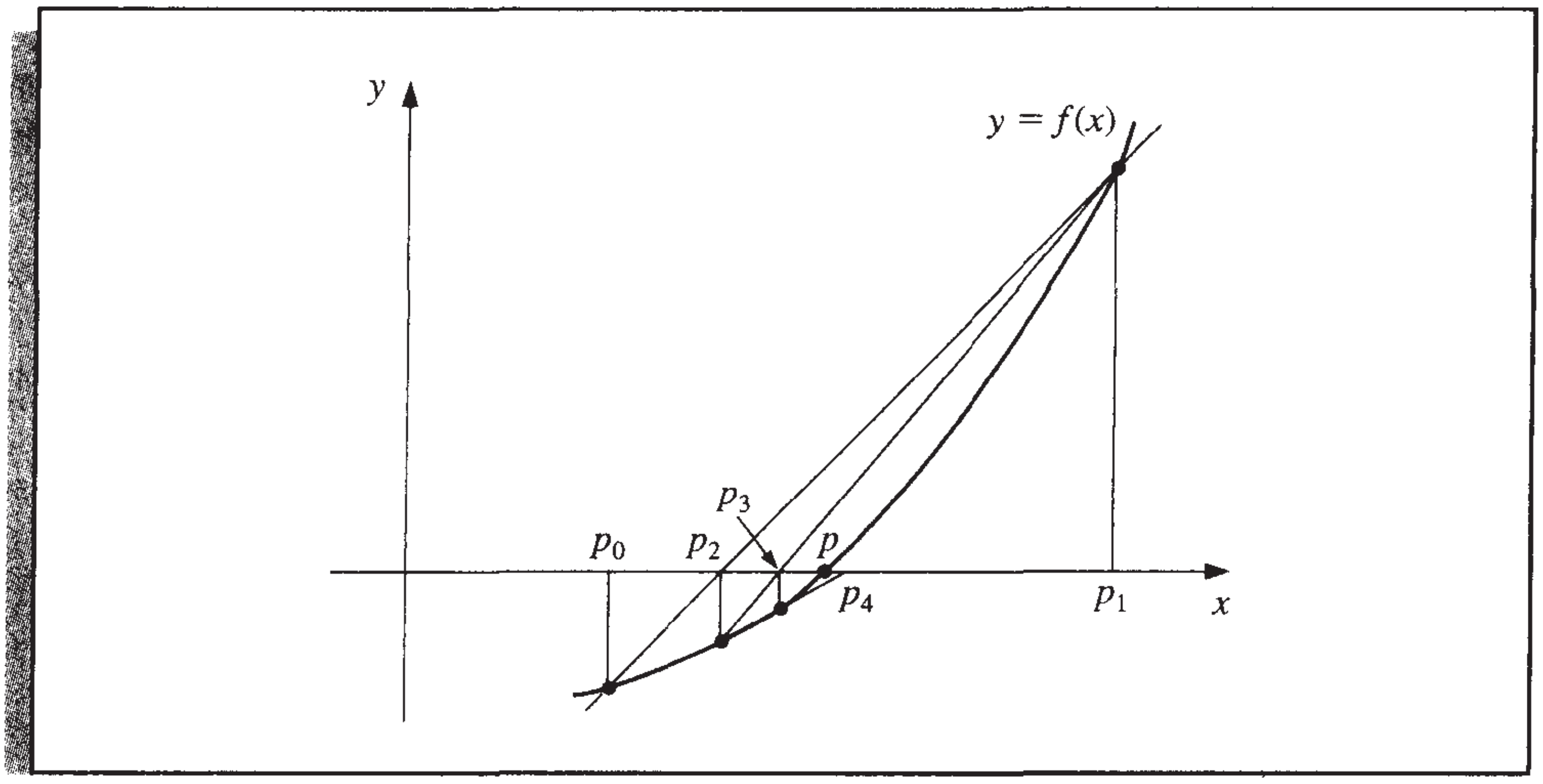
$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

Using this approximation for $f'(p_{n-1})$ in Newton's formula gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}. \tag{2.10}$$

This technique is called the **Secant method** and is presented in Algorithm 2.4. (See Figure 2.9.) Starting with the two initial approximations p_0 and p_1, the approximation p_2 is the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. The approximation p_3 is the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$, and so on.

