Suppose $L = 10$ ft, $r = 1$ ft, and $V = 12.4$ ft^3. Find the depth of water in the trough to within 0.01 ft.

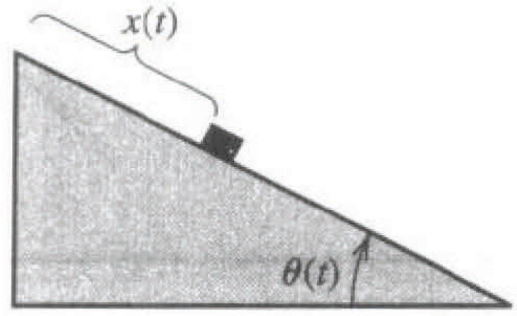
18. A particle starts at rest on a smooth inclined plane whose angle θ is changing at a constant rate

$$\frac{d\theta}{dt} = \omega < 0.$$

At the end of t seconds, the position of the object is given by

$$x(t) = -\frac{g}{2\omega^2}\left(\frac{e^{wt} - e^{-wt}}{2} - \sin \omega t\right).$$

Suppose the particle has moved 1.7 ft in 1 s. Find, to within 10^{-5}, the rate ω at which θ changes. Assume that $g = 32.17$ ft/s^2.



2.2 Fixed-Point Iteration

A number p is a **fixed point** for a given function g if $g(p) = p$. In this section we consider the problem of finding solutions to fixed-point problems and the connection between the fixed-point problems and the root-finding problems we wish to solve.

Root-finding problems and fixed-point problems are equivalent classes in the following sense:

> Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example, as $g(x) = x - f(x)$ or as $g(x) = x + 3f(x)$. Conversely, if the function g has a fixed point at p, then the function defined by $f(x) = x - g(x)$ has a zero at p.

Although the problems we wish to solve are in the root-finding form, the fixed-point form is easier to analyze, and certain fixed-point choices lead to very powerful root-finding techniques.

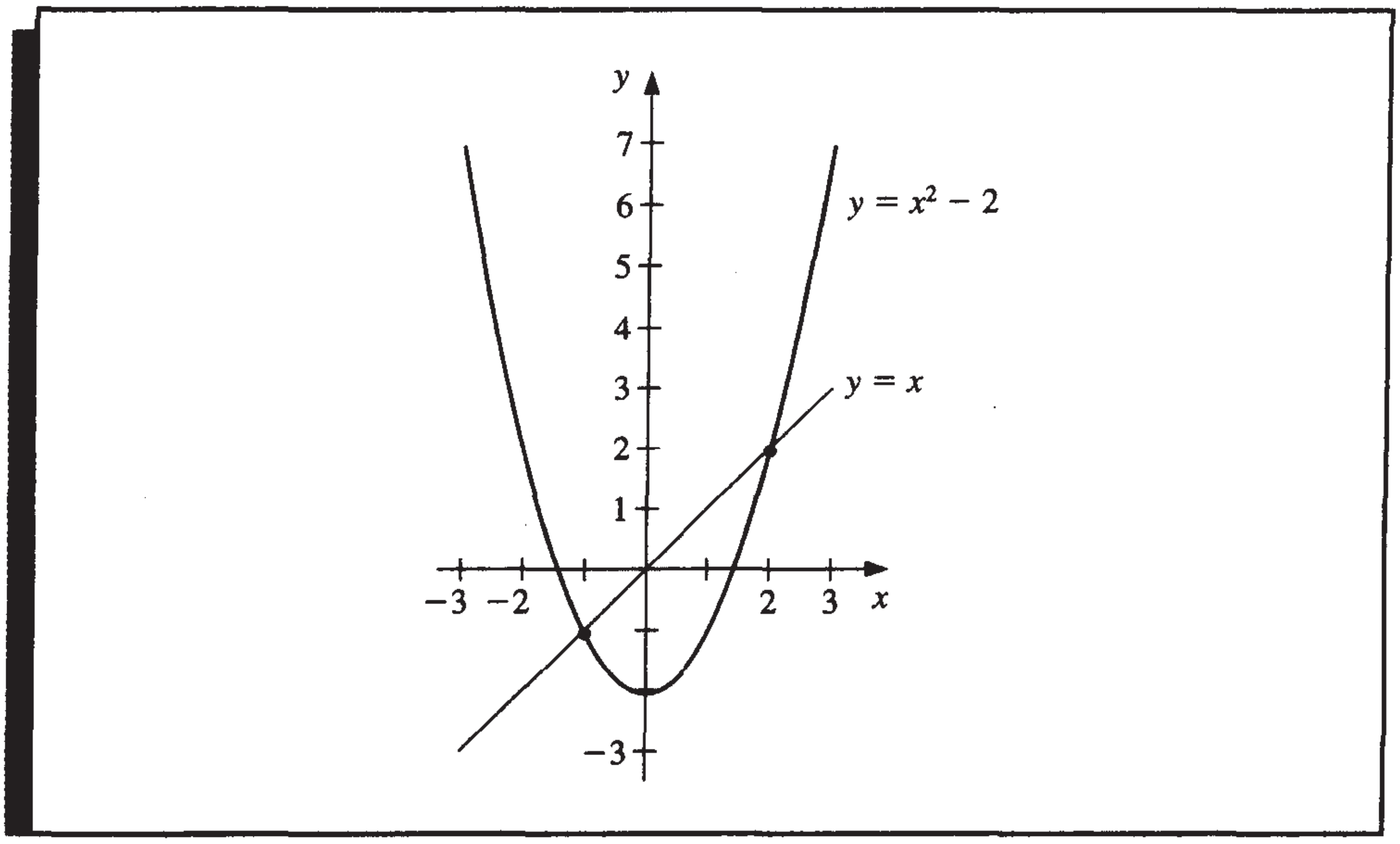
We first need to become comfortable with this new type of problem and to decide when a function has a fixed point and how the fixed points can be approximated to within a specified accuracy.