Figure 2.9



ALGORITHM 2.4

Secant

To find a solution to $f(x) = 0$ given initial approximations p_0 and p_1:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \le N_0$ do Steps 3–6.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. *(Compute p_i.)*

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p_1$; *(Update p_0, q_0, p_1, q_1.)*
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

Step 7 OUTPUT ('The method failed after N_0 iterations, N_0 =', N_0);
(The procedure was unsuccessful.)
STOP. ■

The next example involves a problem considered in Example 1, where we used Newton's method with $p_0 = \pi/4$.

EXAMPLE 2 Use the Secant method to find a solution to $x = \cos x$. In Example 1 we compared functional iteration and Newton's method with the initial approximation $p_0 = \pi/4$. Here we need two initial approximations. Table 2.5 lists the calculations with $p_0 = 0.5$, $p_1 = \pi/4$, and the formula

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_{n-2})(\cos p_{n-1} - p_{n-1})}{(\cos p_{n-1} - p_{n-1}) - (\cos p_{n-2} - p_{n-2})}, \quad \text{for } n \geq 2,$$

from Algorithm 2.4. ■

Table 2.5

n	p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390851493
5	0.7390851332

By comparing the results here with those in Example 1, we see that p_5 is accurate to the tenth decimal place. The convergence of the Secant method is much faster than functional iteration but slightly slower than Newton's method, which obtained this degree of accuracy with p_3. This is generally true. (See Exercise 12 of Section 2.4.)

Newton's method or the Secant method is often used to refine an answer obtained by another technique, such as the Bisection method, since these methods require a good first approximation but generally give rapid convergence.

Each successive pair of approximations in the Bisection method brackets a root p of the equation; that is, for each positive integer n, a root lies between a_n and b_n. This implies that, for each n, the Bisection method iterations satisfy

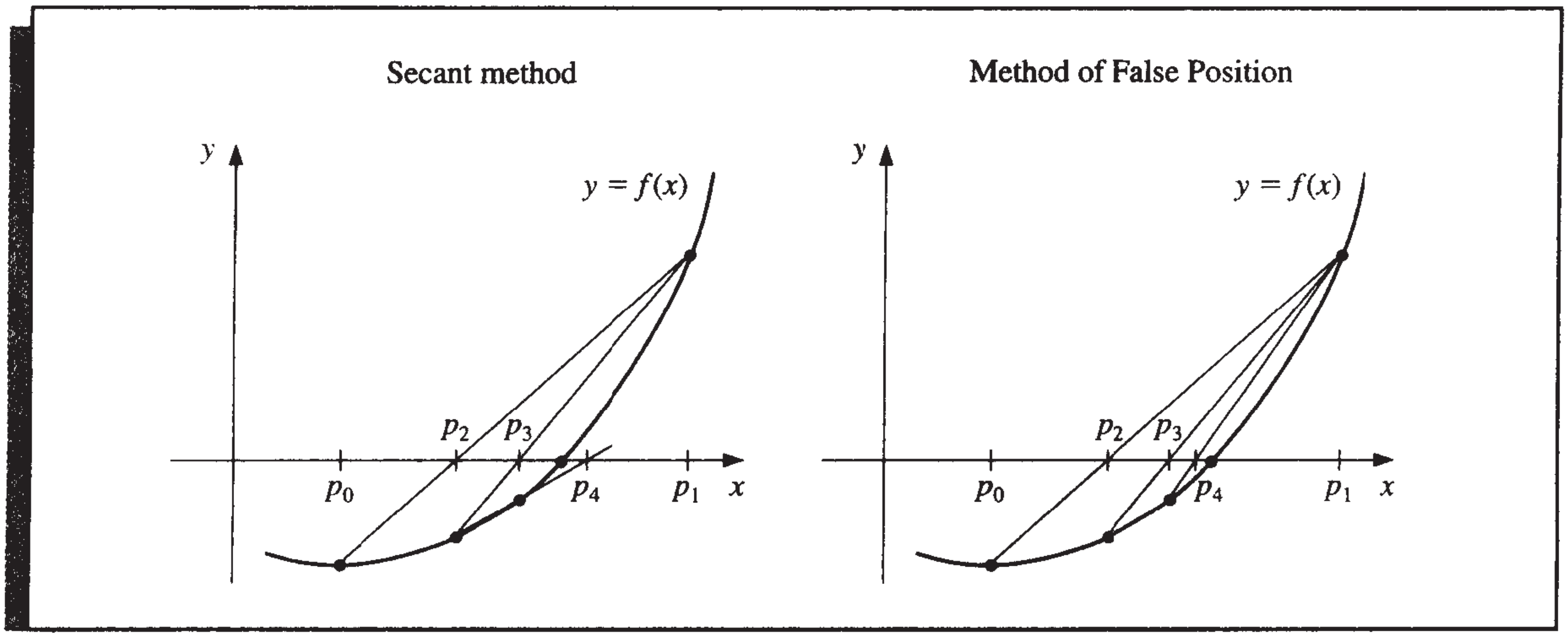
$$|p_n - p| < \frac{1}{2}|a_n - b_n|,$$

which provides an easily calculated error bound for the approximations. Root bracketing is not guaranteed for either Newton's method or the Secant method. Table 2.4 contains results from Newton's method applied to $f(x) = \cos x - x$, where an approximate root was found to be 0.7390851332. Notice that this root is not bracketed by either p_0, p_1 or p_1, p_2. The Secant method approximations for this problem are given in Table 2.5. The initial approximations p_0 and p_1 bracket the root, but the pair of approximations p_3 and p_4 fail to do so.

The **method of False Position** (also called *Regula Falsi*) generates approximations in the same manner as the Secant method, but it includes a test to ensure that the root is bracketed between successive iterations. Although it is not a method we generally recommend, it illustrates how bracketing can be incorporated.

First choose initial approximations p_0 and p_1 with $f(p_0) \cdot f(p_1) < 0$. The approximation p_2 is chosen in the same manner as in the Secant method, as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. To decide which secant line to use to compute p_3, we check $f(p_2) \cdot f(p_1)$. If this value is negative, then p_1 and p_2 bracket a root, and we choose p_3 as the x-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$. If not, we choose p_3 as the x-intercept of the line joining $(p_0, f(p_0))$ and $(p_2, f(p_2))$, and then interchange the indices on p_0 and p_1. In a similar manner, once p_3 is found, the sign of $f(p_3) \cdot f(p_2)$ determines whether we use p_2 and p_3 or p_3 and p_1 to compute p_4. In the latter case a relabeling of p_2 and p_1 is performed. The relabeling ensures that the root is bracketed between successive iterations. The process is described in Algorithm 2.5, and Figure 2.10 shows how the iterations can differ from those of the Secant method. In this illustration, the first three approximations are the same, but the fourth approximations differ.

Figure 2.10



ALGORITHM 2.5

Method of False Position

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[p_0, p_1]$ where $f(p_0)$ and $f(p_1)$ have opposite signs:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \leq N_0$ do Steps 3–7.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. *(Compute p_i.)*

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*
STOP.

[redacted]

Step 5 Set $i = i + 1$;
$q = f(p)$.

Step 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.

Step 7 Set $p_1 = p$;
$q_1 = q$.

Step 8 OUTPUT ('Method failed after N_0 iterations, N_0 =', N_0);
(*The procedure unsuccessful.*)
STOP. ■

EXAMPLE 3 Table 2.6 shows the results of the method of False Position applied to $f(x) = \cos x - x$ with the same initial approximations we used for the Secant method in Example 2. Notice that the approximations agree through p_3 and that the method of False Position requires an additional iteration to obtain the same accuracy as the Secant method. ■

Table 2.6

n	p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390848638
5	0.7390851305
6	0.7390851332

The added insurance of the method of False Position commonly requires more calculation than the Secant method, just as the simplification that the Secant method provides over Newton's method usually comes at the expense of additional iterations. Further examples of the positive and negative features of these methods can be seen by working Exercises 13 and 14.

EXERCISE SET 2.3

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.
2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?
3. Let $f(x) = x^2 - 6$. With $p_0 = 3$ and $p_1 = 2$, find p_3.
   - **a.** Use the Secant method.
   - **b.** Use the method of False Position.
   - **c.** Which of (a) or (b) is closer to $\sqrt{6}$?
4. Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3.
   - **a.** Use the Secant method.
   - **b.** Use the method of False Position.

5. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.

a. $x^3 - 2x^2 - 5 = 0, \quad [1, 4]$ **b.** $x^3 + 3x^2 - 1 = 0, \quad [-3, -2]$

c. $x - \cos x = 0, \quad [0, \pi/2]$ **d.** $x - 0.8 - 0.2 \sin x = 0, \quad [0, \pi/2]$

6. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.

a. $e^x + 2^{-x} + 2\cos x - 6 = 0$ for $1 \le x \le 2$

b. $\ln(x-1) + \cos(x-1) = 0$ for $1.3 \le x \le 2$

c. $2x \cos 2x - (x-2)^2 = 0$ for $2 \le x \le 3$ and $3 \le x \le 4$

d. $(x-2)^2 - \ln x = 0$ for $1 \le x \le 2$ and $e \le x \le 4$

e. $e^x - 3x^2 = 0$ for $0 \le x \le 1$ and $3 \le x \le 5$

f. $\sin x - e^{-x} = 0$ for $0 \le x \le 1$ $3 \le x \le 4$ and $6 \le x \le 7$

7. Repeat Exercise 5 using (i) the Secant method and (ii) the method of False Position.

8. Repeat Exercise 6 using (i) the Secant method and (ii) the method of False Position.

9. Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = x^2$ that is closest to $(1, 0)$. [*Hint:* Minimize $[d(x)]^2$, where $d(x)$ represents the distance from (x, x^2) to $(1, 0)$.]

10. Use Newton's method to approximate, to within 10^{-4}, the value of x that produces the point on the graph of $y = 1/x$ that is closest to (2, 1).

11. The following describes Newton's method graphically: Suppose that $f'(x)$ exists on $[a, b]$ and that $f'(x) \neq 0$ on $[a, b]$. Further, suppose there exists one $p \in [a, b]$ such that $f(p) = 0$, and let $p_0 \in [a, b]$ be arbitrary. Let p_1 be the point at which the tangent line to f at $(p_0, f(p_0))$ crosses the x-axis. For each $n \ge 1$, let p_n be the x-intercept of the line tangent to f at $(p_{n-1}, f(p_{n-1}))$. Derive the formula describing this method.

12. Use Newton's method to solve the equation

$$0 = \frac{1}{2} + \frac{1}{4}x^2 - x \sin x - \frac{1}{2}\cos 2x, \quad \text{with } p_0 = \frac{\pi}{2}.$$

Iterate using Newton's method until an accuracy of 10^{-5} is obtained. Explain why the result seems unusual for Newton's method. Also, solve the equation with $p_0 = 5\pi$ and $p_0 = 10\pi$.

13. The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in $[-1, 0]$ and the other in $[0, 1]$. Attempt to approximate these zeros to within 10^{-6} using the

a. Method of False Position

b. Secant method

c. Newton's method

Use the endpoints of each interval as the initial approximations in (a) and (b) and the midpoints as the initial approximation in (c).

14. The function $f(x) = \tan \pi x - 6$ has a zero at $(1/\pi)\arctan 6 \approx 0.447431543$. Let $p_0 = 0$ and $p_1 = 0.48$, and use ten iterations of each of the following methods to approximate this root. Which method is most successful and why?

a. Bisection method

b. Method of False Position

c. Secant method

15. The iteration equation for the Secant method can be written in the simpler form

$$p_n = \frac{f(p_{n-1})p_{n-2} - f(p_{n-2})p_{n-1}}{f(p_{n-1}) - f(p_{n-2})}.$$

Explain why, in general, this iteration equation is likely to be less accurate than the one given in Algorithm 2.4.

16. The equation $x^2 - 10\cos x = 0$ has two solutions, ± 1.3793646. Use Newton's method to approximate the solutions to within 10^{-5} with the following values of p_0.

   a. $p_0 = -100$ b. $p_0 = -50$ c. $p_0 = -25$
   d. $p_0 = 25$ e. $p_0 = 50$ f. $p_0 = 100$

17. Use Maple to determine how many iterations of Newton's method with $p_0 = \pi/4$ are needed to find a root of $f(x) = \cos x - x$ to within 10^{-100}.

18. Repeat Exercise 17 with $p_0 = \frac{1}{2}$, $p_1 = \frac{\pi}{4}$, and the Secant method.

19. The function described by $f(x) = \ln(x^2 + 1) - e^{0.4x} \cos \pi x$ has an infinite number of zeros.
   a. Determine, within 10^{-6}, the only negative zero.
   b. Determine, within 10^{-6}, the four smallest positive zeros.
   c. Determine a reasonable initial approximation to find the nth smallest positive zero of f. [*Hint:* Sketch an approximate graph of f.]
   d. Use part (c) to determine, within 10^{-6}, the 25th smallest positive zero of f.