EXAMPLE 1 The function $g(x) = x^2 - 2$, for $-2 \le x \le 3$, has fixed points at $x = -1$ and $x = 2$ since

$$g(-1) = (-1)^2 - 2 = -1 \quad \text{and} \quad g(2) = 2^2 - 2 = 2.$$

This can be seen in Figure 2.2. ■

Figure 2.2



The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

Theorem 2.2

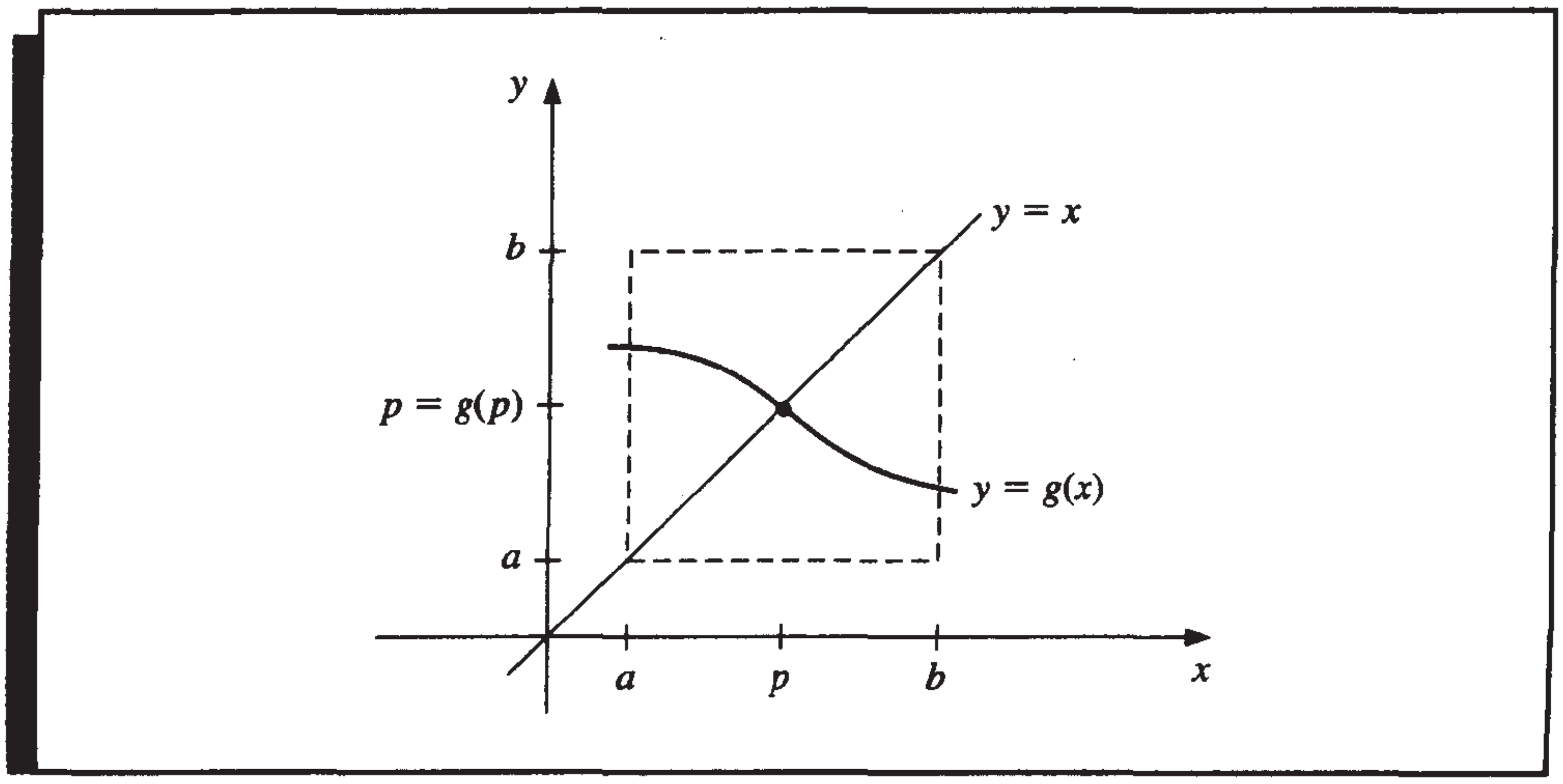
a. If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.

b. If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b),$$

then the fixed point in $[a, b]$ is unique. (See Figure 2.3.) ■

Figure 2.3



Proof

a. If $g(a) = a$ or $g(b) = b$, then g has a fixed point at an endpoint. If not, then $g(a) > a$ and $g(b) < b$. The function $h(x) = g(x) - x$ is continuous on $[a, b]$, with

$$h(a) = g(a) - a > 0 \quad \text{and} \quad h(b) = g(b) - b < 0.$$

The Intermediate Value Theorem implies that there exists $p \in (a, b)$ for which $h(p) = 0$. This number p is a fixed point for g since

$$0 = h(p) = g(p) - p \quad \text{implies that} \quad g(p) = p.$$

b. Suppose, in addition, that $|g'(x)| \leq k < 1$ and that p and q are both fixed points in $[a, b]$. If $p \neq q$, then the Mean Value Theorem implies that a number ξ exists between p and q, and hence in $[a, b]$, with

$$\frac{g(p) - g(q)}{p - q} = g'(\xi).$$

Thus,

$$|p - q| = |g(p) - g(q)| = |g'(\xi)||p - q| \leq k|p - q| < |p - q|,$$

which is a contradiction. This contradiction must come from the only supposition, $p \neq q$. Hence, $p = q$ and the fixed point in $[a, b]$ is unique. ■ ■ ■

EXAMPLE 2

a. Let $g(x) = (x^2 - 1)/3$ on $[-1, 1]$. The Extreme Value Theorem implies that the absolute minimum of g occurs at $x = 0$ and $g(0) = -\frac{1}{3}$. Similarly, the absolute maximum of g occurs at $x = \pm 1$ and has the value $g(\pm 1) = 0$. Moreover, g is continuous and

$$|g'(x)| = \left|\frac{2x}{3}\right| \leq \frac{2}{3}, \quad \text{for all } x \in (-1, 1).$$

So g satisfies all the hypotheses of Theorem 2.2 and has a unique fixed point in $[-1, 1]$.

In this example, the unique fixed point p in the interval $[-1, 1]$ can be determined algebraically. If

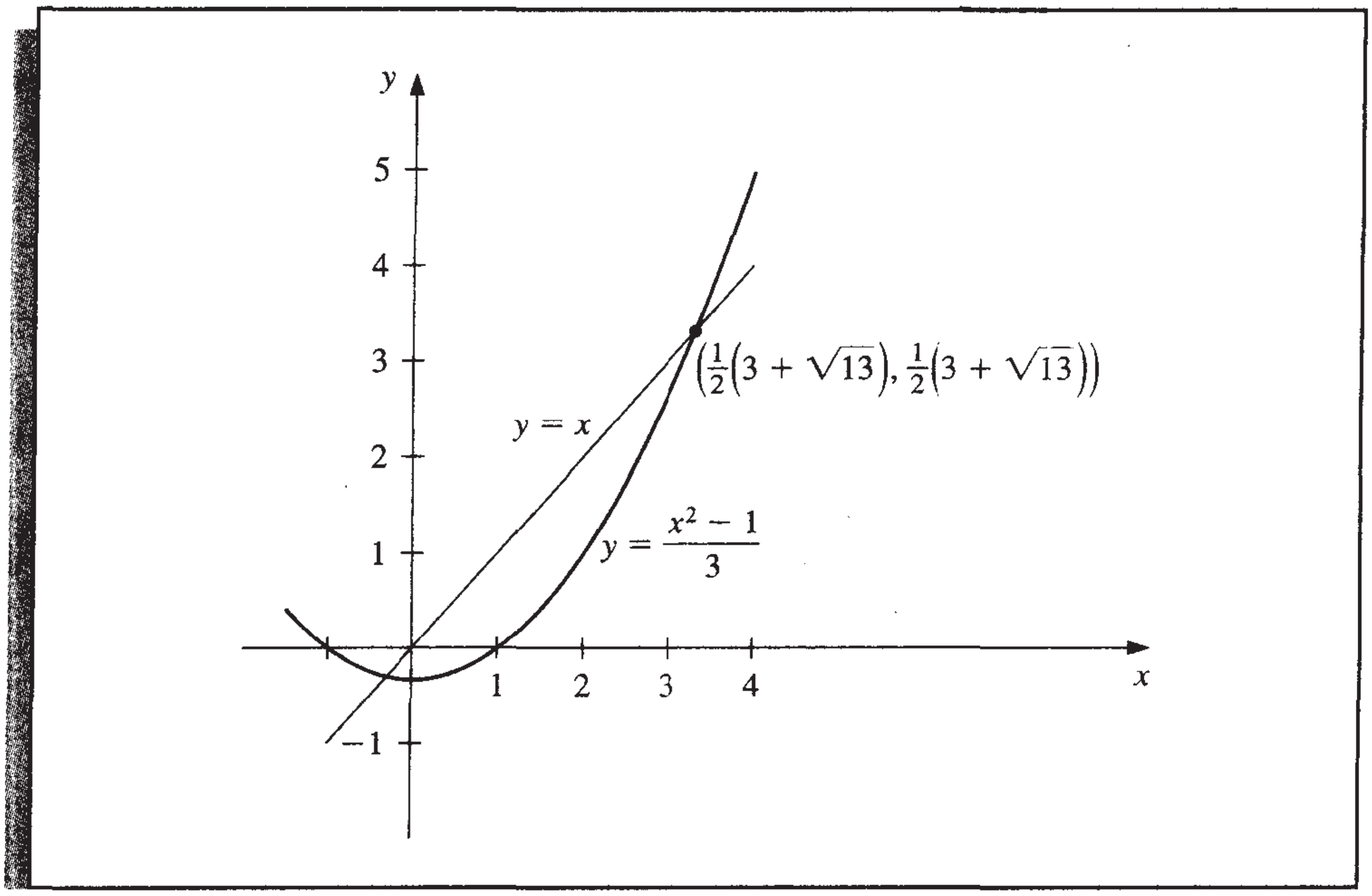
$$p = g(p) = \frac{p^2 - 1}{3}, \quad \text{then} \quad p^2 - 3p - 1 = 0,$$