20. Find an approximation for λ, accurate to within 10^{-4}, for the population equation

$$1{,}564{,}000 = 1{,}000{,}000 e^{\lambda} + \frac{435{,}000}{\lambda}(e^{\lambda} - 1),$$

discussed in the introduction to this chapter. Use this value to predict the population at the end of the second year, assuming that the immigration rate during this year remains at 435,000 individuals per year.

21. The sum of two numbers is 20. If each number is added to its square root, the product of the two sums is 155.55. Determine the two numbers to within 10^{-4}.

22. The accumulated value of a savings account based on regular periodic payments can be determined from the *annuity due equation,*

$$A = \frac{P}{i}[(1+i)^n - 1].$$

In this equation, A is the amount in the account, P is the amount regularly deposited, and i is the rate of interest per period for the n deposit periods. An engineer would like to have a savings account valued at \$750,000 upon retirement in 20 years and can afford to put \$1500 per month toward this goal. What is the minimal interest rate at which this amount can be invested, assuming that the interest is compounded monthly?

23. Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}[1 - (1+i)^{-n}],$$

known as an *ordinary annuity equation.* In this equation, A is the amount of the mortgage, P is the amount of each payment, and i is the interest rate per period for the n payment periods. Suppose that a 30-year home mortgage in the amount of \$135,000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximal interest rate the borrower can afford to pay?

24. A drug administered to a patient produces a concentration in the blood stream given by $c(t) = Ate^{-t/3}$ milligrams per milliliter, t hours after A units have been injected. The maximum safe concentration is 1 mg/ml.

a. What amount should be injected to reach this maximum safe concentration, and when does this maximum occur?

b. An additional amount of this drug is to be administered to the patient after the concentration falls to 0.25 mg/ml. Determine, to the nearest minute, when this second injection should be given.

c. Assume that the concentration from consecutive injections is additive and that 75% of the amount originally injected is administered in the second injection. When is it time for the third injection?

25. Let $f(x) = 3^{3x+1} - 7 \cdot 5^{2x}$.

a. Use the Maple commands `solve` and `fsolve` to try to find all roots of f.

b. Plot $f(x)$ to find initial approximations to roots of f.

c. Use Newton's method to find roots of f to within 10^{-16}.

d. Find the exact solutions of $f(x) = 0$ algebraically.

26. Repeat Exercise 25 using $f(x) = 2^{x^2} - 3 \cdot 7^{x+1}$.

27. The logistic population growth model is described by an equation of the form

$$P(t) = \frac{P_L}{1 - ce^{-kt}},$$

where P_L, c, and $k > 0$ are constants, and $P(t)$ is the population at time t. P_L represents the limiting value of the population since $\lim_{t\to\infty} P(t) = P_L$. Use the census data for the years 1950, 1960, and 1970 listed in the table on page 104 to determine the constants P_L, c, and k for a logistic growth model. Use the logistic model to predict the population of the United States in 1980 and in 2010, assuming $t = 0$ at 1950. Compare the 1980 prediction to the actual value.

28. The Gompertz population growth model is described by

$$P(t) = P_L e^{-ce^{-kt}},$$

where P_L, c, and $k > 0$ are constants, and $P(t)$ is the population at time t. Repeat Exercise 27 using the Gompertz growth model in place of the logistic model.

29. Player A will shut out (win by a score of 21–0) player B in a game of racquetball with probability

$$P = \frac{1+p}{2}\left(\frac{p}{1-p+p^2}\right)^{21},$$

where p denotes the probability A will win any specific rally (independent of the server). (See [Keller, J], p. 267.) Determine, to within 10^{-3}, the minimal value of p that will ensure that A will shut out B in at least half the matches they play.

30. In the design of all-terrain vehicles, it is necessary to consider the failure of the vehicle when attempting to negotiate two types of obstacles. One type of failure is called *hang-up failure* and occurs when the vehicle attempts to cross an obstacle that causes the bottom of the vehicle to touch the ground. The other type of failure is called *nose-in failure* and occurs when the vehicle descends into a ditch and its nose touches the ground.

The accompanying figure, adapted from [Bek], shows the components associated with the nose-in failure of a vehicle. In that reference it is shown that the maximum angle α that