which, by the quadratic formula, implies that

$$p = \frac{1}{2}(3 - \sqrt{13}).$$

Note that g also has a unique fixed point $p = \frac{1}{2}(3 + \sqrt{13})$ for the interval $[3, 4]$. However, $g(4) = 5$ and $g'(4) = \frac{8}{3} > 1$, so g does not satisfy the hypotheses of Theorem 2.2 on $[3, 4]$. Hence, the hypotheses of Theorem 2.2 are sufficient to guarantee a unique fixed point but are not necessary. (See Figure 2.4.)

Figure 2.4



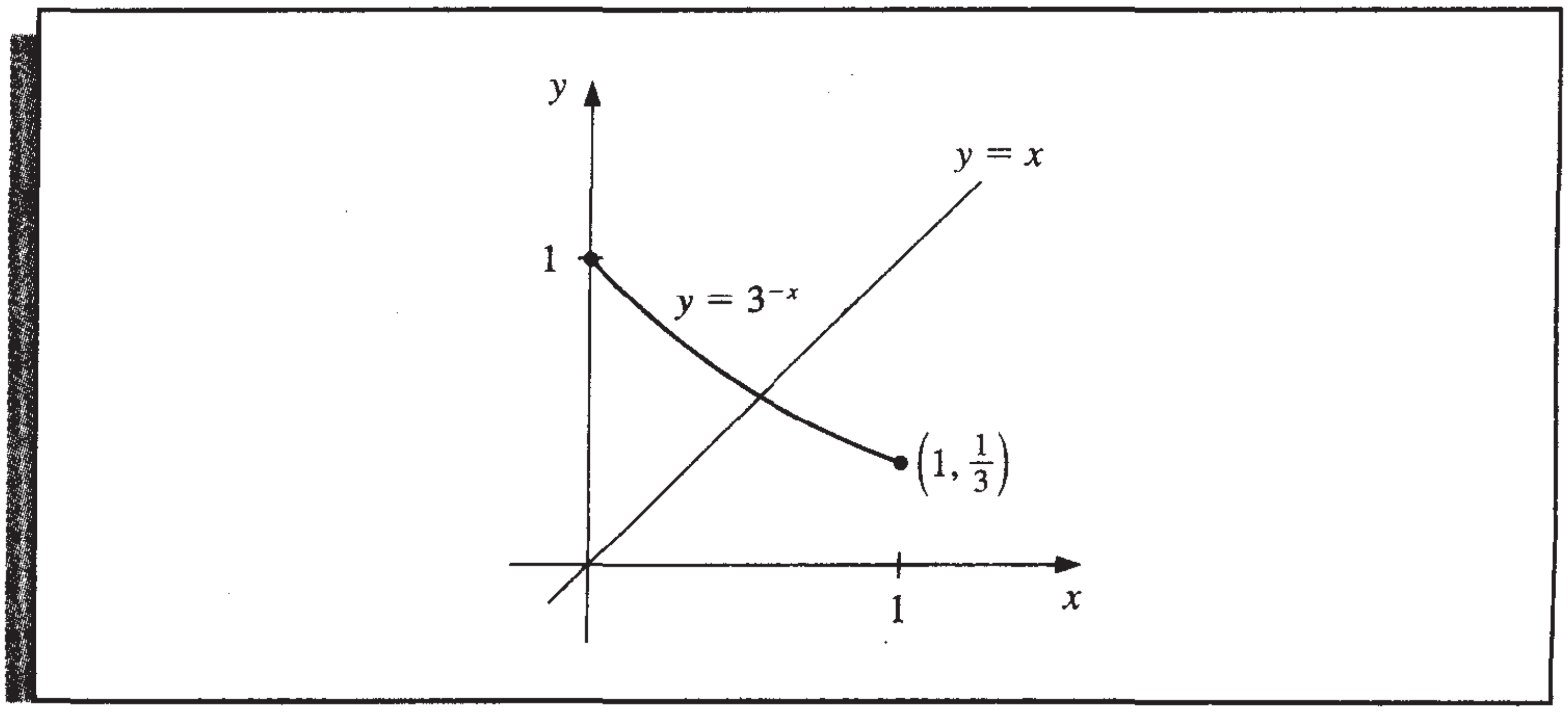
b. Let $g(x) = 3^{-x}$. Since $g'(x) = -3^{-x} \ln 3 < 0$ on $[0, 1]$, the function g is decreasing on $[0, 1]$. So

$$g(1) = \frac{1}{3} \le g(x) \le 1 = g(0), \quad \text{for} \quad 0 \le x \le 1.$$

Thus, for $x \in [0, 1]$, we have $g(x) \in [0, 1]$, and g has a fixed point in $[0, 1]$. Since

$$g'(0) = -\ln 3 = -1.098612289,$$

Figure 2.5



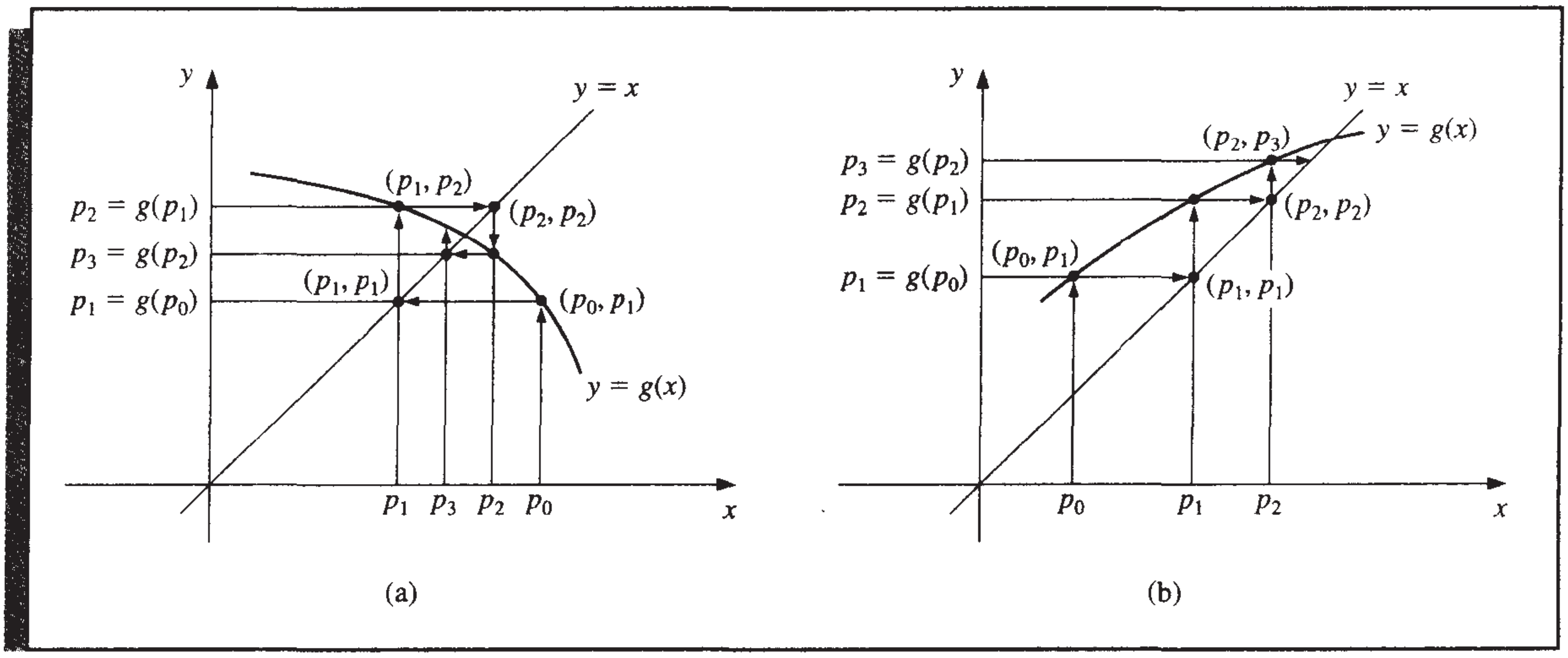
$|g'(x)| \not\leq 1$ on $(0, 1)$, and Theorem 2.2 cannot be used to determine uniqueness. However, g is always decreasing, and it is clear from Figure 2.5 that the fixed point must be unique. ■

To approximate the fixed point of a function g, we choose an initial approximation p_0 and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by letting $p_n = g(p_{n-1})$, for each $n \geq 1$. If the sequence converges to p and g is continuous, then

$$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g\left(\lim_{n\to\infty} p_{n-1}\right) = g(p),$$

and a solution to $x = g(x)$ is obtained. This technique is called **fixed-point iteration**, or **functional iteration**. The procedure is detailed in Algorithm 2.2 and illustrated in Figure 2.6.

Figure 2.6



ALGORITHM **2.2**

Fixed-Point Iteration

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

Step 3 Set $p = g(p_0)$. *(Compute p_i.)*

Step 4 If $|p - p_0| < TOL$ then
OUTPUT (p); *(The procedure was successful.)*

STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, N_0 =', N_0);
(*The procedure was unsuccessful.*)
STOP. ■

The following example illustrates functional iteration.

EXAMPLE 3 The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in $[1, 2]$. There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation. For example, to obtain the function g described in part (c), we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ as follows:

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3),$$

and

$$x = \pm\frac{1}{2}(10 - x^3)^{1/2}.$$

To obtain a positive solution, $g_3(x)$ is chosen. It is not important to derive the functions shown here, but you should verify that the fixed point of each is actually a solution to the original equation, $x^3 + 4x^2 - 10 = 0$.

a. $x = g_1(x) = x - x^3 - 4x^2 + 10$

b. $x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2}$

c. $x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$

d. $x = g_4(x) = \left(\frac{10}{4 + x}\right)^{1/2}$

e. $x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

With $p_0 = 1.5$, Table 2.2 lists the results of the fixed-point iteration for all five choices of g.

The actual root is 1.365230013, as was noted in Example 1 of Section 2.1. Comparing the results to the Bisection Algorithm given in that example, it can be seen that excellent results have been obtained for choices (c), (d), and (e), since the Bisection method requires 27 iterations for this accuracy. It is interesting to note that choice (a) was divergent and that (b) became undefined because it involved the square root of a negative number. ■

Even though the various functions in Example 3 are fixed-point problems for the same root-finding problem, they differ vastly as techniques for approximating the solution to

Table 2.2

n	(a)	(b)	(c)	(d)	(e)
0	1.5	1.5	1.5	1.5	1.5
1	−0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	−469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.367846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

the root-finding problem. Their purpose is to illustrate the true question that needs to be answered:

> How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

The following theorem and its corollary give us some clues concerning the paths we should pursue and, perhaps more importantly, some we should reject.

Theorem 2.3 **(Fixed-Point Theorem)**

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \le k, \quad \text{for all } x \in (a, b).$$

Then, for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \ge 1,$$

converges to the unique fixed point p in $[a, b]$. ∎

Proof Theorem 2.2 implies that a unique fixed point exists in $[a, b]$. Since g maps $[a, b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \ge 0$, and $p_n \in [a, b]$ for all n. Using the fact that $|g'(x)| \le k$ and the Mean Value Theorem, we have, for each n,

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(\xi_n)||p_{n-1} - p| \le k|p_{n-1} - p|,$$

where $\xi_n \in (a, b)$. Applying this inequality inductively gives

$$|p_n - p| \le k|p_{n-1} - p| \le k^2|p_{n-2} - p| \le \cdots \le k^n|p_0 - p|. \tag{2.4}$$

Since $0 < k < 1$, we have $\lim_{n\to\infty} k^n = 0$ and

$$\lim_{n\to\infty} |p_n - p| \le \lim_{n\to\infty} k^n |p_0 - p| = 0.$$

Hence, $\{p_n\}_{n=0}^{\infty}$ converges to p. ■ ■ ■

Corollary 2.4 If g satisfies the hypotheses of Theorem 2.3, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \le k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \le \frac{k^n}{1-k} |p_1 - p_0|, \quad \text{for all} \quad n \ge 1. \qquad ■$$

Proof Since $p \in [a, b]$, the first bound follows from Inequality (2.4):

$$|p_n - p| \le k^n |p_0 - p| \le k^n \max\{p_0 - a, b - p_0\}.$$

For $n \ge 1$, the procedure used in the proof of Theorem 2.3 implies that

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})| \le k|p_n - p_{n-1}| \le \cdots \le k^n |p_1 - p_0|.$$

Thus, for $m > n \ge 1$,

$$\begin{aligned}
|p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - \cdots + p_{n+1} - p_n| \\
&\le |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \cdots + |p_{n+1} - p_n| \\
&\le k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + \cdots + k^n |p_1 - p_0| \\
&= k^n |p_1 - p_0| \left(1 + k + k^2 + \cdots + k^{m-n-1}\right).
\end{aligned}$$

By Theorem 2.3, $\lim_{m\to\infty} p_m = p$, so

$$|p - p_n| = \lim_{m\to\infty} |p_m - p_n| \le \lim_{m\to\infty} k^n |p_1 - p_0| \sum_{i=0}^{m-n-1} k^i \le k^n |p_1 - p_0| \sum_{i=0}^{\infty} k^i.$$

But $\sum_{i=0}^{\infty} k^i$ is a geometric series with ratio k and $0 < k < 1$. This sequence converges to $1/(1-k)$, which gives the second bound:

$$|p - p_n| \le \frac{k^n}{1-k} |p_1 - p_0|. \qquad ■ ■ ■$$

Both inequalities in the corollary relate the rate at which $\{p_n\}_{n=0}^{\infty}$ converges to the bound k on the first derivative. The rate of convergence depends on the factor k^n. The smaller the value of k, the faster the convergence, which may be very slow if k is close to 1. In the following example, the fixed-point methods in Example 3 are reconsidered in light of the results presented in Theorem 2.3 and its corollary.

EXAMPLE 4

a. For $g_1(x) = x - x^3 - 4x^2 + 10$, we have $g_1(1) = 6$ and $g_1(2) = -12$, so g_1 does not map $[1, 2]$ into itself. Moreover, $g_1'(x) = 1 - 3x^2 - 8x$, so $|g_1'(x)| > 1$ for all x in $[1, 2]$. Although Theorem 2.3 does not guarantee that the method must fail for this choice of g, there is no reason to expect convergence.

b. With $g_2(x) = [(10/x) - 4x]^{1/2}$, we can see that g_2 does not map $[1, 2]$ into $[1, 2]$, and the sequence $\{p_n\}_{n=0}^{\infty}$ is not defined when $p_0 = 1.5$. Moreover, there is no interval containing $p \approx 1.365$ such that

$$|g_2'(x)| < 1, \quad \text{since} \quad |g_2'(p)| \approx 3.4.$$

There is no reason to expect that this method will converge.

c. For the function $g_3(x) = \frac{1}{2}(10 - x^3)^{1/2}$,

$$g_3'(x) = -\frac{3}{4}x^2(10 - x^3)^{-1/2} < 0 \quad \text{on } [1, 2],$$

so g_3 is strictly decreasing on $[1, 2]$. However, $|g_3'(2)| \approx 2.12$, so the condition $|g_3'(x)| \leq k < 1$ fails on $[1, 2]$. A closer examination of the sequence $\{p_n\}_{n=0}^{\infty}$ starting with $p_0 = 1.5$ shows that it suffices to consider the interval $[1, 1.5]$ instead of $[1, 2]$. On this interval it is still true that $g_3'(x) < 0$ and g_3 is strictly decreasing, but, additionally,

$$1 < 1.28 \approx g_3(1.5) \leq g_3(x) \leq g_3(1) = 1.5,$$

for all $x \in [1, 1.5]$. This shows that g_3 maps the interval $[1, 1.5]$ into itself. Since it is also true that $|g_3'(x)| \leq |g_3'(1.5)| \approx 0.66$ on this interval, Theorem 2.3 confirms the convergence of which we were already aware.

d. For $g_4(x) = (10/(4 + x))^{1/2}$, we have

$$|g_4'(x)| = \left|\frac{-5}{\sqrt{10}(4 + x)^{3/2}}\right| \leq \frac{5}{\sqrt{10}(5)^{3/2}} < 0.15, \quad \text{for all} \quad x \in [1, 2].$$

The bound on the magnitude of $g_4'(x)$ is much smaller than the bound (found in (c)) on the magnitude of $g_3'(x)$, which explains the more rapid convergence using g_4.

e. The sequence defined by

$$g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x},$$

converges much more rapidly than our other choices. In the next sections we will see where this choice came from and why it is so effective. ■

EXERCISE SET 2.2

1. Use algebraic manipulation to show that each of the following functions has a fixed point at p precisely when $f(p) = 0$, where $f(x) = x^4 + 2x^2 - x - 3$.

   a. $g_1(x) = \left(3 + x - 2x^2\right)^{1/4}$

   b. $g_2(x) = \left(\frac{x + 3 - x^4}{2}\right)^{1/2}$

c. $g_3(x) = \left(\dfrac{x+3}{x^2+2}\right)^{1/2}$ d. $g_4(x) = \dfrac{3x^4+2x^2+3}{4x^3+4x-1}$

2. a. Perform four iterations, if possible, on each of the functions g defined in Exercise 1. Let $p_0 = 1$ and $p_{n+1} = g(p_n)$, for $n = 0, 1, 2, 3$.
   b. Which function do you think gives the best approximation to the solution?
3. The following four methods are proposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.
   a. $p_n = \dfrac{20p_{n-1} + 21/p_{n-1}^2}{21}$ b. $p_n = p_{n-1} - \dfrac{p_{n-1}^3 - 21}{3p_{n-1}^2}$
   c. $p_n = p_{n-1} - \dfrac{p_{n-1}^4 - 21p_{n-1}}{p_{n-1}^2 - 21}$ d. $p_n = \left(\dfrac{21}{p_{n-1}}\right)^{1/2}$
4. The following four methods are proposed to compute $7^{1/5}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.
   a. $p_n = \left(1 + \dfrac{7 - p_{n-1}^3}{p_{n-1}^2}\right)^{1/2}$ b. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{p_{n-1}^2}$
   c. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{5p_{n-1}^4}$ d. $p_n = p_{n-1} - \dfrac{p_{n-1}^5 - 7}{12}$
5. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^4 - 3x^2 - 3 = 0$ on $[1, 2]$. Use $p_0 = 1$.
6. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.
7. Use Theorem 2.2 to show that $g(x) = \pi + 0.5\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration to find an approximation to the fixed point that is accurate to within 10^{-2}. Use Corollary 2.4 to estimate the number of iterations required to achieve 10^{-2} accuracy, and compare this theoretical estimate to the number actually needed.
8. Use Theorem 2.2 to show that $g(x) = 2^{-x}$ has a unique fixed point on $[\frac{1}{3}, 1]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-4}. Use Corollary 2.4 to estimate the number of iterations required to achieve 10^{-4} accuracy, and compare this theoretical estimate to the number actually needed.
9. Use a fixed-point iteration method to find an approximation to $\sqrt{3}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 10 of Section 2.1.
10. Use a fixed-point iteration method to find an approximation to $\sqrt[3]{25}$ that is accurate to within 10^{-4}. Compare your result and the number of iterations required with the answer obtained in Exercise 11 of Section 2.1.
11. For each of the following equations, determine an interval $[a, b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5}, and perform the calculations.
    a. $x = \dfrac{2 - e^x + x^2}{3}$ b. $x = \dfrac{5}{x^2} + 2$
    c. $x = (e^x/3)^{1/2}$ d. $x = 5^{-x}$
    e. $x = 6^{-x}$ f. $x = 0.5(\sin x + \cos x)$
12. For each of the following equations, determine a function g and an interval $[a, b]$ on which fixed-point iteration will converge to a positive solution of the equation.
    a. $3x^2 - e^x = 0$ b. $x - \cos x = 0$

    Find the solutions to within 10^{-5}.

13. Find all the zeros of $f(x) = x^2 + 10\cos x$ by using the fixed-point iteration method for an appropriate iteration function g. Find the zeros accurate to within 10^{-4}.

14. Use a fixed-point iteration method to determine a solution accurate to within 10^{-4} for $x = \tan x$, for x in $[4, 5]$.

15. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $2\sin \pi x + x = 0$ on $[1, 2]$. Use $p_0 = 1$.

16. Let A be a given positive constant and $g(x) = 2x - Ax^2$.

  a. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = 1/A$, so the inverse of a number can be found using only multiplications and subtractions.

  b. Find an interval about $1/A$ for which fixed-point iteration converges, provided p_0 is in that interval.

17. Find a function g defined on $[0, 1]$ that satisfies none of the hypotheses of Theorem 2.2 but still has a unique fixed point on $[0, 1]$.

18. **a.** Show that Theorem 2.2 is true if the inequality $|g'(x)| \le k$ is replaced by $g'(x) \le k$, for all $x \in (a, b)$. [*Hint:* Only uniqueness is in question.]

  b. Show that Theorem 2.3 may not hold if inequality $|g'(x)| \le k$ is replaced by $g'(x) \le k$. [*Hint:* Show that $g(x) = 1 - x^2$, for x in $[0, 1]$, provides a counterexample.]

19. **a.** Use Theorem 2.3 to show that the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \quad \text{for } n \ge 1,$$

  converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

  b. Use the fact that $0 < (x_0 - \sqrt{2})^2$ whenever $x_0 \ne \sqrt{2}$ to show that if $0 < x_0 < \sqrt{2}$, then $x_1 > \sqrt{2}$.

  c. Use the results of parts (a) and (b) to show that the sequence in (a) converges to $\sqrt{2}$ whenever $x_0 > 0$.

20. **a.** Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \ge 1,$$

  converges to $\sqrt{A}$ whenever $x_0 > 0$.

  b. What happens if $x_0 < 0$?

21. Replace the assumption in Theorem 2.3 that "a positive number $k < 1$ exists with $|g'(x)| \le k$" with "g satisfies a Lipschitz condition on the interval $[a, b]$ with Lipschitz constant $L < 1$." (See Exercise 25, Section 1.1.) Show that the conclusions of this theorem are still valid.

22. Suppose that g is continuously differentiable on some interval (c, d) that contains the fixed point p of g. Show that if $|g'(p)| < 1$, then there exists a $\delta > 0$ such that if $|p_0 - p| \le \delta$, then the fixed-point iteration converges.

23. An object falling vertically through the air is subjected to viscous resistance as well as to the force of gravity. Assume that an object with mass m is dropped from a height s_0 and that the height of the object after t seconds is

$$s(t) = s_0 - \frac{mg}{k}t + \frac{m^2 g}{k^2}(1 - e^{-kt/m}),$$

where $g = 32.17$ ft/s^2 and k represents the coefficient of air resistance in lb-s/ft. Suppose $s_0 = 300$ ft, $m = 0.25$ lb, and $k = 0.1$ lb-s/ft. Find, to within 0.01 s, the time it takes this quarter-pounder to hit the ground.